Renormalization of an SU(n) linear σ model in the one-loop approximation for $n \geq 4$

H. B. Geddes
Department of Physics, Carleton University, Ottawa, Canada
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A linear SU(n) σ model with spin-zero mesons is demonstrated to be renormalizable in the one-loop approximation for $n \geq 4$. The mesons are assigned to the $(n,n^*)\oplus(n^*,n)$ representation of SU(n) $\times$ SU(n). The model incorporates both Nambu-Goldstone symmetry breaking and explicit symmetry-breaking terms linear in the fields. The counterterms are evaluated. As an additional four-point coupling is available in the SU(4) case, the SU(4) model is considered separately. A number of useful relations among the SU(n) f and d tensors are presented.

I. INTRODUCTION

The σ model has been actively studied for almost twenty years.[1] This sustained interest originates primarily from its ability to incorporate a large number of general theoretical ideas and its diversity of linear and nonlinear forms. As a result it has proved to be a useful laboratory for both theoretical and phenomenological investigations.

For many years the phenomenological studies employing this model used it only in the tree approximation, in an effective Lagrangian approach.[2] In the tree approximation the model can reproduce most of the results of current algebra and partial conservation of axial-vector current (PCAC).[3] The vector and axial-vector currents in the model obey an SU(n) $\times$ SU(n) current algebra. The current divergences can be constructed to be proportional to the fields. More recently, renormalizable models have been considered and the calculations have been carried out in the one-loop approximation. Numerical work with the linear SU(2) (Refs. 4 and 5) and SU(3) (Ref. 6) models in the one-loop approximation indicates that the second-order corrections are relatively small (10–15%) and that the resulting spin-zero mass spectrum agrees quite well with experiments.

With the discovery of the fourth quark flavor[7] (charm) and the likelihood of additional flavors,[8] it is interesting to generalize the linear σ model to include these cases. In this paper we consider the linear SU(n) σ model with $n \geqslant 4$ containing spin-zero mesons. The model incorporates both Nambu-Goldstone[9] and explicit linear symmetry breaking. We undertake an explicit renormalization in the one-loop approximation and isolate the required counterterms. This will enable numerical work at the one-loop level.

Lee[10] and Symanzik[11] have independently shown that the SU(2) σ model with linear symmetry breaking is renormalizable and that only the counterterms of the symmetric Lagrangian acquire divergent parts. Crater[12] investigated the SU(3) model with spontaneous symmetry breaking in the one-loop approximation. Chan and Haymaker[13] considered the SU(3) model with the addition of linear explicit symmetry-breaking terms in the one-loop approximation. Our approach parallels that of the latter authors.

In Sec. II we outline the general structure of a linear SU(n) σ model containing spin-zero mesons and incorporating symmetry-breaking terms that are linear in the fields. The renormalization of the model with $n>4$ in the one-loop approximation is considered in Sec. III. The renormalization procedure in the special SU(4) case is summarized in Sec. IV. Our conclusions are presented in Sec. V. A number of useful identities for the f_{ijk} and d_{ijk} tensors of SU(n) are given in the Appendix.

II. THE SU(n) LINEAR σ MODEL

The SU(n) σ model is constructed from the basic fields $M^a_{\bar b}$ and $M^{\bar a}_b$ (a, $b=1, \ldots, n$), where the upper (lower) index denotes the n (n^*) representation of SU(n) and the unbarred (barred) indices denote the left- (right-) hand space of chiral SU(n) $\times$ SU(n). These operators obey the linear equal-time commutation relations

$$[F^+_i, M^a_{\bar b}] = -\tfrac{1}{2}(\lambda^i)_{ac} M^c_{\bar b} \quad (i=1, \ldots, n^2-1)\,, \tag{2.1}$$

$$[F^-_i, M^a_{\bar b}] = \tfrac{1}{2}(\lambda^i)_{cd} M^a_{\bar c}\,, \tag{2.2}$$

$$[F^+_i, M^{\bar b}_a] = \tfrac{1}{2}(\lambda^i)_{ca} M^{\bar b}_c\,, \tag{2.3}$$

and

$$[F^-_i, M^{\bar b}_a] = -\tfrac{1}{2}(\lambda^i)_{bc} M^{\bar c}_a \tag{2.4}$$

with F^+ and F^- the generators of SU(n) $\times$ SU(n) which act on the left- and right-hand spaces, respectively. These generators are related to F and F^5, the vector and axial-vector charges, respectively, via

$$F^{\pm} = \tfrac{1}{2}(F \pm F^5)\,. \tag{2.5}$$

The field operators also obey the Hermiticity relation

$$(M_{\bar{a}}^{b})^{\dagger}=M_{b}^{\bar{a}} \tag{2.6}$$

and transform under parity as

$$PM_{\bar{a}}^{b}(\vec{x},t)P^{-1}=M_{a}^{\bar{b}}(-\vec{x},t). \tag{2.7}$$

These operators of mixed parity can therefore be decomposed as

$$M_{\bar{a}}^{b}=\Sigma_{a}^{b}+i\Phi_{a}^{b} \tag{2.8}$$

and

$$M_{a}^{\bar{b}}=\Sigma_{a}^{b}-i\Phi_{a}^{b}, \tag{2.9}$$

where Σ_a^b and Φ_a^b denote multiplets of scalar and pseudoscalar fields, respectively. For matrix notation we identify

$$M_{\bar{b}}^{a}=M_{ab} \tag{2.10}$$

and

$$M_{b}^{\bar{a}}=M_{ab}^{\dagger}. \tag{2.11}$$

Thus the $n\times n$ matrices M and $M^\dagger$ belong to the (n,n^*) and (n^*,n) representations of $\mathrm{SU}(n)\times\mathrm{SU}(n)$, respectively.

The number of even-parity, chiral-invariant, renormalizable operators that can be constructed from M and $M^\dagger$ depends on n. For $n>4$, the only such invariants are

$$I_1=\mathrm{Tr}(MM^\dagger), \tag{2.12}$$

$$I_1{}^2=[\mathrm{Tr}(MM^\dagger)]^2, \tag{2.13}$$

and

$$I_2=\mathrm{Tr}(MM^\dagger MM^\dagger). \tag{2.14}$$

These couplings are invariant under $\mathrm{U}(n)\times\mathrm{U}(n)$. In the SU(3) and SU(4) cases the additional renormalizable coupling

$$I_3=\det M+\det M^\dagger \tag{2.15}$$

can be used. In these cases it is an n-point coupling and is invariant only under $\mathrm{SU}(n)\times\mathrm{SU}(n)$.

The chiral invariants can be rewritten in n^2-component notation using the reduction

$$M=\frac{1}{\sqrt{2}}\lambda^i(\sigma_i+i\phi_i), \tag{2.16}$$

where ϕ_i and σ_i are the n^2-plets $(1\oplus n^2-1)$ of Hermitian pseudoscalar and scalar fields, respectively. Latin indices will be summed from 0 to n^2-1 unless otherwise noted. The σ_i and ϕ_i fields transform as the $(n,n^*)\oplus(n^*,n)$ representation of $\mathrm{SU}(n)\times\mathrm{SU}(n)$ and have the linear commutation relations[14]

$$[F_i,\phi_j]=if_{ijk}\phi_k \quad (i=1,\ldots,n^2-1), \tag{2.17}$$

$$[F_i,\sigma_j]=if_{ijk}\sigma_k, \tag{2.18}$$

$$[F_i^5,\phi_j]=id_{ijk}\sigma_k, \tag{2.19}$$

and

$$[F_i^5,\sigma_j]=-id_{ijk}\phi_k. \tag{2.20}$$

We now restrict our discussion to the case $n>4$. The bare, chiral-invariant, Lagrangian density is then

$$\mathcal{L}_0=\tfrac{1}{2}\mathrm{Tr}(\partial_\mu M\partial^\mu M^\dagger)-\tfrac{1}{2}\mu^2 I_1+f_1 I_1{}^2+f_2 I_2. \tag{2.21}$$

We assume that the symmetry-breaking Lagrangian density transforms as the $(n,n^*)\oplus(n^*,n)$ representation. The simplest form available is then

$$\mathcal{L}_{SB}=-\epsilon_i\sigma_i, \tag{2.22}$$

where ϵ_i is nonvanishing only for $I=Y=0$ operators.

Employing the SU(n) identities given in the Appendix, the bare Lagrangian can be rewritten in the n^2-component notation as

$$\begin{aligned}\mathcal{L}=&\tfrac{1}{2}\partial_\mu\sigma_i\partial^\mu\sigma_i+\tfrac{1}{2}\partial_\mu\phi_i\partial^\mu\phi_i-\tfrac{1}{2}\mu^2(\sigma_i\sigma_i+\phi_i\phi_i)\\&+\tfrac{1}{3}F_{ijkl}(\sigma_i\sigma_j\sigma_k\sigma_l+\phi_i\phi_j\phi_k\phi_l)\\&+2\hat{F}_{ij,kl}\sigma_i\sigma_j\phi_k\phi_l-\epsilon_i\sigma_i,\end{aligned} \tag{2.23}$$

where

$$F_{ijkl}=f_1 J^1_{ijkl}+\tfrac{1}{2}f_2 J^2_{ijkl}, \tag{2.24}$$

$$\hat{F}_{ij,kl}=f_1\delta_{ij}\delta_{kl}+\tfrac{1}{2}f_2 J^3_{ijkl}, \tag{2.25}$$

and

$$J^1_{ijkl}=\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}, \tag{2.26}$$

$$J^2_{ijkl}=d_{ijm}d_{mkl}+d_{ikm}d_{mjl}+d_{ilm}d_{mjk}, \tag{2.27}$$

and

$$J^3_{ijkl}=d_{ijm}d_{mkl}+f_{ikm}f_{mjl}+f_{ilm}f_{mjk}. \tag{2.28}$$

To permit a Nambu-Goldstone symmetry realization we define the vacuum expectation value of the scalar fields using

$$\langle 0|\sigma_i|0\rangle=\xi_i. \tag{2.29}$$

A new scalar field with vanishing vacuum expectation value is then defined as

$$S_i=\sigma_i-\xi_i. \tag{2.30}$$

Owing to the difficulties inherent in this translation, we do not normal-order the translated Lagrangian.[10]

The Lagrangian after translation can be written as

$$\begin{aligned}\mathcal{L}=&\tfrac{1}{2}\partial_\mu S_i\partial^\mu S_i+\tfrac{1}{2}\partial_\mu\phi_i\partial^\mu\phi_i-\tfrac{1}{2}m^{s2}_{ij}S_iS_j-\tfrac{1}{2}m^{\phi 2}_{ij}\phi_i\phi_j\\&+\tfrac{1}{3}F_{ijkl}(S_iS_jS_kS_l+\phi_i\phi_j\phi_k\phi_l)+2\hat{F}_{ij,kl}S_iS_j\phi_k\phi_l\\&+G^s_{ijk}S_iS_jS_k-3G^\phi_{ij,k}\phi_i\phi_jS_k-E_iS_i,\end{aligned} \tag{2.31}$$

where

$$m^{s2}_{ij} = \mu^2\delta_{ij} - 4F_{ijkl}\xi_k\xi_l\,, \tag{2.32}$$

$$m^{\phi 2}_{ij} = \mu^2\delta_{ij} - 4\hat{F}_{ij,kl}\xi_k\xi_l\,, \tag{2.33}$$

$$G^s_{ijk} = \tfrac{4}{3}F_{ijkl}\xi_l\,, \tag{2.34}$$

$$G^\phi_{ij,k} = -\tfrac{4}{3}\hat{F}_{ij,kl}\xi_l\,, \tag{2.35}$$

and

$$E_i = \epsilon_i + \mu^2\xi_i - \tfrac{4}{3}F_{ijkl}\xi_j\xi_k\xi_l\,. \tag{2.36}$$

Perturbation theory is defined as an expansion in the powers of λ defined via

$$\mathcal{L}(M,\ \lambda) = \frac{1}{\lambda^2}\mathcal{L}(\lambda M)\,. \tag{2.37}$$

This is effectively an expansion in the number of closed loops. The variable λ is used only for power counting and is set equal to unity at the end of the calculation. This expansion preserves the symmetry of the Lagrangian order by order.[15]

As the final step in the restructuring of the Lagrangian, we introduce the λ factors and the second-order counterterms giving

$$\begin{aligned}\mathcal{L} &= \tfrac{1}{2}\partial_\mu S_i\partial^\mu S_i + \tfrac{1}{2}\partial_\mu\phi_i\partial^\mu\phi_i - \tfrac{1}{2}(m^2 + \lambda^2\delta m^2)^s_{ij}S_iS_j \\ &\quad - \tfrac{1}{2}(m^2+\lambda^2\delta m^2)^\phi_{ij}\phi_i\phi_j \\ &\quad + \tfrac{1}{3}\lambda^2(F + \lambda^2\delta F)_{ijkl}(S_iS_jS_kS_l + \phi_i\phi_j\phi_k\phi_l) \\ &\quad + 2\lambda^2(\hat{F} + \lambda^2\delta\hat{F})_{ij,kl}S_iS_j\phi_k\phi_l \\ &\quad + \lambda(G + \lambda^2\delta G)^s_{ijk}S_iS_jS_k - 3\lambda(G+\lambda^2\delta G)^\phi_{ij,k}\phi_i\phi_jS_k \\ &\quad - \frac{1}{\lambda}(E + \lambda^2\delta E)_iS_i\,,\end{aligned} \tag{2.38}$$

where the second-order counterterms are denoted by the δ. These counterterms can be separated into divergent (D) and finite (Δ) components, i.e.,

$$\delta = D + \Delta\,. \tag{2.39}$$

Naturally this separation is arbitrary; however, the finite parts of the divergent Feynman integrals encountered in the theory can be chosen in a natural way. Once this is done the Δ terms are well defined. The divergent parts of the counterterms are used to cancel the divergent parts of the integrals. All physical quantities are then finite. This renormalization procedure is considered in detail in the one-loop approximation in Sec. III.

In the definitions of the masses and coupling constants of the translated Lagrangian, all basic Lagrangian constants appeared linearly, except the ξ_i. To ensure that the symmetry of the Lagrangian is maintained, only terms to a given order of λ can be retained in the counterterms. Consequently, to second order, the counterterms of Eq. (2.38) are related to the basic Lagrangian constants by

$$\delta F = F(\delta f_1,\ \delta f_2)\,, \tag{2.40}$$

$$\delta\hat{F} = \hat{F}(\delta f_1,\ \delta f_2)\,, \tag{2.41}$$

$$\delta G^s_{ijk} = G^s_{ijk}(\delta f_1,\ \delta f_2) + \tfrac{4}{3}F_{ijkl}\delta\xi_l\,, \tag{2.42}$$

$$\delta G^\phi_{ij,k} = G^\phi_{ij,k}(\delta f_1,\ \delta f_2) - \tfrac{4}{3}\hat{F}_{ij,kl}\delta\xi_l\,, \tag{2.43}$$

$$\delta m^{s2}_{ij} = m^{s2}_{ij}(\delta_\mu{}^2,\ \delta f_1,\ \delta f_2) - 8F_{ijkl}\xi_k\delta\xi_l\,, \tag{2.44}$$

$$\delta m^{\phi 2}_{ij} = m^{\phi 2}_{ij}(\delta_\mu{}^2,\ \delta f_1,\ \delta f_2) - 8\hat{F}_{ij,kl}\xi_k\delta\xi_l\,, \tag{2.45}$$

and

$$\delta E_i = E_i(\delta_\mu{}^2,\ \delta f_1,\ \delta f_2,\ \delta\epsilon) + m^{s2}_{ij}\delta\xi_j\,. \tag{2.46}$$

We conclude the general case by stating the currents and their divergences. After translation the vector and axial-vector currents are

$$V^\mu_k = \tfrac{1}{2}f_{kij}(S_i\overleftrightarrow{\partial}^\mu S_j + \phi_i\overleftrightarrow{\partial}^\mu\phi_j) + f_{kij}\xi_i\partial^\mu S_j \tag{2.47}$$

and

$$A^\mu_k = d_{kij}\phi_i\overleftrightarrow{\partial}^\mu S_j - d_{kij}\xi_i\partial^\mu\phi_j\,, \tag{2.48}$$

respectively. Employing the Gell-Mann divergence relation[16] one finds

$$\partial_\mu V^\mu_i = f_{ijk}\epsilon_jS_k \tag{2.49}$$

and

$$\partial_\mu A^\mu_i = -d_{ijk}\epsilon_j\phi_k\,. \tag{2.50}$$

The SU(4) case requires some special consideration as the additional I_3 four-point coupling is allowed. The bare, symmetric Lagrangian is

$$\mathcal{L}_0 = \tfrac{1}{2}\mathrm{Tr}(\partial_\mu M\partial^\mu M^\dagger) - \tfrac{1}{2}\mu^2 I_1 + f_1 I_1{}^2 + f_2 I_2 + gI_3\,. \tag{2.51}$$

As $\mathcal{L}_{\rm SB}$ is unchanged, the only major effect of I_3 is to modify the F and $\hat{F}$ couplings. In this case

$$\begin{aligned}F_{ijkl} &= \tfrac{1}{2}gA_{ijkl} + (f_1 + \tfrac{1}{4}g)J^1_{ijkl} \\ &\quad + \tfrac{1}{2}(f_2 - \tfrac{1}{2}g)J^2_{ijkl}\end{aligned} \tag{2.52}$$

and

$$\begin{aligned}\hat{F}_{ij,kl} &= -\tfrac{1}{2}gA_{ijkl} + f_1\delta_{ij}\delta_{kl} + \tfrac{1}{2}f_2J^3_{ijkl} \\ &\quad - \tfrac{1}{4}g(J^1_{ijkl} - J^2_{ijkl})\,,\end{aligned} \tag{2.53}$$

where

$$\begin{aligned}A_{ijkl} &= 8\delta_{i0}\delta_{j0}\delta_{k0}\delta_{l0} \\ &\quad - 2(\delta_{i0}\delta_{j0}\delta_{kl} + 5\text{ symmetric terms}) \\ &\quad + \sqrt{2}(\delta_{i0}d_{jkl} + 3\text{ symmetric terms})\,.\end{aligned} \tag{2.54}$$

The remainder of the analysis parallels that given for the general case. Equations (2.49) and (2.50) are valid only for $i \neq 0$ in the SU(4) model. The renormalization in the one-loop approximation for this case is considered briefly in Sec. IV.

III. ONE-LOOP RENORMALIZATION FOR THE GENERAL CASE

In this section we demonstrate that the linear SU(n) [$n > 4$] σ model is renormalizable in the one-loop approximation. As indicated above, we incorporate both Nambu-Goldstone symmetry breaking and explicit symmetry-breaking terms linear in the fields and transforming according to the $(n, n^*) \oplus (n^*, n)$ representation of SU(n)×SU(n). It is shown that all the divergences can be canceled by employing only the counterterms of the symmetric Lagrangian. These divergent counterterms are evaluated.

Consequently, in this demonstration we assume that only the terms of the symmetric Lagrangian require divergent counterterms to cancel all divergences to second order and set

$$D\xi_i = D\epsilon_i = 0\,. \tag{3.1}$$

At the conclusion it is clear that $D\mu^2$, Df_1, and Df_2 are, in fact, sufficient to cancel all second-order divergences. The Feynman rules for the Lagrangian of Eq. (2.38) are given in Fig. 1.

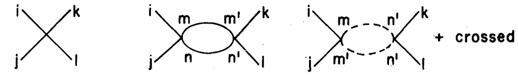

FIG. 2. Feynman graph representations of the divergent contributions to the four-point proper scalar vertex to second order.

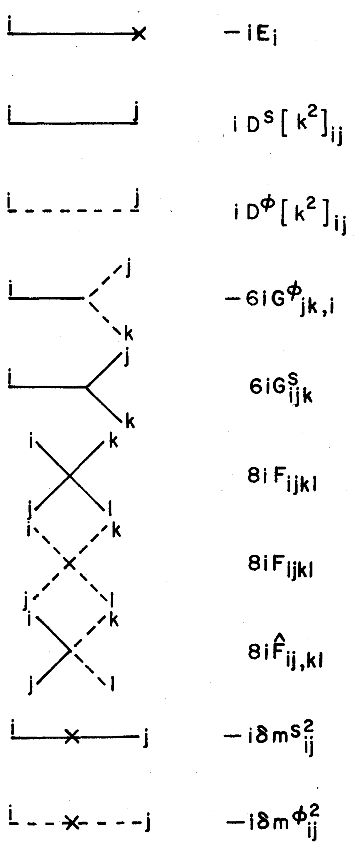

FIG. 1. Feynman rules for the Lagrangian of Eq. (2.38). Solid lines represent scalar fields and dashed lines pseudoscalars.

To begin, consider the four-point proper scalar vertex. The diagrams to be evaluated to second order containing divergent terms are given in Fig. 2. Straightforward evaluation of these diagrams gives the relation

$$\begin{aligned} DF_{ijkl} = {} & 4F_{ijmn}F_{klm'n'}Di\int \frac{d^4l}{(2\pi)^4} D^s[(l-p)^2]_{mm'} D^s[l^2]_{nn'} \\ & + 4\hat{F}_{ij,mn}\hat{F}_{kl,m'n'} \\ & \times Di\int \frac{d^4l}{(2\pi)^4} D^\phi[(l-p)^2]_{mm'} D^\phi[l^2]_{nn'} \\ & + \text{crossed terms.} \end{aligned} \tag{3.2}$$

To isolate the divergent parts of the Feynman integrals we expand the propagators about the point $p^2 = 0$ and the arbitrary chiral-invariant mass $m^2 = \nu^2$ giving

$$D[(l-p)^2]_{ij} = \frac{\delta_{ij}}{l^2 - \nu^2} + \frac{m^2{}_{ij} - (p^2 - 2l\cdot p + \nu^2)\delta_{ij}}{(l^2-\nu^2)^2} + \cdots\,. \tag{3.3}$$

This expansion is valid whether or not there is particle mixing and allows the divergent parts of integrals to be easily identified.

Equation (3.2) contains the logarithmically divergent integral

$$B_{ij,kl}(p^2) = i\int \frac{d^4l}{(2\pi)^4} D[(l-p)^2]_{ij} D[l^2]_{kl}\,. \tag{3.4}$$

Using Eq. (3.3), one finds

$$DB_{ij,kl}(p^2) = \delta_{ij}\delta_{kl}\, i\int \frac{d^4l}{(2\pi)^4}\frac{1}{(l^2-\nu^2)^2} \tag{3.5}$$

$$= \delta_{ij}\delta_{kl} B(\nu^2)\,. \tag{3.6}$$

The convergent integral $\overline{B}_{ij,kl}(p^2)$ is defined as

$$\overline{B}_{ij,kl}(p^2) = B_{ij,kl}(p^2) - \delta_{ij}\delta_{kl}B(\nu^2)\,. \tag{3.7}$$

Including the crossed terms explicitly, Eq. (3.2) can now be rewritten as

$$\begin{aligned} DF_{ijkl} = 4(& F_{ijmn}F_{klmn} + F_{ikmn}F_{jlmn} + F_{ilmn}F_{jkmn} \\ & + \hat{F}_{ij,mn}\hat{F}_{kl,mn} + \hat{F}_{ik,mn}\hat{F}_{jl,mn} \\ & + \hat{F}_{il,mn}\hat{F}_{jk,mn})B(\nu^2)\,. \end{aligned} \tag{3.8}$$

Employing the identities in the Appendix, $F_{ijmn}F_{klmn}$ and $\hat{F}_{ij,mn}\hat{F}_{kl,mn}$ can be reduced to

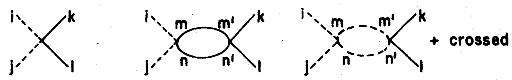

FIG. 3. Feynman graph representations of the second-order, divergent contributions to the four-point proper scalar-pseudoscalar vertex.

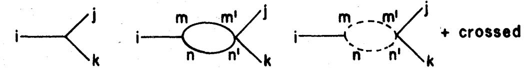

FIG. 4. The divergent diagrams in the three-point proper scalar vertex to second order.

$$F_{ijmn}F_{klmn}=f_1^2[(n^2+4)\delta_{ij}\delta_{kl}+2(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})]+\tfrac{1}{2}f_1f_2[8n\delta_{ij}\delta_{kl}+2n(\delta_{ij}\delta_{k0}\delta_{l0}+\delta_{kl}\delta_{i0}\delta_{j0})+4J^2_{ijkl}]$$
$$+\tfrac{1}{4}f_2^2[4nd_{ijm}d_{mkl}+8\delta_{ij}\delta_{kl}+2(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})+2\sqrt{2n}(\delta_{i0}d_{jkl}+\delta_{j0}d_{ikl}+\delta_{k0}d_{ijl}+\delta_{l0}d_{ijk})] \quad (3.9)$$

and

$$\hat{F}_{ij,mn}\hat{F}_{kl,mn}=f_1^2n^2\delta_{ij}\delta_{kl}+\tfrac{1}{2}f_1f_2[8n\delta_{ij}\delta_{kl}-2n(\delta_{ij}\delta_{k0}\delta_{l0}+\delta_{kl}\delta_{i0}\delta_{j0})]$$
$$+\tfrac{1}{4}f_2^2[4nd_{ijm}d_{mkl}+8\delta_{ij}\delta_{kl}+2(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})-2\sqrt{2n}(\delta_{i0}d_{jkl}+\delta_{j0}d_{ikl}+\delta_{k0}d_{ijl}+\delta_{l0}d_{ijk})]. \quad (3.10)$$

Equation (3.8) can next be rewritten as

$$DF_{ijkl}=8J^1_{ijkl}[f_1^2(n^2+4)+4nf_1f_2+3f_2^2]B(\nu^2)+8J^2_{ijkl}(3f_1f_2+nf_2^2)B(\nu^2). \quad (3.11)$$

From Eqs. (2.24) and (2.40) we also have

$$DF_{ijkl}=Df_1J^1_{ijkl}+\tfrac{1}{2}Df_2J^2_{ijkl}. \quad (3.12)$$

Consequently, the required counterterms are

$$Df_1=8[f_1^2(n^2+4)+4nf_1f_2+3f_2^2]B(\nu^2) \quad (3.13)$$

and

$$Df_2=16f_2(3f_1+nf_2)B(\nu^2). \quad (3.14)$$

With these values for the second-order counterterms, the four-point proper scalar vertex is finite in the one-loop approximation. Similarly, these counterterms remove the second-order divergences in the four-point proper pseudoscalar vertex in an identical calculation.

The four-point proper scalar-pseudoscalar vertex requires separate consideration. The diagrams containing divergences to be evaluated to second order are given in Fig. 3. Requiring the divergent parts of the amplitude to vanish, one obtains

$$D\hat{F}_{ij,kl}=4(\hat{F}_{ij,mn}F_{klmn}+F_{ijmn}\hat{F}_{kl,mn}+2\hat{F}_{im,kn}\hat{F}_{jm,ln}+2\hat{F}_{im,ln}\hat{F}_{jm,kn})B(\nu^2). \quad (3.15)$$

Employing the identities in the Appendix,

$$F_{ijmn}\hat{F}_{kl,mn}=f_1^2(n^2+2)\delta_{ij}\delta_{kl}+\tfrac{1}{2}f_1f_2[8n\delta_{ij}\delta_{kl}-2n(\delta_{ij}\delta_{k0}\delta_{l0}-\delta_{kl}\delta_{i0}\delta_{j0})+2J^3_{ijkl}]$$
$$+\tfrac{1}{4}f_2^2[4nd_{ijm}d_{mkl}+8\delta_{ij}\delta_{kl}-2(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})+2\sqrt{2n}(\delta_{i0}d_{jkl}+\delta_{j0}d_{ikl}-\delta_{k0}d_{ijl}-\delta_{l0}d_{ijk})] \quad (3.16)$$

and

$$\hat{F}_{im,kn}\hat{F}_{jm,ln}=f_1^2\delta_{ij}\delta_{kl}+f_1f_2J^3_{ijkl}+\tfrac{1}{4}f_2^2[4nf_{ikm}f_{mjl}+2(\delta_{ij}\delta_{kl}+\delta_{il}\delta_{jk})]. \quad (3.17)$$

Equation (3.15) can now be rewritten as

$$D\hat{F}_{ij,kl}=8[(n^2+4)f_1^2+4nf_1f_2+3f_2^2]\delta_{ij}\delta_{kl}B(\nu^2)+8f_2(3f_1+nf_2)J^3_{ijkl}B(\nu^2). \quad (3.18)$$

From Eqs. (2.25) and (2.41) we also have

$$D\hat{F}_{ij,kl}=Df_1\delta_{ij}\delta_{kl}+\tfrac{1}{2}Df_2J^3_{ijkl}. \quad (3.19)$$

Consequently, the counterterms of Eqs. (3.13) and (3.14) also render this four-point vertex finite.

The three-point proper scalar and scalar-pseudoscalar vertex calculations reduce to those given above. First consider the scalar vertex. The diagrams containing second-order divergences are given in Fig. 4. Requiring the divergent part of the amplitude to vanish, one has

$$DG^s_{ijk}=4(G^s_{imn}F_{jkmn}-G^\phi_{mn,i}\hat{F}_{jk,mn}+G^s_{jmn}F_{ikmn}-G^\phi_{mn,j}\hat{F}_{ik,mn}+G^s_{kmn}F_{ijmn}-G^\phi_{mn,k}\hat{F}_{ij,mn})B(\nu^2). \quad (3.20)$$

Substituting for G^s and G^ϕ using Eqs. (2.34) and (2.35), and using

$$DG^s_{ijk}=\tfrac{4}{3}\xi_l DF_{ijkl} \quad (3.21)$$

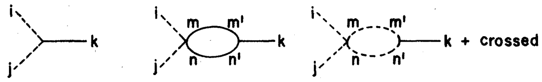

FIG. 5. The divergent diagrams in the three-point proper scalar-pseudoscalar vertex to second order.

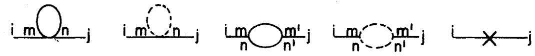

FIG. 6. Feynman graph representations of the second-order contributions to the scalar mass.

[setting $D\xi_i=0$ in Eq. (2.42)], Eq. (3.20) reduces to Eq. (3.8). Consequently, the divergent part vanishes by employing the above counterterms.

The diagrams to be considered for the three-point proper scalar-pseudoscalar vertex are given in Fig. 5. Setting the divergent part of this amplitude equal to zero gives

$$DG^{\phi}_{ij,k}=4(F_{ijmn}G^{\phi}_{mn,k}-\hat{F}_{ij,mn}G^{s}_{kmn}+2G^{\phi}_{im,n}\hat{F}_{jm,kn}+2G^{\phi}_{jm,n}\hat{F}_{im,kn})B(\nu^2)\,. \tag{3.22}$$

Again, substituting for G^ϕ and G^s and using

$$DG^{\phi}_{ijk}=-\tfrac{4}{3}\xi_l D\hat{F}_{ij,kl}\,, \tag{3.23}$$

this relation reduces to Eq. (3.15). This amplitude does not require any additional counterterms.

The Feynman diagrams for the scalar mass to second order are given in Fig. 6. Requiring that the divergent part of this amplitude vanish, one has

$$Dm^{s2}_{ij}=4F_{ijmn}Di\int\frac{d^4l}{(2\pi)^4}D^s[l^2]_{mn}+4\hat{F}_{ij,mn}Di\int\frac{d^4l}{(2\pi)^4}D^\phi[l^2]_{mn}$$
$$-18G^s_{imn}G^s_{jm'n'}Di\int\frac{d^4l}{(2\pi)^4}D^s[(l-p)^2]_{mm'}D^s[l^2]_{nn'}-18G^{\phi}_{mn,i}G^{\phi}_{m'n',j}Di\int\frac{d^4l}{(2\pi)^4}D^\phi[(l-p)^2]_{mm'}D^\phi[l^2]_{nn'}\,. \tag{3.24}$$

The quadrically divergent integral

$$A_{ij}=i\int\frac{d^4l}{(2\pi)^4}D[l^2]_{ij} \tag{3.25}$$

is separated into its finite and divergent parts using Eq. (3.3). Setting

$$A(\nu^2)=i\int\frac{d^4l}{(2\pi)^4}\frac{1}{l^2-\nu^2} \tag{3.26}$$

one has

$$DA_{ij}=A(\nu^2)\delta_{ij}+(m^2_{ij}-\nu^2\delta_{ij})B(\nu^2)\,. \tag{3.27}$$

This reduction gives

$$Dm^{s2}_{ij}=4F_{ijmn}[A(\nu^2)\delta_{mn}+(m^{s2}_{mn}-\nu^2\delta_{mn})B(\nu^2)]$$
$$+4\hat{F}_{ij,mn}[A(\nu^2)\delta_{mn}+(m^{\phi 2}_{mn}-\nu^2\delta_{mn})B(\nu^2)]$$
$$-18(G^s_{imn}G^s_{jmn}+G^{\phi}_{mn,i}G^{\phi}_{mn,j})B(\nu^2)\,. \tag{3.28}$$

Substituting for m^{s2}, $m^{\phi 2}$, G^s, and G^ϕ and comparing the result with Eq. (3.8), one finds

$$Dm^{s2}_{ij}=4(F_{ijmm}+\hat{F}_{ij,mm})[A(\nu^2)+(\mu^2-\nu^2)B(\nu^2)]$$
$$-4\xi_p\xi_q DF_{ijpq}\,. \tag{3.29}$$

From Eqs. (2.44) and (2.32)

$$Dm^{s2}_{ij}=D\mu^2\delta_{ij}-4\xi_p\xi_q DF_{ijpq}\,. \tag{3.30}$$

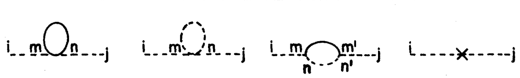

FIG. 7. Feynman diagrams for the second-order pseudoscalar mass contributions.

Employing the identities in the Appendix and comparing terms gives

$$D\mu^2=8[(n^2+1)f_1+2nf_2][A(\nu^2)+(\mu^2-\nu^2)B(\nu^2)]\,. \tag{3.31}$$

The pseudoscalar mass calculation parallels that above. From the diagrams of Fig. 7

$$Dm^{\phi 2}_{ij}=4\hat{F}_{ij,mn}[A(\nu^2)\delta_{mn}+(m^{s2}_{mn}-\nu^2\delta_{mn})B(\nu^2)]$$
$$+4F_{ijmn}[A(\nu^2)\delta_{mn}+(m^{\phi 2}_{mn}-\nu^2\delta_{mn})B(\nu^2)]$$
$$-36G^{\phi}_{im,n}G^{\phi}_{jm,n}B(\nu^2)\,. \tag{3.32}$$

This can be reduced to

$$Dm^{\phi 2}_{ij}=4(F_{ijmm}+\hat{F}_{ij,mm})[A(\nu^2)+(\mu^2-\nu^2)B(\nu^2)]$$
$$-4\xi_p\xi_q D\hat{F}_{ij,pq}\,. \tag{3.33}$$

Using Eq. (2.33) this relation can easily be shown to give the same constraints as above. Thus, the divergences encountered in the second-order mass calculations are canceled using only the counterterms of the symmetric Lagrangian.

We next consider the vacuum expectation value of the scalar field. Setting the divergent part of the

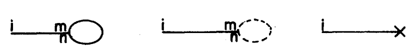

FIG. 8. Feynman diagrams for the scalar one-point vertex to second order.

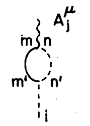

FIG. 9. A Feynman diagram that contributes to the second-order axial-vector-current–pseudoscalar-field vertex.

amplitude from Fig. 8 to zero gives

$$DE_i = 3G^s_{imn}[A(\nu^2)\delta_{mn} + (m^{s^2}_{mn} - \nu^2\delta_{mn})B(\nu^2)] - 3G^{\phi}_{mn,i}[A(\nu^2)\delta_{mn} + (m^{\phi 2}_{mn} - \nu^2\delta_{mn})B(\nu^2)]. \qquad (3.34)$$

Following the program outlined above, this reduces to

$$DE_i = 4(F_{ilmm} + \hat{F}_{il,mm})[A(\nu^2) + (\mu^2 - \nu^2)B(\nu^2)]\xi_l - \tfrac{4}{3}\xi_j\xi_k\xi_l DF_{ijkl}. \qquad (3.35)$$

From Eqs. (2.46) and (2.36) DE_i is also given by

$$DE_i = \xi_i D\mu^2 - \tfrac{4}{3}\xi_j\xi_k\xi_l DF_{ijkl}. \qquad (3.36)$$

Combining these equations and comparing them with the mass calculations, the symmetric counterterms are clearly seen to cancel all divergences.

An additional potential divergence must be considered when employing current-field vertices. Consider, for example, the axial-vector-current–pseudoscalar-field vertex diagram in Fig. 9. The amplitude for this diagram contains the integral

$$R^{\mu}(p,x^2,y^2) = i\int \frac{d^4 l}{(2\pi)^4}\frac{(2l-p)^{\mu}}{[l^2 - x^2][(l-p)^2 - y^2]}. \qquad (3.37)$$

Formally this integral is linearly divergent. However, explicit evaluation of the integral gives a finite result. The only manifestation of the formal divergence is a surface term that contributes to the finite result. This term cannot be retained as it violates the Ward-Takahashi identities. A regularization procedure is used to remove the surface term.[10,17] Consequently, this type of vertex does not introduce a new divergence and can be handled in the above scheme.

In summary, we have demonstrated that in the linear SU(n) σ model only the counterterms of the symmetric Lagrangian are needed to renormalize the theory in the one-loop approximation. Neither the implicit Nambu-Goldstone symmetry breaking nor the explicit linear symmetry breaking altered the values of those counterterms.

IV. THE SU(4) CASE

The program in the SU(4) case follows that of Sec. III. We need only consider the effect of the additional $\det M + \det M^{\dagger}$ term.

Consider the four-point scalar vertex. In the SU(4) case Eq. (3.11) becomes

$$DF_{ijkl} = [\tfrac{1}{2}g^2(2J^1_{ijkl} - J^2_{ijkl}) + 40f_1^2 J^1_{ijkl} + f_2^2(3J^1_{ijkl} + 4J^2_{ijkl}) + 3(f_1 - f_2)g(2A_{ijkl} + J^1_{ijkl} - J^2_{ijkl}) + 2f_1f_2(16J^1_{ijkl} + 3J^2_{ijkl})]B(\nu^2) \qquad (4.1)$$

and Eq. (3.12) is

$$DF_{ijkl} = \tfrac{1}{2}DgA_{ijkl} + (Df_1 + \tfrac{1}{4}Dg)J^1_{ijkl} + \tfrac{1}{2}(Df_2 - \tfrac{1}{2}Dg)J^2_{ijkl}. \qquad (4.2)$$

In this case the required counterterms are

$$Df_1 = 4(40f_1^2 + 32f_1f_2 + 6f_2^2 + g^2)B(\nu^2), \qquad (4.3)$$

$$Df_2 = 4[4f_2(4f_2 + 3f_1) - g^2]B(\nu^2), \qquad (4.4)$$

and

$$Dg = 48g(f_1 - f_2)B(\nu^2). \qquad (4.5)$$

The scalar mass calculation proceeds exactly as before. The counterterm $D\mu^2$ is not affected by the g term and is

$$D\mu^2 = 8(17f_1 + 8f_2)[A(\nu^2) + (\mu^2 - \nu^2)B(\nu^2)]. \qquad (4.6)$$

As in the general case, only the counterterms of the symmetric Lagrangian are required for one-loop renormalization.

V. CONCLUSION

We have demonstrated the renormalizability of the SU(n) linear σ model with mesons in the one-loop approximation for $n \geq 4$. The model incorporates both spontaneous symmetry breaking and explicit linear symmetry-breaking terms. In all cases, only the counterterms of the symmetric Lagrangian acquire divergent parts.

ACKNOWLEDGMENT

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APPENDIX

In this appendix we present the SU(n) identities used throughout this paper. These identities involve the standard λ matrices[18] and f_{ijk} and d_{ijk} tensors.[14]

The algebra of SU(n) consists of all $n \times n$ traceless Hermitian matrices.[19] The standard basis is chosen as the set of $n^2 - 1$ matrices λ^i such that

$$\mathrm{Tr}(\lambda^i\lambda^j) = 2\delta_{ij}. \qquad (A1)$$

These matrices correspond to the Pauli matrices for SU(2) and the standard λ matrices for SU(3) and SU(4).

To this set we adjoin the matrix

$$\lambda^0=\left(\frac{2}{n}\right)^{1/2} I . \tag{A2}$$

Consequently, we now have a set of n^2 matrices obeying Eq. (A1) with

$$\mathrm{Tr}(\lambda^i)=\sqrt{2n}\,\delta_{i0} , \tag{A3}$$

$$\lambda^i\lambda^j=(d_{ijk}+if_{ijk})\lambda^k , \tag{A4}$$

$$[\lambda^i,\lambda^j]=2if_{ijk}\lambda^k \tag{A5}$$

and

$$[\lambda^i,\lambda^j]=2d_{ijk}\lambda^k . \tag{A6}$$

The f_{ijk} and d_{ijk} are the usual totally antisymmetric and symmetric SU(n) tensors, respectively. The extension to the index 0 and U(n) is straightforward:

$$f_{0ij}=0 , \tag{A7}$$

$$d_{0ij}=\left(\frac{2}{n}\right)^{1/2}\delta_{ij} , \tag{A8}$$

$$d_{iij}=n\sqrt{2n}\,\delta_{j0} . \tag{A9}$$

The f and d tensors obey the following basic identities:

$$f_{ilm}f_{mjk}+f_{jlm}f_{imk}+f_{klm}f_{ijm}=0 , \tag{A10}$$

$$f_{ilm}d_{mjk}+f_{jlm}d_{imk}+f_{klm}d_{ijm}=0 , \tag{A11}$$

$$f_{ijm}f_{klm}=d_{ikm}d_{jlm}-d_{jkm}d_{ilm} , \tag{A12}$$

$$f_{ijk}f_{ljk}=n(\delta_{il}-\delta_{i0}\delta_{l0}) , \tag{A13}$$

$$d_{ijk}d_{ljk}=n(\delta_{il}+\delta_{i0}\delta_{l0}) , \tag{A14}$$

and

$$d_{ijk}f_{ljk}=0 . \tag{A15}$$

It is convenient to define the matrices

$$(F_j)_{ik}=if_{ijk} \tag{A16}$$

and

$$(D_j)_{ik}=d_{ijk} . \tag{A17}$$

Equations (A10) through (A15) can now be rewritten as

$$[F_i,F_j]=if_{ijk}F_k , \tag{A18}$$

$$[F_i,D_j]=if_{ijk}D_k , \tag{A19}$$

$$[D_i,D_j]=if_{ijk}F_k , \tag{A20}$$

$$\mathrm{Tr}(F_iF_j)=n(\delta_{ij}-\delta_{i0}\delta_{j0}) , \tag{A21}$$

$$\mathrm{Tr}(D_iD_j)=n(\delta_{ij}+\delta_{i0}\delta_{j0}) , \tag{A22}$$

and

$$\mathrm{Tr}(F_iD_j)=0 , \tag{A23}$$

respectively.

We also need the additional relations

$$\mathrm{Tr}(F_iF_jF_k)=i\tfrac{1}{2}nf_{ijk} , \tag{A24}$$

$$\mathrm{Tr}(D_iF_jF_k)=\tfrac{1}{2}nd_{ijk}+\left(\frac{n}{2}\right)^{1/2}(\delta_{i0}\delta_{jk}-\delta_{j0}\delta_{ik}-\delta_{k0}\delta_{ij}) , \tag{A25}$$

$$\mathrm{Tr}(D_iD_jF_k)=i\tfrac{1}{2}nf_{ijk} , \tag{A26}$$

$$\mathrm{Tr}(D_iD_jD_k)=\tfrac{1}{2}nd_{ijk}+\left(\frac{n}{2}\right)^{1/2}(\delta_{i0}\delta_{jk}+\delta_{j0}\delta_{ik}+\delta_{k0}\delta_{ij}) , \tag{A27}$$

$$\begin{aligned}\mathrm{Tr}(D_iD_jD_kD_l)=&\tfrac{1}{4}n(d_{ijm}d_{mkl}-f_{ijm}f_{mkl})\\&+\tfrac{1}{2}(\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})\\&+\frac{\sqrt{2n}}{4}(\delta_{i0}d_{jkl}+\delta_{j0}d_{ikl}\\&+\delta_{k0}d_{ijl}+\delta_{l0}d_{ijk}) ,\end{aligned} \tag{A28}$$

$$\begin{aligned}\mathrm{Tr}(D_iD_jF_kF_l)=&\tfrac{1}{4}n(d_{ijm}d_{mkl}-f_{ijm}f_{mkl})\\&+\tfrac{1}{2}(\delta_{ij}\delta_{kl}-\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk})\\&+\frac{\sqrt{2n}}{4}(\delta_{i0}d_{jkl}+\delta_{j0}d_{ikl}\\&-\delta_{k0}d_{ijl}-\delta_{l0}d_{ijk}) ,\end{aligned} \tag{A29}$$

and

$$\begin{aligned}\mathrm{Tr}(F_iF_jF_kF_l)=&\tfrac{1}{4}n(d_{ijm}d_{mkl}-f_{ijm}f_{mkl})\\&+\tfrac{1}{2}(\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})\\&-\frac{\sqrt{2n}}{4}(\delta_{i0}d_{jkl}+\delta_{j0}d_{ikl}\\&+\delta_{k0}d_{ijl}+\delta_{l0}d_{ijk}) .\end{aligned} \tag{A30}$$

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