Comments and Addenda

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Equivalence of two approaches to noninertial observers

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An equivalence relation between our exact formula for special-relativity noninertial acceleration and the special-relativity part of a general-relativity approximate expression by Ni and Zimmerman is given.

Recently, while working on very different physical problems using entirely different mathematical techniques, Ni and Zimmerman[1] and DeFacio, Dennis, and Retzloff[2] derived remarkably similar formulas for the noninertial coordinate acceleration. Unfortunately, the two expressions seemed to differ by a single factor of 2 in one term. Considerable effort exposed no error in either derivation. This comment will show that, in fact, no such error exists and that a suitable generalization[3] of Ni and Zimmerman is exactly equivalent to our expression.

Ni and Zimmerman[1] were using local coordinate methods to study the class of astrophysical problems involving both inertial and gravitational effects. Local coordinate methods such as they used are useful to authors formulating astrophysical models, especially those involving considerations of general relativity.[4] It is necessary for Ni and Zimmerman to make the approximation of restricting their spacelike coordinate $\vec{r}$ to small values. They found that inertial and gravitational effects completely decouple through second order in $\vec{r}$, in their local coordinate calculation. Li and Ni[5] carried this analysis through third order in r and found coupling between inertial and gravitational effects at that order.

DeFacio, Dennis, and Retzloff[2] used modern invariant-differential-geometry methods to *geometrize* the presymmetry of Ekstein and Avishai,[6-9] for the case of classical special-relativity particle mechanics. Since Ref. 2 was published, we have found a nice paper by Estabrook and Wahlquist[10] which also gives a more general treatment of representing the covariant derivative ∇ for formulating both special- and general-relativity problems. There are, however, some differences in our respective approaches beyond the fact that we used presymmetry and Estabrook and Wahlquist did not.

The recent paper by Li and Ni[5] showed more than just inertial-gravitational coupling. *They also showed that there are no higher-order terms which are independent* of $\vec{a}\cdot\vec{r}$. However, the difference in our expression and theirs in the last term of our equation, or Eq. (1) here, is that they have $2\vec{a}\cdot(\vec{\omega}\times\vec{r})$, whereas we have $\vec{a}\cdot(\vec{\omega}\times\vec{r})$. A careful reading of Li and Ni[5] (see also Ref. 3) shows that higher-order corrections *cannot* provide the desired equivalence because the 1 term in the binominal expansion is already different. Therefore, either

(i) Ni and Zimmerman[1] or DeFacio, Dennis, and Retzloff[2] are wrong, or

(ii) the relevant parts of the two formulas are equivalent to each other.

The general expression obtained in Ref. 2 for noninertial acceleration $\ddot{\vec{r}}$ is

$$\ddot{\vec{r}} = -\vec{a}\left(1 + \frac{\vec{a}\cdot\vec{r}}{c^2}\right) - 2(\vec{\omega}\times\dot{\vec{r}}) - \vec{\omega}\times(\vec{\omega}\times\vec{r}) - \dot{\vec{\omega}}\times\vec{r} + \frac{(\dot{\vec{r}}+\vec{\omega}\times\vec{r})}{c^2(1+\vec{a}\cdot\vec{r}/c^2)}[\dot{\vec{a}}\cdot\vec{r} + 2\vec{a}\cdot\dot{\vec{r}} + \vec{a}\cdot(\vec{\omega}\times\vec{r})], \tag{1}$$

where $\vec{r}$ is the noninertial coordinate, $\vec{\omega}$ is the angular velocity of rotation, $\vec{a}$ is the inertial acceleration, and c is the velocity of light *in vacuo*. As mentioned in Ref. 3, we can expand the term $(1+\vec{a}\cdot\vec{r})^{-1}$ in the second line of Eq. (1) to lowest order in $\vec{r}$ but not $\dot{\vec{r}}$ or $\vec{a}$ to get (in units $c=1$)

$$\ddot{\vec{r}} = -\vec{a}(1+\vec{a}\cdot\vec{r}) - 2\vec{\omega}\times\dot{\vec{r}} - \vec{\omega}\times(\vec{\omega}\times\vec{r}) - \dot{\vec{\omega}}\times\vec{r} + \dot{\vec{r}}[\dot{\vec{a}}\cdot\vec{r} + \vec{a}\cdot(\vec{\omega}\times\vec{r})] + 2(\vec{a}\cdot\dot{\vec{r}})(1-\vec{a}\cdot\vec{r})\dot{\vec{r}} + 2(\vec{a}\cdot\dot{\vec{r}})\vec{\omega}\times\vec{r}. \tag{2}$$

Upon rearranging, Eq. (3) becomes

$$\ddot{\vec{r}} = -(1+\vec{a}\cdot\vec{r})\vec{a} - \vec{\omega}\times(\vec{\omega}\times\vec{r}) - 2\vec{\omega}\times\dot{\vec{r}} - \dot{\vec{\omega}}\times\vec{r} + 2(\vec{a}\cdot\dot{\vec{r}})\vec{\omega}\times\vec{r} + \dot{\vec{r}}[2\vec{a}\cdot(\vec{\omega}\times\vec{r}) + 2(\vec{a}\cdot\dot{\vec{r}})(1-\vec{a}\cdot\vec{r}) + (\dot{\vec{a}}+\vec{\omega}\times\vec{a})\cdot\vec{r}]. \tag{3}$$

Now in the notation of Ni and Zimmerman[1]

$$\vec{r}\ (\text{ours}) = \vec{x}\ (\text{theirs}),$$

$$\dot{\vec{r}}\ (\text{ours}) = \vec{\omega}\ (\text{theirs}),$$

$$\dot{\vec{\omega}}\ (\text{ours}) = \vec{\eta}\ (\text{theirs}),$$

and

$$(\dot{\vec{a}}+\vec{\omega}\times\vec{r})\ (\text{ours}) = \vec{b}\ (\text{theirs}), \tag{4}$$

which with Eqs. (4) inserted into Eq. (3) gives exactly the special-relativity terms of Ni and Zimmerman's Eq. (20). This completes the demonstration of equivalence of our two approaches in the sense of Ref. 3.

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[1]W.-T. Ni and M. Zimmerman, Phys. Rev. D 17, 1473 (1978).

[2]B. DeFacio, P. W. Dennis, and D. G. Retzloff, Phys. Rev. D 18, 2813 (1978).

[3]The suitable generalization is that one isolates a complicated coefficient called $f(\vec{\omega},\vec{a},\vec{r})$, and sums their binominal series on the right to obtain our exact denominator $(1+\vec{a}\cdot\vec{r})^{-1}$ on the left, i.e.,

$$[(1+\vec{a}\cdot\vec{r})^{-1}]f(\dot{\vec{r}},\vec{\omega},\vec{a}) = [1-\vec{a}\cdot\vec{r}+(\vec{a}\cdot\vec{r})^2-\cdots]\times f(\dot{\vec{r}},\vec{\omega},\vec{a}).$$

[4]B. E. Brumbaugh, Phys. Rev. D 18, 1335 (1978).

[5]W.-Q. Li and W.-T. Ni, J. Math. Phys. (to be published).

[6]H. Ekstein, Phys. Rev. 153, 1937 (1967).

[7]H. Ekstein, Phys. Rev. 184, 1315 (1969); Phys. Rev. D 1, 1851(E) (1970).

[8]Y. Avishai and H. Ekstein, Commun. Math. Phys. 37, 193 (1974).

[9]Y. Avishai and H. Ekstein, Found. Phys. 2, 257 (1972).

[10]F. B. Estabrook and H. D. Wahlquist, J. Math. Phys. 5, 1629 (1964).