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Convergent polynomial expansion, lines of zeros, and slopes of diffraction scattering. II. Oscillation of the slope parameter in $\pi^- p$ scattering

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It is shown that the simple polynomial approximation of transversity amplitudes using complex zeros of Barrelet type cannot account for the slope-parameter data at any of the energy ranges except for providing only a qualitative explanation of the bump structure around $s \simeq 3$ GeV^2. At high energies experimentally observed linearly increasing or decreasing trajectories can never account for the slope-parameter data. These results reflect the fact that the sharp forward peak cannot be represented by finite-order polynomials in the conventional scheme of parametrization. Using Mandelstam analyticity and information on the real and complex zeros we propose a new scheme of parametrization to understand the energy dependence of slope-parameter data for all energies such that the problem of a dynamical understanding of diffraction scattering reduces to the problem of providing a dynamical origin of zero trajectories. This leads to a formula that relates the slope parameter with equations to boundaries of spectral functions and real and complex zero trajectories. An excellent description of the data is obtained for all energies with $s > 1.44$ GeV^2 using theoretical elastic boundaries of spectral functions, experimentally determined real and complex zero trajectories in a limited energy range, and their realistic extrapolations for higher and lower energies. The slope-parameter data are found to be very sensitive to small variations of the real trajectory around its extrapolated value at low energies. Using our formula, real zero trajectory in the range $1.33 \leq s \leq 1.62$ GeV^2 has been determined from the slope-parameter data. From the structures observed in the slope-parameter data we make some qualitative predictions of complex trajectories at high energies. Contrary to the general supposition that the forward peak cannot be represented by true zeros of the amplitude, our new scheme of parametrization strongly suggests that the diffractive parts of transversity amplitudes possess all the complex zeros and the real zero existing away from the forward-peak region.

I. INTRODUCTION

Although many theories have been proposed[1] to explain hadron-hadron collision processes only a few have been successful in explaining shrinkage-antishrinkage of forward peaks in diffraction scattering. A brief review of experimental data on slope parameters and the present status of theory has been reported in Ref. 2 (hereinafter referred as paper I). In particular the oscillatory pattern of the slope-parameter data[3] in $\pi^\pm p$ and $K^- p$ scattering and the antishrinkage of $\bar{p}p$ scattering have eluded predictions of many models. The only fit to the oscillatory pattern, proposed by Barger and Cline,[4] by an empirical merger of Regge and geometrical models, has been strongly criticized by Weare[5] who has pointed out that the model is applicable at energies much higher than the range where the oscillatory pattern is observed. Many other theories[6,7,8] which have been developed for asymptotic energies, but attempt to explain the slope-parameter data at lower energies suffer, more or less, from the same type of criticism advanced by Weare.[5]

Apart from providing very important informations from data analysis,[9] the optimized polynomial expansion (OPE)[10] has been used to propose phenomenological formulas for diffraction scattering.[2,6,7,11,12] But the formula for the slope parameter proposed in Ref. 7 could account for shrinkage only. However, in I a convergent polynomial expansion (CPE) was developed for all energies. The fundamental requirement for the realization of CPE for all energies was shown to be the existence of at least one real zero on the physical region in the $x = \cos\theta$ plane. This led to a new formula which related the slope parameter with equations to real zero trajectories[13] and boundaries of spectral functions. Such a formula has provided a unified description of the slope-parameter data for several diffraction scattering processes. Whereas the criticism of the type advanced by Weare[5] holds against the theories of Barger and Cline,[4] Deo and Parida,[6,7] and Leader and Pennington,[8] CPE of I is applicable for all energies and is free of such criticism. But the formula in I could not describe adequately the oscillatory pattern,[3] although it reproduced a good average of the data for all energies. In particular the observed oscillations at low and intermediate energies including the bump around $s = 3$ GeV^2 could not be fitted by the formula even after using four parameters one of which yields an effective boundary of spectral function retreating away from the theoretical elastic boundary, a feature unwanted by the S-matrix theory. However, it was conjectured[2] intuitively that the use of complex zero trajectories[14] might yield a correct account of the oscillatory pattern.

In this paper we attempt to understand the observed variation of slope parameter in $\pi^- p$ scattering in terms of real and complex zero trajectories and theoretical elastic boundaries, such that understanding diffraction scattering reduces to the problem of understanding the dynamical origin of zero trajectories, the dynamical origin of the boundaries of spectral functions being already known. While determining the complex trajectories from the cross-section and polarization data by Barrelet's moment analysis, the sharp forward peak is usually excluded from data analysis.[15] This is further reflected in the fact, which has been demonstrated in Sec. II, that the simple polynomial approximation using recently known complex zero trajectories[16] does not account for the slope-parameter data. Further, it is evident from the expression for slope parameter resulting from such polynomial approximation that the experimentally observed zero trajectories, which are either decreasing or increasing linearly with energy at high energies can never account for the slope-parameter data.

Using CPE of I we propose a new scheme of parametrization for the transversity amplitudes near forward angles such that they possess the same complex zeros of Barrelet type and the real zero in the backward hemisphere, but still exhibit the forward-peak structure and satisfy Mandelstam analyticity in the x plane. This leads to a new formula that relates the slope parameter to boundaries of spectral functions and real and complex zero trajectories. Using the available data on the real and complex zero trajectories[16] in a limited energy range and their realistic extrapolations to higher and lower energies, and theoretical elastic boundaries of spectral functions, the slope-parameter data can be well explained at lower, intermediate, and high energies with one parameter only. At lower energies the formula is very much sensitive to the type of extrapolation used for the real trajectory only. Thus using the slope-parameter data we determine the real zero trajectory for $1.33 \leq s \leq 1.62$ GeV^2, the path of which can be verified by other methods. From the second bump structure around $s \simeq 5.2$ GeV^2 the present analysis predicts emergence of new trajectories with small imaginary parts and/or the existence of critical points in the imaginary parts of two of the already existing trajectories around this region. The long-standing discrepancy of the slope-parameter data in the range $20 < s < 50$ GeV^2 from smooth extrapolation of the data at other energies can be resolved if similarly new trajectories emerge around this energy region. Contrary to the conventional belief that zeros are associated only with nondiffractive parts of the amplitude, our analysis strongly favors the idea that the diffractive parts also possess the same real and complex zeros as the nondiffractive parts.

In Sec. II we briefly review the problem of associating the diffraction peak with zeros of the amplitude. We calculate the contribution due to Barrelet zeros using simple polynomial approximation and demonstrate its disagreement with the data. We also obtain some useful conclusions on the relation of the zero trajectories to the slope-parameter data. Section III deals with the new scheme of parametrization and the derivation of the new formula for slope parameter. In Sec. IV we compare the results of calculation in the new scheme with the slope-parameter data and determine zero trajectories from the data at low energies. Here we also make qualitative predictions about zero trajectories for high energies. In Sec. V we summarize our results and state our conclusions and limitations of this approach

II. BARRELET ZEROS AND THE SLOPE-PARAMETER DATA

As against the real zeros of Odorico type[13] the existence of which, in some cases, get strong support from the Veneziano model, Barrelet[14] established the extence of complex zero trajectories initiated by Gersten.[17] It may be pointed out that in his analysis Odorico[13] attributes the presence of zeros as due to the nondiffractive parts. To obtain zero trajectories from the differential cross section and the polarization data, Barrelet's method of moment analysis[14-18] is applied to the partial angular interval $a < x = \cos\theta < b$ with a weighting norm $n_p(x)$. Barrelet[14] employs a set of pseudopolynomials, such that

$$\int_a^b p_l(x) p_{l'}(x) n_p(x) dx = \delta_{ll'} . \quad (1)$$

The differential cross section in this interval is then represented as

$$\frac{d\sigma}{d\Omega} = \sum_{l=0}^{N_1} A_l p_l(x) , \quad (2)$$

where the moment A_l is determined from the formula

$$A_l = \int_a^b \frac{d\sigma}{d\Omega}(x) p_l(x) n_p(x)\, dx . \quad (3)$$

A similar polynomial expansion as (2) but with different coefficients is also adopted for fitting the data on $P d\sigma/d\Omega$

$$P \frac{d\sigma}{d\Omega} = \sum_{l=0}^{N_2} B_l p_l(x) , \quad (4)$$

where P is the experimental data on polarization.

Here N_1 and N_2 are truncation points in the series such that $A_l(B_l)$ smoothly goes to zero when $l \geq N_1(N_2)$. Combining (2) and (4) a polynomial of degree determined by $\max(N_1, N_2)$ in x is obtained for the transversity corss sections,

$$\Sigma^{\pm} = \frac{d\sigma}{d\Omega}(1 \pm P). \tag{5}$$

The roots of this polynomial in the x plane are then determined numerically which in general turn out to be complex. The loci of real and imaginary parts of these zeros lying inside the Lehmann-Martin ellipse for various energies yield zero-trajectories. Barrelet[14] originally developed his method for individual experiments that cover a finite portion of the total physical interval in x. Barrelet also stressed that the nearby zeros may be unambigously established even on the basis of data which may not have been adequate for a full amplitude analysis.[14,15] Since the forward peak in diffraction scattering depends exponentially on t for any fixed energy, one does not expect it to be represented by finite-order polynomials.[14,15,18] In Barrelet's program[18] based on moment analysis it has been recommended that the extreme forward peak be ignored. It has been[15,18] argued that the sharp forward peak has no relation to the true amplitude zeros and inclusion of this peak in the data analysis can only obscure information residing in other angular regions. Barrelet,[14,15,18] however, also stressed the possibility of a norm that assigns comparable importance to all angular intervals. If such a norm is used for constructing orthogonal polynomials, the diffraction peak may not be an impediment in obtaining informations on zeros from the data analysis. Recently the existence of such a norm has been shown by Chew[15] for π^+p scattering at $P_{1ab} = 1.77$ GeV/c. But even after chosing the norm properly, the degree of polynomial and hence the number of complex zeros have increased from six in Barrelet's work to ten in the work of Chew.[15] Following Barrelet's method of data analysis[14] in a partial interval in x, which explicitly excludes the near forward and the backward peak regions Barrelet *et al.*[16] have obtained two complex zero trajectories for each of the transversity cross sections in the range $2.5 \leq s \leq 4.5$ GeV^2, as shown in Figs. 1 and 2. These authors have also obtained a real zero trajectory of the differential cross section corresponding to the dip position near the backward direction. To find out the real zero trajectory the authors have not used moment analysis, but adopted a local parametrization of the differential cross section near the dip region. Another genuine difficulty for excluding the extreme forward and backward regions from moment analysis may be due to lack of polarization

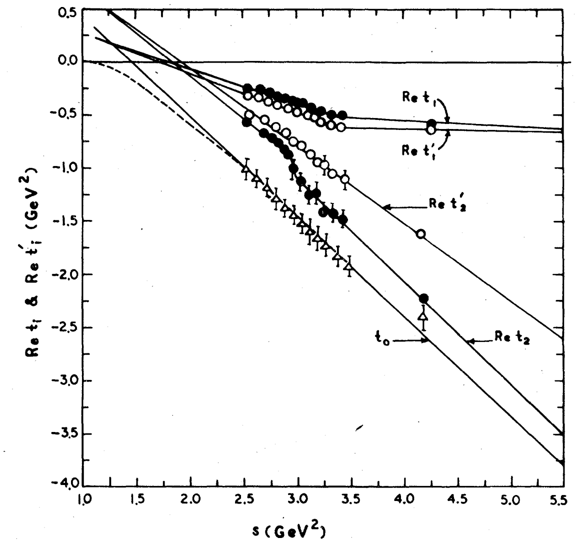

FIG. 1. Real parts of zero trajectories as a function of s. The data points are from Barrelet *et al.* (Ref. 16) and the solid lines are the extrapolated curves used in the calculation. The dotted line is the physical region boundary corresponding to the backward direction.

data in these regions. With the knowledge of the zeros of an analytic function one can construct a polynomial approximation for it. As stated above if some portion of the forward peak has been excluded from moment analysis, the simple polynomial approximation constructed out of complex zeros determined by moment analysis, may not adequately extrapolate well into the forward peak region. To check this we calculate the contribution to the slope-parameter data by the polynomial approximations using complex zeros obtained by Barrelet *et al.*[16] Recently Carter[19] has deter-

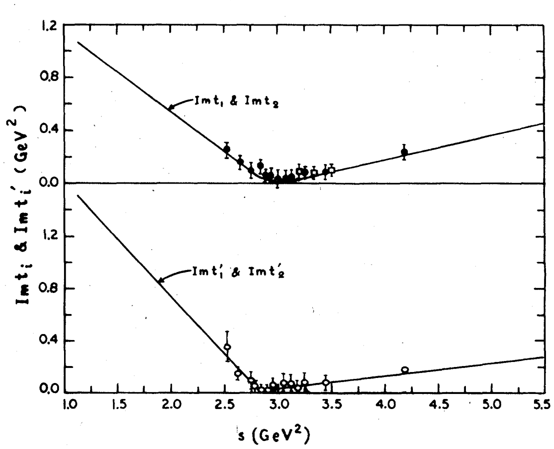

FIG. 2. Imaginary parts of zero trajectories as a function of s. The data points are from Barrelet *et al.* (Ref. 16) and the solid lines are the extrapolated curves used in the calculation. The meaning of different data points has been clarified in Ref. 16.

mined complex zero trajectories of differential cross sections for several processes. At present we will use only Barrelet-type complex zeros which suit our purpose best.

Defining $F^{\pm}(s,x)$ to be the transversity amplitudes[14,16] which are related to $\Sigma^{\pm}$ by the relations

$$\Sigma^{\pm}(s,x)=|F^{\pm}(s,x)|^2, \tag{6}$$

and using only the complex zero trajectories[20] of Ref. 16, we write

$$F^{+}(s,x)=A(s)\prod_{i=1}^{2}(x-x_i), \tag{7}$$

$$F^{-}(s,x)=B(s)\prod_{i=1}^{2}(x-x_i'), \tag{8}$$

where $x_i(x_i')$, with $i=1,2$, are the complex zeros of $F^{+}(F^{-})$ in the x plane whose positions are dependent upon s. For simplicity we have neglected possible phase factors which, however, do not affect our analysis. The simple polynomial approximations given by (7) and (8) represent the data on Σ^{+} and Σ^{-} correctly in the partial angular interval.[20] The slope of the forward peak is defined as

$$b(s)=\frac{d}{dt}\ln\frac{d\sigma}{dt}\bigg|_{t=0} \tag{9}$$

Using formulas (6)–(9), the formula

$$\frac{d\sigma}{dt}=\frac{\pi}{2q^2}(\Sigma^{+}+\Sigma^{-}) \tag{10}$$

and the normalization condition

$$\Sigma^{+}|_{t=0}=\Sigma^{-}|_{t=0}=\frac{d\sigma}{d\Omega}\bigg|_{t=0} \tag{11}$$

leads to the following simple expression for the slope parameter:

$$b(s)=-\sum_{i=1}^{2}\left[\frac{\mathrm{Re}\,t_i(s)}{|t_i(s)|^2}+\frac{\mathrm{Re}\,t_i'(s)}{|t_i'(s)|^2}\right] \tag{12}$$

where we have used

$$\begin{aligned}\mathrm{Re}\,t_i(s)&=-2q^2[1-\mathrm{Re}\,x_i(s)],\\ \mathrm{Im}\,t_i(s)&=2q^2\,\mathrm{Im}\,x_i(s),\end{aligned} \tag{13}$$

and similar expressions for $\mathrm{Re}\,t_i(s)$ and $\mathrm{Im}\,t_i(s)$. In the analysis of Barrelet *et al.*[16] there is an ambiguity regarding the sign of the imaginary parts of the trajectories. Formula (12) is not affected by such ambiguity. Using zero trajectories of Barrelet *et al.*[16] we now calculate the right-hand side of (12) for different values of s. Since in the analysis of Ref. 16 zero trajectories have been determined in a limited energy range, we use linear extrapolations of trajectories for higher and lower energies as shown in Figs. 1 and 2. Judging from the trend of the data these extrapolations

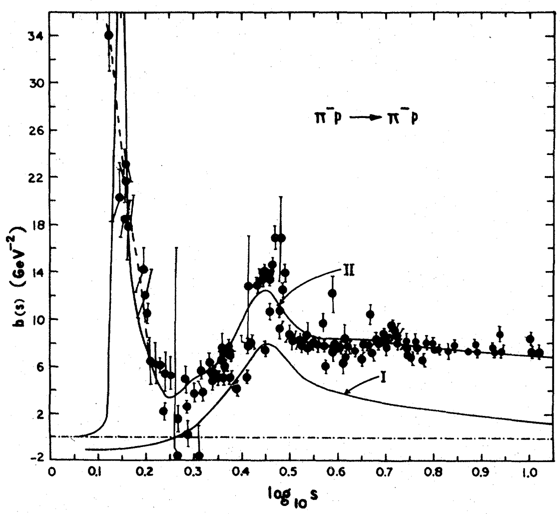

FIG. 3. Fit to the slope-parameter data in the range $1.33<s<10$ GeV^2. Curve I shows the fit by the formula (12), and curve II is the fit by (21) described in the text. The dotted curve in the range $1.33<s<1.62$ GeV^2 is the continuation of the curve II which has been used to extract the real trajectory $t_0(s)$ plotted as curve II in Fig. 4.

are not unrealistic. Contribution of (12) has been plotted as curve I in Figs. 3 for different values of s. The discrepancy of the calculated values from the experimental data exists for all energies and is prominent for high and low energies. Near the bump region around $s\simeq 3$ GeV^2 there is only a qualitative agreement. From this result we conclude that the simple polynomial approximation, using Barrelet-type complex zeros obtained from moment analysis of cross-section data residing in partial angular intervals, does not represent the forward peak correctly.

Looking to the formula (12) it is easy to find out the reason for such discrepancy at high energies. In different diffraction scattering processes the slope parameter at high energies is either constant or increasing logarithmically with energy.[3] At least the real parts of zero trajectories by Barrelet[14] for π^+p scattering, Barrelet *et al.*[16] for π^-p scattering are linearly decreasing functions of s for high energies. The quantity $\mathrm{Re}\,t_i(s)/|t_i(s)|^2$ will decrease as s^{-1} for large s if either of the real and imaginary parts varies linearly as s or both vary linearly as s. Thus linear trajectories as observed experimentally can never account for the slope-parameter data at high energies in the Barrelet type of parametrization.

Although the formula (12) fails to account for the quantitative description of the slope-parameter data, it is very clear from Fig. 2 that curve I has the potentialities to reproduce the bump structure around $s\simeq 3$ GeV^2. In the next section we propose a new scheme of parametrization which apart from exploiting such a potentiality yields a very good description of the data for all energies.

III. CONVERGENT POLYNOMIAL EXPANSION FOR ALL ENERGIES AND A NEW SCHEME OF PARAMETRIZATION

Since the simple polynomial approximations (7) and (8) fail to account for the slope-parameter data, it is natural to suppose that the functions A and B possess additional x dependence at least in the forward peak region. We propose to construct the functions $A(s,x)$ and $B(s,x)$ by the theory of analytic approximations specifically developed in I using Mandelstam analyticity to account for the forward peak structure at all energies.[2] Analytic properties of scattering amplitudes are due to nature of forces responsible for scattering of hadrons. S-matrix theory places maximum emphasis on the nearest singularity. The ratio of the real to the imaginary part of the forward amplitude is known to be small at least for high energies and the absorptive part is mainly responsible for diffraction. In the x plane the domain of analyticity of the absorptive part is different from that of the real part and is decided by the theoretical boundaries of spectral functions computed from box diagrams. Series expansion in orthogonal polynomials[14,16] in x, given by (2) and (4) and also the formulas (7) and (8), do not possess the desired branch-point structures in the x plane.

Being closer to the forward direction the right-hand cut should have more influence over diffraction scattering than the left-hand cut. Moreover, since the diffraction peak is observed even at low energies a convergent polynomial expansion is necessary for all energies. Unlike the earlier works[6,7] CPE in I has been developed satisfying these criteria. The primary requirement for the realization of CPE for all energies was found to be the existence of at least one real zero on the physical region in the x plane. Further the convergence of the polynomial expansion was shown to be faster in the case where the position of the real zero occurs closer to the backward direction, and the position of the left-hand cut lies farther away than the right-hand cut. The existence of the real zero very near the backward direction[16] not only ensures existence of CPE for all energies in $\pi^- p$ scattering, but also the convergence is accelerated because of the closeness of the position of zero to the backward direction and the asymmetry of the cut plane of analyticity. Thus assuming that the diffractive part of transversity amplitudes possess a real zero in the backward hemisphere in addition to possessing the same complex zeros of Ref. 16 and following the method of I, we approximate A and B functions of (7) and (8) by CPE in terms of Laguerre polynomials in the mapped variable z,

$$A(s)\rightarrow A(s,x)=\exp[-\alpha z/2]\sum_n a_n(s)L_n(\alpha z)\,, \tag{14a}$$

$$B(s)\rightarrow B(s,x)=\exp[-\alpha z/2]\sum_n b_n(s)L_n(\alpha z)\,, \tag{14b}$$

where

$$z=g(x)z_0\,, \tag{15a}$$

$$g(x)=\left[\frac{1+x}{x+x_0(s)}\right]^2, \tag{15b}$$

$$z_0=(\cosh^{-1}\sqrt{w_0})^2\,, \tag{15c}$$

$$w_0=\left(\frac{x_-+1}{x_+-1}\right)\left(\frac{x_+-x}{x_-+x}\right). \tag{15d}$$

Here α is any real constant which may, in general, depend upon s, x_+ $(-x_-)$ is the start of the right- (left-) hand cut in the x plane related to the elastic boundaries of spectral functions ρ_{st} (ρ_{su}) by the relations

$$x_+=1+\frac{t_R}{2q^2}\,,\qquad x_-=1+\frac{t_L}{2q^2}-\frac{\Delta}{2q^2s}\,, \tag{16}$$

with $\Delta=(m^2-m_\pi{}^2)^2$. In (16)

$$t_R=\min(t_{1R},t_{2R})\,,\qquad t_L=\min(t_{1L},t_{2L})\,, \tag{17}$$

where t_{iR} (t_{iL}) are the theoretical boundaries of spectral functions ρ_{st} (ρ_{su}), computed from box graphs.[2] It is evident from Eqs. (14) and (15) that $A(s,x)$ and $B(s,x)$ have a common real zero at $x=-x_0(s)$. In the present case we are interested for scattering near forward angles. It is to be noted that even for lower energies and the values of $|t|$ with $|t|\ll t_R$, $z\sim t/t_R$ and there is an additional convergence of the series in (14). Thus for small t we can write

$$A(s,x)\simeq a'(s)\exp(-\alpha z/2)\,, \tag{18a}$$

$$B(s,x)\simeq b'(s)\exp(-\alpha z/2)\,, \tag{18b}$$

where $a'(s)$ and $b'(s)$ are unknown functions of s. Now using (18) in Eqs. (7) and (8), using the relations (6) and (10), we write in the forward-peak region

$$\frac{d\sigma}{dt}\simeq\frac{\pi}{2q^2}|a'(s)|^2\Big[\,|(x-x_1)(x-x_2)|^2+d(s)\,|(x-x_1')(x-x_2')|^2\Big]\times\exp(-\alpha z)\,, \tag{19}$$

where $d(s) = |b'(s)/a'(s)|^2$. Using the normalization condition (11) we determined $d(s)$ in terms of zero trajectories

$$d(s) = \left| \frac{t_1(s)t_2(s)}{t_1'(s)t_2'(s)} \right|^2 . \tag{20}$$

Formulas (19), (20), and the definition (9) then lead to the following expression for the slope parameter in the forward direction:

$$b(s) = \frac{16\alpha q^4}{t_R[t_0(s)]^2}\left(1 + \frac{t_R}{4q^2 + t_L - \Delta/s}\right) - \sum_{i=1}^{2}\left(\frac{\mathrm{Re}\,t_i(s)}{|t_i(s)|^2} + \frac{\mathrm{Re}\,t_i'(s)}{|t_i'(s)|^2}\right). \tag{21}$$

Formula (21) relates the slope parameter to the real and complex zero trajectories and the boundaries of spectral functions. First term in (21) was used in I to give a unified description of the slope parameters for pp, $\bar{p}p$, $K^{\pm}p$, and $\pi^{\pm}p$ scattering with the knowledge[13] or certain assumptions on the real trajectory $t_0(s)$, and using theoretical or effective boundaries of spectral functions wherever necessary. The real zero trajectory enters into the expression (21) along with the cut contribution because of the minimal requirement for the existence of CPE for all energies. We will see in the next section that formula (21) agrees very well with the data at lower, intermediate, and high energies with only one parameter.

IV. ZERO TRAJECTORIES AND THE SLOPE-PARAMETER DATA

A. Comparison with the data

Using experimental information on zero trajectories in the energy range $2.5 \leqslant s \leqslant 4.5$ GeV^2 and their linear but realistic extrapolations for higher and lower energies as shown in Figs. 1 and 2, we tried to examine how far the formula (21) can fit the data for all energies. For the best fit we found $\alpha = 0.35$. The fit has been denoted by curve II in Fig. 3. Except for some disagreement for $s < 1.44$ GeV^2, a plausible explanation for which will be suggested later in this section, it is very clear that curve II gives a very good fit for low, intermediate, and high energies. For the linearly increasing and decreasing trajectories, the contribution of the second term in (21) will be zero for very high energies. If the real trajectory $t_0(s)$ is a fixed-u-type zero lying nearer to the backward direction, the first term in (21) will be a constant for $s \to \infty$.

It is very important from the point of view of S-matrix theory and Mandelstam analyticity that for the best fit we have used only the theoretical boundaries of spectral functions. Use of as many as four free parameters in the formula of I and an effective shape of spectral function boundary for t_R, retreating away from the nearest boundary allowed by the S-matrix theory, could describe only an average of the data. Also the simple polynomial approximation of Barrelet type, using information on zeros determined experimentally, could not yield even a qualitative description of the slope-parameter data for all energies, although a qualitative description of the data only near the bump region was obtained. In the present case, however, use of only one free parameter, theoretical elastic boundary, experimentally determined real and complex zero trajectories, and quite realistic extrapolations of them for higher and lower energies has yielded a spectacular description of the data. Such a good fit has emerged because of our assumption that diffractive parts of amplitudes possess the same zeros as the nondiffractive part, including the real zero close to the backward direction. The first term in (21) becomes a constant as $s \to \infty$. But actually the slope-parameter data show a $\ln s$ type of increase for large values of s. It is possible to account for this type of logarithmic increase by considering energy dependence of α. An analytic approximation for α by means of a convergent polynomial expansion is possible by means of a conformal mapping which uses Mandelstam analyticity of the s plane. This type of parametrization has been used to estimate asymptotic behavior of slope parameters and suggest new scaling variables for diffraction scattering.[11,12] The same type of analysis[12] to take into account energy dependence of α can be repeated here. It turns out that,[12] in place of α, there is a function of s involving two unknown parameters which account for the high-energy behavior beautifully well.

One objection against our fit may be that the zero trajectories have not been determined from theory, but taken from the analysis of the differential cross section and polarization data. Even then, in addition to many other good qualities mentioned earlier and subsequently, we emphasize the following merits of the present analysis:

(1) Our analysis suggests a new form of parametrization which can include the forward peak to get useful information regarding true zeros of the amplitude, whereas in Barrelet's method the forward peak was thought to be an impediment in getting information on zeros. This point has been further clarified later in this section while determining the zero trajectory and making predictions about zero trajectories from the slope-parameter data.

(2) We have given an explanation of the observed

behavior of the slope-parameter data in terms of theoretically known elastic boundaries of spectral functions and the zero trajectories, such that understanding the dynamics of diffraction scattering reduces to understanding zero trajectories. At present a dynamical explanation of some of the real zero trajectories exist in the Veneziano model.[13] Although no consistent dynamical theory for the origin of complex zero trajectories has emerged yet; we hope some may emerge in near future.

B. Zero trajectory from the slope-parameter data

One bad feature of the fit given by curve II in Fig. 3 is that at lower energies there is an infinity at $s \simeq 1.41$ GeV^2, a feature not wanted by the experimental data. At low energy the first term in formula (21) is very much sensitive to the real trajectory $t_0(s)$ and completely dominates the total contribution to the slope parameter. In this energy region our results of calculation are not sensitive to small variations of complex trajectories occurring in the second term in (21). The infinity arises at $s = 1.41$ GeV^2 because our linear extrapolation of the data on $t_0(s)$ crosses the s axis at this point.

Now we ask the following question: Can the slope-parameter data determine at least the trajectory $t_0(s)$ at low energies? To answer such a question we fix the value of α at 0.35 which has been determined from fit to the data at all other energies. Since as it has been emphasized in I the CPE holds at all energies, we now use the formula (21) to determine the trajectory $t_0(s)$ at low energies. At present the only possible reason for disagreement with the data at low energies appears to lie in a suitable extrapolation of $t_0(s)$ in this region. It is quite probable that our extrapolated trajectories at low energies may not be good approximations to the actual trajectories. Since at low energy our formula is not affected by small variations of complex trajectories, but very much affected by such variations in the real trajectory $t_0(s)$, we take the complex trajectories to be the same as shown in Fig. 1 and vary the trajectory $t_0(s)$ for $s < 1.62$ GeV^2 until full agreement with the slope-parameter data in this energy region is obtained. The best-fit curve used to obtain $t_0(s)$ in this manner has been shown by the dotted line in Fig. 3 for $1.33 \leq s \leq 1.62$ GeV^2. The zero trajectory $t_0(s)$ determined from the best fit to the slope-parameter data in this energy region has been shown by curve II in Fig. 4. Curve I in Fig. 4 is the linear extrapolation of $t_0(s)$ as given in Fig. 1. The dotted curve in Fig. 4 shows the boundary corresponding to the backward direction $\mathrm{Re}\,t = -4q^2$. We see from this figure that the real zero trajectory obtained from the slope-parameter data bends from the linear extrapolation and towards the backward direction at first gradually and then rapidly as the energy approaches the threshold values. Because of the lack of experimental data on the slope parameter close to the threshold energy, the zero trajectory $t_0(s)$ could not be continued to the threshold. As described here our method of parametrization has determined the real zero trajectory $t_0(s)$ from the slope-parameter data for the first time for $\pi^- p$ scattering. The existence of such trajectories can be verified by other means by following the method of Barrelet[14] or by others.[19] In view of the rapidly growing interest in the phenomena of zeros,[13-19] determination of zero trajectory by such a simple scheme of parametrization is important.

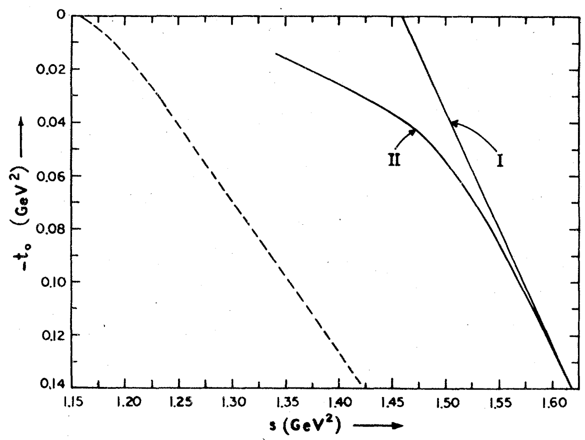

FIG. 4. Real zero trajectory t_0 as a function of s. Curve I is the linear extrapolation from the data. Curve II has been computed using the dotted-line fit to the slope-parameter data for $1.33 < s < 1.62$ GeV^2 as shown in Fig. 3. The dotted line is the physical-region boundary corresponding to the backward direction.

C. Some qualitative predictions about zero trajectories from the high-energy data

Looking to the pattern of the slope-parameter data at high energies and the manner in which the trajectories have contributed to account for the data at other energies, it is now possible to make some qualitative predictions about zero trajectories. To make such predictions let us examine how the bump structure around $s \simeq 3$ GeV^2 could be explained. Experimentally observed complex trajectories t_1 and t_1' which emerge roughly at $s \simeq 1.75$ GeV^2 continue to have the magnitudes of their real parts small even at $s \simeq 2.75$ GeV^2 around which there exist critical points reducing the imaginary parts close to zero. The contribution to the second term in (21) due to a complex trajectory, which has the form $-\mathrm{Re}\,t_i(s)/|t_i|^2$, is more the closer $\mathrm{Re}\,t_i$ is to zero, provided that

$|\mathrm{Im}t_i| \ll |\mathrm{Re}t_i|$. Such a condition is seen to be fulfilled by the trajectories t_1 and t_1' near $s \simeq 3$ GeV^2 and to some extent by the trajectory t_2' which give rise to the bump structure. It may be noted that whereas a qualitative fit to the bump is obtained by the second term containing complex trajectories, the rest of the data is well fitted by the first term in formula (21) around the bump region. Thus in the presence of the first term in formula (21), a bump in the slope-parameter data arises when $|\mathrm{Re}t_i| \ll 1$ and $|\mathrm{Im}t_i| \ll |\mathrm{Re}t_i|$. It is to be noted that $\mathrm{Re}t_i$ is to be negative for a substantial contribution.

From the experimental data plotted in Fig. 3 and also by Lassinski *et al.*,[3] it is very clear that there is a bump around $s \simeq 5.2$ GeV^2. Such a bump structure could not be accounted for by the present formula which contains continuously and linearly rising or falling zero trajectories. Using our observation mentioned above, the bump structure can be explained by the present parametrization only in the following ways: (a) Since we have $-\mathrm{Re}t_1$ and $-\mathrm{Re}t_1' \simeq 0.6$ GeV^2, the bump can arise if $\mathrm{Im}t_1$ and $\mathrm{Im}t_1'$ possess critical points for $s \simeq 5.2$ GeV^2. In that case these imaginary parts will not be linear as shown in Fig. 1. However, such deviations of the imaginary parts from linearity will not cause substantial changes to the fit at other energies except around the desired region near $s \simeq 5.2$ GeV^2. (b) With or without condition (a) being fulfilled, the bump can also be explained if new trajectories emerge near $s \simeq 5.2$ GeV^2 with very small but negative real parts and very small imaginary parts. The same structure will be observed if the trajectories emerge earlier, but the new trajectories continue to have their real and imaginary parts small with $|\mathrm{Im}t_i| \ll |\mathrm{Re}t_i|$. This will give rise to critical points in the new trajectories around $s \simeq 5.2$ GeV^2.

Thus from the observed bump structure around $s \simeq 5.2$ GeV^2, our analysis predicts the existence of critical points in the trajectories t_1 and t_1'. If these trajectories do not possess such critical points, but their imaginary parts continuously rise as shown in Fig. 2, new trajectories must emerge around $s \simeq 5.2$ GeV^2 with small but negative real parts. Critical points in the imaginary parts of at least some of these new trajectories must occur very near $s \simeq 5.2$ GeV^2. At present zero trajectories for such higher energies have not been determined.

There is a remarkable enhancement of the slope-parameter data of Foley *et al.*[21] at still higher energies for $20 < s < 50$ GeV^2. These data have been used along with others for higher and lower energies for data fitting taking into account Mandelstam analyticity in both s and $\cos\theta$ planes.[12] It has been found[12] that the data of Foley *et al.*[21] deviate remarkably from the best-fit curve which fits all other data at higher and lower energies. Such a conclusion regarding a remarkable discrepancy of these data from what would be expected of a smooth extrapolation of the remaining data at lower energies has also been obtained by Höhler and Staudenmaier,[22] Krubasik,[23] and Ambats *et al.*[24] No explanation has been furnished yet about the possible cause of such discrepancy. Continuing our argument we predict that new zero trajectories emerge, and the imaginary parts of at least some of the new trajectories possess critical points in this energy range. Such qualitative predictions as described here around $s \simeq 5.2$ GeV^2 and in the range $20 < s < 50$ GeV^2 can be verified by using Barrelet's moment analysis[14] or following the method of Carter.[19] In view of the rapidly growing interest in the phenomena of zeros, these qualitative predictions from such a simple scheme of parametrization is nevertheless important.

V. SUMMARY, CONCLUSION, AND LIMITATIONS

Prior to this work and paper I there did not exist even a qualitative explanation of the shrinkage-antishrinkage pattern of the forward peak as a function of s. The only work (Barger and Cline[14]) which attempts to understand the energy dependence of the slope parameter for $\pi^{\pm}p$ scattering in the region of oscillation has been strongly criticized by Weare.[5] The range of applicability of the model[4,5] is much higher than the energy range where the oscillatory pattern of the data exists. On the other hand, along with providing a unified description of the slope-parameter data for several diffractive processes, the CPE of paper I could reproduce an average of the data for $\pi^{\pm}p$ scattering for all energies. But this was possible for $\pi^- p$ scattering only after using four free parameters one of which yields an effective boundary of spectral function ρ_{st} retreating away from the theoretical elastic boundary, a feature objectionable from the point of view of S-matrix theory. In spite of this basic flaw the CPE of I was the only one in the literature which provided a good account for the slope-parameter data for different processes. The basic requirement for the realization of CPE for all energies has been shown to be the existence of one real zero in the physical region of the x plane. Recent analysis of the data on the transversity cross section and differential cross section by Barrelet *et al.*[16] shows that there exist, in addition to such real zeros of the differential cross section, complex zeros of transversity cross sections. In the present paper we attempt to understand shrinkage-antishrinkage of forward peak for $\pi^- p$ scattering in terms of real and complex zero trajectories such that the problem of understanding diffraction scattering reduces to the problem of understanding the origin of zero trajectories. In the method of moment analysis proposed by Barrelet[14] and used by Barrelet *et al.*[16] to obtain zero trajectories from data analysis, only transversity-cross-section data in a partial angular interval are used excluding the sharp-forward-peak region. It has been remarked that the inclusion of the forward peak may be an impediment in getting information on true zeros by moment analysis.[14,15,18] This leads to the suggestion that the forward peak may not be associated with true zeros of the amplitude.[14,15,18] One may also argue that the variation of the forward-peak slope may reflect, fully or in part, changes in the angular dependence which in turn are determined by zeros. If zeros of an analytic function are known, one can construct the function easily.

To verify how far the simple polynomial approximation obtained from the experimental information on complex zeros of the Barrelet type accounts for the energy dependence of slope-parameter data, we have first calculated its contribution for all energy ranges. It is found that there is not even a qualitative explanation at low and high energies, although there is a qualitative description of the slope-parameter data only around the bump region. It is noted that in the Barrelet-type parametrization, zero trajectories with linearly increasing or decreasing real and/or imaginary parts, which happens to be the observed trend, can never account for the slope-parameter data at high energies. Also the slope parameter is not affected by sign ambiguity.

Using CPE of I we propose a new scheme of parametrization assuming that the amplitudes near forward angles possess the same real and complex zeros as obtained by Barrelet *et al.*[16] In Ref. 7 the slope parameter was related to boundaries of spectral function. In paper I it was related to boundaries of the spectral function and the real zero trajectory. The new scheme of parametrization proposed here leads to a formula which relates the slope parameter to the boundaries of spectral functions and real and complex zero trajectories. Using the data on zero trajectories by Barrelet *et al.*[16] in a limited energy range and their linear but realistic extrapolations for higher and lower energies, an excellent description of the slope-parameter data is obtained for all energies, except for $s < 1.44$ GeV^2, with one free parameter and a theoretical elastic boundary of spectral function. Thus the objectionable feature existing in earlier analyses[2] is removed. Further since the CPE has been specifically[2] developed for all energies, no such criticisms as raised by Weare[5] apply to the present analysis. The logarithmic increase observed at very high energy can be accounted using Mandelstam analyticity as it has been done in Ref. 12.

The formula is found to be very sensitive to small variations of the real zero trajectory from its linear extrapolation, but unaffected by such variations in the complex trajectories. Thus a possible reason for disagreement at low energy has been attributed to be due to deviation of our linearly extrapolated real trajectory from actual trajectory. Using the available slope-parameter data for $1.33 < s < 1.62$ GeV^2, we have computed the real trajectory in this energy region. The computed trajectory deviates from the extrapolated trajectory and approaches the backward direction at first slowly and then rapidly, as energy decreases and approaches threshold.

From the observed bump structure in the data near $s \simeq 5.2$ GeV^2, we also make some qualitative predictions about zero trajectories. We predict that the imaginary parts of two zero trajectories t_1 and t_1' possess critical points around $s \simeq 5.2$ GeV^2 and/or new trajectories with small real and imaginary parts exist around this region. Similarly from the data of Foley *et al.*[21] we predict further new trajectories with small real and imaginary parts to exist in the range $20 < s < 50$ GeV^2. At present this appears to be the only explanation for the unusually high values of the experimental data.

Determination of a real zero trajectory using the slope-parameter data for low energies and such qualitative predictions about complex trajectories as described here are important in view of the rapidly growing interest on the phenomena of zeros.[13-19] Such predictions can be verified following other methods adopted for analysing the transversity-cross-section data[14,16] and the data on differential cross sections.[19] Contrary to the conventional view[13,15] that dips and zeros are associated only with the nondiffractive part and inclusion of the forward peak spoils information on zeros, our analysis strongly suggests that the diffractive part possesses all the observed zeros occurring away from the forward peak region. Barrelet's method has been used to determine complex zeros[14-16] from the data on transversity cross sections. Recently Carter[19] has determined zero trajectories using differential cross-section data. The present analysis shows how to obtain real zero trajectory in a limited energy region at low energy using slope-parameter data.

In the future if a dynamical understanding of the existing zero trajectories emerges, the present paper will help us in understanding the diffraction scattering at least for $\pi^- p$ scattering. The boundaries of spectral functions have been fully understood in terms of box diagrams. Out of the existing complex and real zero trajectories, the Veneziano model provides some dynamical explanations only for real zero trajectories in some cases.[13] In view of the increasing interest in the phenomena of zeros,[13-19] we hope some dynamical explanation of all the zero trajectories may emerge in near future.

Before concluding, we must point out certain limitations of the present scheme of parametrization. Following Barrelet,[14] we have assumed the functions $A(s,x)$ and $B(s,x)$ to possess the same analytic structures as the invariant amplitudes and complex and real zeros as observed experimentally. But it is well known that helicity amplitudes possess additional zeros and singularities of kinematical origin. In the present work no method has been prescribed to subtract such possible kinematical effects. We have also assumed the dominance of the absorptive part for the contribution of transversity amplitudes near forward angles and neglected possible contributions due to poles which may play a significant role at low energy. Our incorporation of the real zero at $x=-x_0(s)$ through the function $g(x)$ is not unique; a functional form $g(x)=[(c+x)/(x+x_0)]^2$ would serve the purpose equally well with c different from unity. This modification, however, can be shown[12] to result in a readjustment of the constant in the forward direction, and hence does not introduce more parameters into the theory. The mapping function z has been shown[2] to introduce a spurious cut in the mapped plane completely overlapping the physical region. This cut does not affect analyticity property[2] since the function z possesses only those singularities allowed by dynamics. In the work of Ciulli[10] and Barrelet[14] results on the convergence of polynomial expansion in mapped planes have been taken to hold in the presence of "artificial" cuts which completely overlap the images of the physical region. Such conformal mappings explicitly affect analyticity property introducing branch points at unwanted points. Although the effects of such branch points are removed by Ciulli[10] using a suitable combination of the mapped variables for expansion, they exist in the polynomial expansion of Barrelet.[14] In Barrelet's work the transversity amplitudes have been taken to be analytic in the upper- and lower-half planes, separately. In the present case we suppose that the spurious cut introduced by z will not affect convergence of representations[2,12] proposed here so long as the cut is confined onto the physical region in the mapped plane.

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Barrelet-type complex zeros and excluded the real zero while making polynomial approximations.

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