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Hadronic wave equations from lattice gauge theories: The breathing modes on a restricted lattice

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A continuum limit of the lattice gauge theory restricted to the hadron sector is proposed. Details of the appropriateness of formulating such a limit within a specific scheme of approximation are discussed for the quarkless hadron sector. The general method is then applied to the gauge fields, with the restriction that motions occur only along one spatial direction. This leads to a continuum eigenvalue equation for the spectrum of internal excitations associated with fluctuations in the length of the electric flux line. In addition, a nontrivial renormalization condition is encountered if one insists on a relativistic form for the spectrum. A speculative method for dealing with this condition is proposed—the method eliminates the lattice spacing and removes the coupling constant as a free parameter, as in dimensional transmutation. Alternative possibilities for restoring covariance in the hadronic sector of the theory are discussed.

I. INTRODUCTION

In most current theories regarding the nature of confinement, the quantum field responsible for confinement is on equal footing as a dynamical entity with the hadronic constituents it acts to confine.[1] Obvious though this circumstance may seem, it is seldom taken into account in hadron spectroscopy. There are at least two reasons for this neglect. The first is that one might naively expect the excitation spectrum of the gluon degrees of freedom within hadrons with flavor to start at relatively large masses. The second is that attempts at verifying such expectations by calculation have been few[2]—the calculations are not easy. In this Introduction, we will first indicate specific reasons why the calculations are not easy even within a framework where they can be attempted. Next, a plan of attack to make such calculations tractable will be sketched. The body of the paper is a report on progress made by using the strategy on a simplified model where the full complexity of the problem does not overwhelm.

A. Spectral calculations that include gluons

There are three methods currently available for attempting calculations in hadronic spectroscopy, including gluons. The first of these is the string model and its variants.[3] The trouble with this type of model appears to be that one gets all or one gets nothing. It has long been recognized that for model building along these lines to progress, a certain infusion of qualitative ingredients from more traditional field theories might be required. Rather than trying to guess those ingredients and how they should be incorporated into a consistent mathematical framework, one could attempt to construct a string picture systematically from an underlying field theory.[4] This we try to do in our approach.

Another approach, which has been remarkably successful in describing the spectrum of the low-lying hadronic states containing flavor, is the MIT bag model.[5] This model has also been applied to study the spectrum of states of large angular momentum, and has been successful there as well.[6] Unfortunately, however, results on the spectrum of low-lying quarkless excitations (gluon bound states) have not yet been published for this model.

A third approach to the problem under consideration is the lattice regularization scheme for gauge field theories.[7] This whole approach is subject to a host of fundamental criticisms which we need not bother to repeat. Let us instead revive a certain criticism[8] which is more pragmatic than fundamental, and consequently more tractable.

The Hamiltonian version of lattice perturbation theory in the inverse of the gauge field coupling constant starts off with the identification of eigenstates of the electric part of the Hamiltonian. For example, a quark and an antiquark are joined by an electric flux line across the links of the lattice, and the energy carried by the flux line is proportional to its length. If the length of the flux line were in some sense a variational parameter for the problem, it could be argued that minimizing the length of the flux line minimizes the energy. In the lowest states, the energy would be proportional to the spatial separation of the quark and antiquark.

However, the lattice theory is not defined in this manner. One is required to deal systematically with the enormous degeneracy of eigenstates

of the electric part of the Hamiltonian—one degenerate state for each place the antiquark can be located in the lattice relative to the quark, for fixed path length. The potential lifts a large part of the degeneracy, but this entails supplying new labels for the states which indicate how the original degenerate set was split.

What can the meaning of such additional labels characterizing the states of the system possibly be? Inasmuch as the only labels allowed by the Poincaré group are masses, momenta, spins, and spin projections, one must hope that if the lattice ultimately describes relativistic particles, the additional labels refer to the spin content of the states of the system.[9]

A problem with studying states of gluonic excitation in the lattice theory, then, is that the process of classification is exceedingly tedious for "long" configurations, i.e., configurations where the electric flux line spans many links. However, this statement of the problem suggests a method for its resolution. Since the classification problem is severe for *long* lattice strings, perhaps the variations of string configurations over one or a few lattice spacings can be treated differentially and approximated accurately by taking a continuum limit. We arrive by this logic to the notion that a continuum limit of the lattice theory *restricted to the hadronic sector* may be most appropriate for describing states of gluonic excitation.

The most difficult step in implementing such a continuum limit lies in ensuring that the limiting theory is relativistically covariant. There is nothing in the lattice-theory Lagrangian which even suggests relativistic invariance. The task is complicated further because the theory must be solved approximately, and *a priori* there is no way to assess the effect of a given systematic approximation procedure on the question of covariance. In the absence of a formal structure within which covariance can be maintained systematically, we are forced to rely on heuristic arguments to guide us as far as possible.

B. An outline of a program

Those readers who are not familiar with the details of lattice gauge theory dynamics may wish to read Sec. II A before continuing along the lines of this discussion. Basically, however, the qualitative feature of lattice perturbation theory that we need to draw on now is that the "magnetic" piece of the gauge field energy contains pieces which cancel or add bits of electric flux. Thus, a given string of electric flux gets stretched or contracted (among other things) by the action of the magnetic term in the Hamiltonian. One of the other things that can occur is that a closed loop of electric string in the x-y plane (for example) can suddenly develop a small protrusion into the z direction.

Even an analysis as scant as the above suggests that three classes of motions are generated by repeated action of the magnetic perturbation.

(1) *String length fluctuations*. An exact eigenstate of the gauge field Hamiltonian will contain a superposition of eigenstates of the electric Hamiltonian of different-length electric strings. The concept of "the length" of the exact eigenstate is well defined only when the coupling constant $(1/g^2)$ is small, so there is an overlap of the exact eigenstate with a particular bare eigenstate which is much larger than all other such overlaps. The spread in lengths of the exact eigenstate may be considered to be a quantum-mechanical broadening of the width of the electric string. When perturbation theory fails, presumably the smeared object is an incoherent "whisp" rather than a well-defined "smoke-ring."

(2) *Center-of-mass motions*. By repeated action of the potential, an entire path of electric flux may be annihilated from one set of links and duplicated on another set of links in such a manner that the motion corresponds to a displacement of the configuration on the lattice.

(3) *Orbital angular momentum*. The same local string fluctuations which are responsible for the motions described above give rise to twisting and contorting motions of the original electric string. String bits at different links move relative to one another in three-dimensional space, so these motions contain what in a continuum would be called angular rotations. Since the string bits carry energy and, beyond zeroth-order perturbation theory, momentum, internal angular momentum is being generated.

There is a fourth class of motions which arises from the fact that the lattice theory describes vector gauge fields. The electric flux lines span links by virtue of this directionality inherent in the theory. Spin interactions will not be discussed in this paper.

Let us now pretend that only the first class of motions was present in the real problem. Then if one were to consider a rather long initial configuration (eigenstate of H_0) and treat it in perturbation theory, at low orders the fluctuations in its length would be small in comparison with its overall length. This suggests that a differential approach to the problem of length fluctuation might be appropriate. Differential in what? The Hamiltonian operator causes transitions between the bare states of the system. *The aim will be to introduce*

an effective operator on which passage to differentials is well defined.

In a similar fashion, small changes in the shape of a loop configuration will induce small shifts in the center of mass of the configuration. Thus, we may hope to treat these motions differentially as well, subject to the complications of relativity alluded to earlier.

The third and fourth classes of motion are more difficult to deal with. If we succeed in introducing an overall length variable for the configuration of interest into the problem, we may consider it an integral over a differential length parameter defined along the curve. The third and fourth classes of motion involve "local" variations of the oriented differential length parameter. We propose to study these local variations by recognizing that the differential length parameter itself provides a parametrization of the curve in the three-dimensional space in which the curve is embedded.

Not all of the above classes of motion are discussed in this paper. Rather, we focus on the dilatation modes and center-of-mass modes, neglecting the complications of angular momentum and spin. We do this by analyzing the full theory within a restricted kinematical regime.

Even within this restricted analysis, however, one of the central problems of lattice theories must be contended with. An interesting aspect of the present work is that it casts the problem of eliminating the lattice constant in a form which resembles in some aspects the traditional problems of renormalizing continuum field theories. Hopefully, some of the traditional techniques of field theory may be brought to bear on this approximation to the lattice theory—some first steps in this direction are discussed in the paper.

C. An outline of this paper

In Sec. II, the Hamiltonian lattice gauge theory is reviewed to fix our conventions. We also present the general outline of the approximation scheme concretely, and explain the specific objects in the theory on which a continuum limit is to be defined. In brief, on a spatial lattice Schrödinger's equation is a finite-order difference equation in the basis of eigenstates of H_0. Passage to a differential equation defines a specific continuum limit of the theory.

Section III illustrates these procedures for a simplified model which amounts to restricting the motions of the closed electric flux loops to occur along one spatial dimension.

Having done this, we discuss the problems encountered in passing to the continuum limit in Sec. IV. It is at this stage that we tackle the problem of eliminating the lattice parameter by a variant of dimensional transmutation.[10] There is no guarantee that the hadronic states of the lattice-regulated theory satisfy relativistic dispersion formulas. A nontrivial renormalization condition arises if we insist that the dispersion law for the gluonic excitations has the correct relativistic form. The interpretation of this condition will be discussed in detail.

In Sec. V we summarize our findings, and comment on the implications for the later stages of the program.

II. GENERAL REMARKS ON THE FORMALISM

In this section, we will record our conventions for the lattice Hamiltonian gauge theory, and then present the general outline of our approach for deriving hadronic wave equations within the zero-width and planar approximations.

A. Lattice Hamiltonian gauge theory

1. Operators

In the present work, quark degrees of freedom will not be discussed. Nonetheless, the gauge field degrees of freedom are a nontrivially interacting system. Their dynamics is governed by a Hamiltonian operator

$$H=H_0+V, \tag{2.1a}$$

$$H_0=\frac{g^2}{2a}\sum_{\vec{x},n+,C}[\pi^C(\vec{x},\hat{n})]^2, \tag{2.1b}$$

$$V=-\frac{1}{2ag^2}\sum_{\text{plaq}}(U_P+U_P^\dagger), \tag{2.1c}$$

$$U_P=\mathrm{Tr}\,U(\vec{x},\hat{n})U(\vec{x}+\hat{n},\hat{m})U^\dagger(\vec{x}+\hat{m},\hat{n})U^\dagger(\vec{x},\hat{m}). \tag{2.1d}$$

In this equation, g is the gauge field coupling constant, and a is the lattice spacing. The index C refers to the color degree of freedom, and the Tr in (2.1d) is over these indices on the U's, which are matrix representations of the color group.

For all the developments of the theory which follow, we will delete the indices C and the Tr over U's. This is not simply for notational simplicity, but reflects an important aspect of the approximation scheme to be discussed in the next subsection. The "planar" approximation amounts to rules for calculation in the non-Abelian theory which are identical to those of the Abelian theory. Thus, we need only record the canonical commutation relations of the Abelian theory,

$$[\pi(\vec{x},\hat{n}),U(\vec{y},\hat{m})]=\delta_{\vec{x},\vec{y}}\delta_{\hat{n},\hat{m}}U(\vec{x},\hat{n})\,. \tag{2.2}$$

Here as in Eq. (2.1b), the canonical momenta $\pi(\vec{x},\hat{n})$ are defined on links joining $\vec{x}$ to $\vec{x}+\hat{n}$ *with* $\hat{n}$ *a positive unit vector.*

Finally, Gauss's law is

$$G(\vec{x})\,|P\rangle=\sum_{\hat{n}+}[\pi(\vec{x}-\hat{n},\hat{n})-\pi(\vec{x},\hat{n})]\,|P\rangle$$
$$=0\,, \tag{2.3}$$

where the operator $G(\vec{x})$ generates infinitesimal time-independent local gauge transformations, and $|P\rangle$ is a state in the physical subspace.

2. *Naive continuum limit*

The rules of correspondence with the continuum bare theory are as follows:

$$U(\vec{x},\hat{n})\equiv e^{i\theta_n(\vec{x})}\,, \tag{2.4a}$$

$$\theta_n(\vec{x})=agA_n^{(\mathrm{cont})}(\vec{x})\,, \tag{2.4b}$$

$$\pi(\vec{x},\hat{n})=\frac{a^2}{g}E_n^{(\mathrm{cont})}(\vec{x})\,. \tag{2.4c}$$

3. *Space of states*

It is convenient to work in a Schrödinger representation,

$$H\,|\psi t\rangle=i\frac{\partial}{\partial t}\,|\psi t\rangle\,. \tag{2.5}$$

Moreover, the rules for the actions of the operators take a simple graphic form if we refer our states to a coordinate basis defined by

$$\langle\theta\,|\,U(\vec{x},\hat{n})\,|\,\theta'\rangle=\delta_P(\theta-\theta')e^{i\theta_n(\vec{x})}\,, \tag{2.6a}$$

$$\langle\theta\,|\,\pi(\vec{x},\hat{n})\,|\,\theta'\rangle=i\frac{\partial}{\partial\theta_n'(\vec{x})}\delta_P(\theta-\theta')\,, \tag{2.6b}$$

$$\langle\theta\,|\,\theta'\rangle=\delta_P(\theta-\theta')$$
$$=\prod_{\mathrm{links}(L)}\left[\sum_{n_L}e^{in_L(\theta_L-\theta_L')}\right], \tag{2.6c}$$

$$1=\int\mathfrak{D}\theta\,|\,\theta\rangle\langle\theta\,|$$
$$\equiv\left(\sum_L\int_0^{2\pi}\frac{d\theta_L}{2\pi}\right)(|\cdots\theta_L\theta_L'\cdots\rangle\langle\cdots\theta_L\theta_L'\cdots|)\,. \tag{2.6d}$$

Thus, a wave function in the coordinate representation is

$$\psi_s(\theta)=\langle\theta\,|\,S\rangle\,, \tag{2.7}$$

and its norm is

$$\langle S\,|\,S\rangle=\int\mathfrak{D}\theta\psi_s^*(\theta)\psi_s(\theta)=1\,. \tag{2.8}$$

4. *Perturbation theory*

The general problem of this Hamiltonian theory is to solve Eq. (2.5) subject to the constraint (2.3). It is convenient for purposes of calculation to rescale the time variable so that Eq. (2.5) reads

$$H\psi(\theta,t)=i\frac{\partial}{\partial t}\psi(\theta,t)$$
$$=\left\{-\sum_L\frac{\delta^2}{\delta\theta_L^{\,2}}-\lambda\sum_P[U_P(\theta)+U_P^\dagger(\theta)]\right\}\psi(\theta,t) \tag{2.5'}$$

with $\lambda=1/g^4$. Then, as is well known, for "strong coupling" in g, $\lambda\ll1$ and (2.5') can be solved approximately with, e.g., Rayleigh-Schrödinger perturbation theory.

The constraint condition (2.3) requires dressed physical states to be those which are generated by perturbation theory from closed loop of U's. These closed-loop wave functions are eigenstates of H_0. One class of these eigenstates has eigenvalues proportional to their length,

$$H_0\left[\prod_\Gamma U(\theta\in\Gamma)\right]=L_\Gamma\left[\prod_\Gamma U(\theta\in\Gamma)\right] \tag{2.9}$$

with L_Γ equal to the number of links on the path.

However, Eq. (2.9) does not encompass all the eigenstates of H_0. Two further possibilities can occur. For example, a single plaquette can be multiply occupied:

$$H_0(U_P)^n=4n^2(U_P)^n\,. \tag{2.10a}$$

Alternately, several disconnected plaquettes can be occupied simultaneously, e.g.,

$$H_0U_{P_1}^{\,n_1}\cdots U_{P_j}^{\,n_j}=4(n_1^{\,2}+\cdots+n_j^{\,2})U_{P_1}^{\,n_1}\cdots U_{P_j}^{\,n_j}\,. \tag{2.10b}$$

In the non-Abelian theory, Eq. (2.9) becomes modified in a trivial manner:

$$H_0^{(\mathrm{NONAB})}\left[\prod_\Gamma\mathrm{Tr}\,U(\alpha^c\in\Gamma)\right]=\kappa L_\Gamma\left[\prod_\Gamma\mathrm{Tr}\,U\right], \tag{2.9'}$$

where κ is the eigenvalue of the Casimir operator $(\sum_c T^cT^c)$, with T^c the cth representation matrix in the fundamental representation.[8] For example, for SU_3 of color, $\kappa=\frac{4}{3}$.

Unfortunately, the non-Abelian analog of Eq. (2.10a) is more complicated.[8] When, for example, two unit plaquettes overlap on links, the group-theoretic product of the two U's on each individual link must be taken:

$$U_P^{\,2}\xrightarrow{(\mathrm{NONAB})}\prod_{j=1}^{4}[U(\alpha_{L_j}^c)\otimes U(\alpha_{L_j}^c)]\,. \tag{2.11}$$

Then, to satisfy the non-Abelian gauge condition, which is a generalization of (2.3), the overall group singlet configuration on each vertex of the plaquette must be extracted.

The upshot is that the eigenvalue of H_0 on the n-compounded configuration is not of the simple form $4n^2$, but involves the eigenvalues of the above-mentioned Casimir operator on the direct-product representations that contribute to the overall singlet.

In spite of these complications, the non-Abelian version of (2.10a) does share one important feature with the Abelian version. It is that the eigenvalue of H_0 on a compounded configuration is *larger* than the sum of the eigenvalues of the plaquettes which are being compounded. Thus, the single occupancy configurations which satisfy Eq. (2.9) are *not bosonic*, in the sense that the energy of a multiply occupied (compounded) configuration is not additive.[11]

This feature allows us to distinguish singly occupied states from multiply occupied states qualitatively. Furthermore, quantitatively, the contributions of states satisfying (2.10) or (2.10′) will always be smaller than the contributions due to states satisfying (2.9) or (2.9′) in Rayleigh-Schrödinger perturbation theory. This is simply because the energy denominators will be larger for the (2.10)-type states than for the (2.9)-type states.

Suppose now that we attempt a certain truncation scheme which goes beyond ordinary perturbation theory, and amounts to neglecting all compounded states.[12] We expect their contributions to any given matrix element to be smaller than the contributions due to "simple" loop states for the reasons we just discussed. Nevertheless, one is inviting trouble with unitarity, and it will require some luck for the scheme to work. If we tentatively proceed within this truncation scheme, however, we enjoy a tremendous reduction of our labor for the non-Abelian theory. This reduction stems from Eq. (2.9′). We can compute the non-Abelian theory as though it were an Abelian theory.[13] It is for this reason that we only bothered to remind the reader of the rules of the Abelian lattice gauge theory.

In what follows, we shall call the restriction to states of the type satisfying Eq. (2.9) the "planar approximation." The reason for this terminology is that a spacetime picture of the evolution of our specially selected states would look like a tubular sheet of rubber or hose. States of the excluded type would modify this picture in two possible ways. One way would be to locally alter the "thickness" of our rubber sheet, so it would not remain uniform. Another would be to cause the single channel of the hose to bifurcate. Figure 1 illustrates these possibilities. Our approximation is, therefore, a concrete realization of the topological expansion of Veneziano and collaborators, although the specific types of contributions to this expansion which we have been discussing arise *from* the lattice theory, and do not appear in the considerations of those investigators.[14]

The more familiar nonleading contributions to the topological expansion are the nonplanar contributions illustrated in Fig. 1(d). These represent multiparticle states, of the type described by Eq. (2.10b) and its non-Abelian counterpart. We shall neglect these states as well, and denote this the "zero-width" approximation for the obvious reasons.

B. Hadronic wave equations

Although we have been discussing perturbation theory as it applies to the lattice gauge theory, our aim is to construct hadronic wave equations which go beyond perturbation theory. In a very general sense, what is sought is a solution of the constraint equation (2.3), which permits a rewriting of the Hamiltonian in terms of operators which couple *only* to physical states. There are many possible ways to approach this general problem. The present approach has certain advantages over perturbation theory, which will be elucidated as the approach is developed.

Consider our Schrödinger equation (2.5). Let the eigenstates of H_0 be denoted by $|\hat{n}\rangle$, with eigenvalues $\tilde{\epsilon}_n$. Let those eigenstates which are single-particle and "planar" in the sense discussed in Sec. II A be denoted by $|n\rangle$, with eigenvalues ϵ_n. Finally, let $|\psi_m, t\rangle$ be an exact eigenstate of H, with energy E_m, and which leads in perturbation theory as $|\psi_m, 0\rangle = |m\rangle + \lambda\delta\psi_m$ +higher-order terms. Clearly,

$$\langle n|H|\psi_m, 0\rangle = E_m\langle n|\psi_m, 0\rangle . \tag{2.12}$$

If we now truncate the intermediate states,

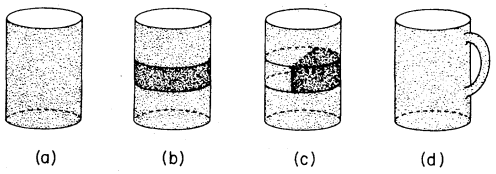

FIG. 1. (a) A rubber tube swept out by a smoke-ring gluon bound state in space-time. (b) A multiply occupied state has a larger energy per unit length at fixed time, i.e., it is denser, than the minimally occupied state. Hence, it is depicted with cross hatches in the off-energy-shell evolution. (c) A dense piece of the tube can coexist with the minimal density tube. (d) A coffee-mug graph, with more than one smoke ring during a period of time.

$$1=\sum_{\tilde{n}}|\tilde{n}\rangle\langle\tilde{n}|\approx\sum_{n}|n\rangle\langle n|\,, \tag{2.13}$$

we obtain a new, approximate equation,

$$\sum_{k}H_{nk}\langle k|\psi_m\rangle\approx E_m\langle n|\psi_m\rangle\,. \tag{2.14}$$

Inasmuch as $|\psi_m\rangle$ has a zeroth-order overlap with $|m\rangle$, we can restrict our attention to the "diagonal" difference equation

$$E_m\langle m|\psi_m\rangle\approx(\epsilon_m\delta_{mk}+V_{mk})\langle k|\psi_m\rangle\,. \tag{2.15}$$

To justify our reference to (2.15) as a difference equation, let us reflect on what form the wave functions $\langle k|\psi_m\rangle$ actually take in the lattice gauge theory.

From Sec. II A, we have

$$\langle\theta|k\rangle=\prod_{\Gamma(k)}U(\theta\in\Gamma)\,. \tag{2.16}$$

More specifically, we can choose any starting point on the contour Γ and denote it by $\vec{x}$. Then the next point on Γ is $x+\hat{m}_1$, the point after that is $x+\hat{m}_1+\hat{m}_2$, etc. If Γ is k links along, we will require $\vec{x}$ and at least k unit vectors to label the points on Γ. However, since Γ must be a closed contour $\sum_{j=1}^{k}\hat{m}_j=0$. Furthermore, any site may be visited more than once, but $\langle\theta|k\rangle$ will be of the desired planar class only if no link gets retraced in either direction.

Consequently, a more detailed specification of $\langle\theta|k\rangle$ is

$$\langle\theta|k\rangle=\exp\left\{i\left[\sum_{j=1}^{k}\theta_{n_j}(\vec{x}+\hat{n}_{j-1})\right]\right\}, \tag{2.16'}$$

with $\vec{x}+\hat{n}_{k-1}\equiv\vec{x}$, and $\tilde{\sum}$ enforcing the required restriction on the $\hat{n}_j$.

The huge degeneracy of $|k\rangle$ referred to in Sec. I stems from two sources. The most easily manageable source is the location of the reference point $\vec{x}$ in the entire lattice. In spite of the fact that $\vec{x}$ is in no way the center of energy of the configuration, it is possible to deal with this degeneracy provisionally by forming plane-wave states, $\sum_{\vec{x}}e^{i\vec{p}\cdot\vec{x}}\langle\theta|k\rangle$.

Much more serious than this $\vec{x}$ problem is the degeneracy due to all the possible ways of choosing the $\hat{n}$'s. Even for an elementary plaquette, in a three-dimensional lattice there are eight boxes all labeled by $\vec{x}$ and a pair of unit vectors. On each of these, we can further choose whether the path circulates in accordance with the right-hand rule, or in the opposite direction.

So really, one should write symbolically

$$\langle\theta|k\rangle_j(\vec{p})=\sum_{\vec{x}}e^{i\vec{p}\cdot\vec{x}}\psi(\vec{x},j\in\{\hat{n}\})\,, \tag{2.16''}$$

where the index j labels a particular configuration of $\hat{n}$'s in the set of all possible $\hat{n}$'s that lead to a closed loop Γ of length k.

In principle, the $|\psi_m\rangle$ are calculable in perturbation theory from the $|m\rangle$. Consequently, the coordinate-space wave functions $\langle\theta|\psi_m\rangle$ also require additional labels of the kind displayed in Eq. (2.16''). In view of these observations, Eq. (2.15) for our specific theory can be written in the symbolic form

$$E_m\varphi_j^{(m)}(\vec{p})=\sum_{m',j',\vec{p}'}\mathcal{K}_{(j,j')}^{(m,m')}(\vec{p},\vec{p}')\varphi_{j'}^{(m')}(\vec{p}')\,. \tag{2.17}$$

That is:

(i) Since the theory is displacement invariant, we expect *some* center-of-mass $\vec{p}$ to be conserved, even though it may not be the $\vec{p}$ introduced in Eq. (2.16'').

(ii) The perturbation V can cause transitions from paths of length m to paths of length m'.

(iii) The perturbation V can change a given configuration of $\hat{n}$'s into another configuration in the same set of total path length m.

We are now in a position to attempt to guess a continuum form of the Schrödinger equation for purposes of discussion. In the equation below, $\vec{X}$ refers to the center of mass of the extended hadron, and $(\partial/\partial\vec{x})$ refers to a lattice difference operation which passes to a derivative in the continuum limit. The derivatives $\partial/\partial y'$ are defined in the same way with respect to each y' of sets $\{y'\}$ of relative coordinates among the constituents of the composite hadron. The index m on the wave function ϕ remains to indicate the zeroth-order energy of the state:

$$i\frac{\partial}{\partial t}\varphi^{(m)}(\vec{X},\{\vec{y}\})\stackrel{?}{=}\int\mathfrak{D}y'\left[-\frac{\delta(\vec{y}-\vec{y}')}{2M}\frac{\partial^2}{\partial\vec{X}^2}+\mathfrak{V}\left(\vec{y}',\frac{\partial}{\partial\vec{y}'};\vec{y}\right)\right]\varphi^{(m)}(\vec{X},\{\vec{y}'\})\,. \tag{2.18}$$

We have explicitly *assumed* a nonrelativistic $(M+P^2/2M)$ spectral form. But this equation is clearly problematic. Evidently, the operator

$$\int\mathfrak{D}\vec{y}'\mathfrak{V}\left(\vec{y}',\frac{\partial}{\partial\vec{y}'};\vec{y}\right)\varphi^{(m)}(\vec{X},\{\vec{y}'\})\equiv\sum_{n}\int\cdots\int d\vec{y}_1'd\vec{y}_2'\cdots d\vec{y}_n'\mathfrak{V}\left(\vec{y}_1',\vec{y}_2',\ldots,\vec{y}_n';\frac{\partial}{\partial\vec{y}_1'},\ldots;\vec{y}_1,\cdots\right)\varphi^{(m)}(\vec{X};\vec{y}_1',\ldots) \tag{2.19}$$

refers to the internal degrees of excitation of the system, and itself poses an eigenvalue problem. Pictorially, our lattice loops on paths Γ might be expected to go over into "smoke rings" of assorted shapes in the continuum limit, with a quantized spectrum of energies, and quantum wave functions describing the shapes of the smoke rings.

This in itself is comprehensible. What is not clear is what the mass parameter M in the center-of-mass kinetic energy refers to. That is, how does one consistently obtain an equation of the form (2.19), with M bearing a definite relationship to the eigenvalues of the internal operator $\mathfrak{V}$? In hope of circumventing this problem, we turn to examine an alternative procedure.

C. Second-order hadronic wave equations

In the preceding subsection, we developed a wave equation based on the actual theory under consideration, and speculated on the form it might take if the naive continuum limit existed. We continue in this vein, but present two approaches which are superficially of relativistic form. Thus, we obtain Klein-Gordon instead of Schrödinger wave equations.

The first approach is simply to "square" Eq. (2.18). Notice that even if the spectrum of the theory is Galilean, $E_\phi = m + \vec{P}^2/2m$, then $E_\phi^2 \approx m^2 + \vec{P}^2$ to lowest order in $\vec{P}^2$. Thus, by using H^2, one separates the center-of-mass momentum from the mass operator. As will be seen, the theory will not actually give a spectrum of this form. Nevertheless, it is easier to compare the spectral relation that is obtained with conventional propagators if H^2 is employed. That is, for any time-independent H, Eq. (2.5) also implies that

$$H^2 |\psi t\rangle = -\frac{\partial^2}{\partial t^2} |\psi t\rangle . \tag{2.20}$$

It is only necessary to parallel the arguments leading to Eq. (2.18) to obtain

$$-\frac{\partial^2}{\partial t^2}\varphi^{(m)}(\vec{X}, \{\vec{y}\}) = \int \mathfrak{D}\vec{y}' \left(-\pi(\vec{y}', \vec{y}) \frac{\partial^2}{\partial \vec{X}^2} + M^2(\vec{y}', \vec{y}) \right) \varphi^{(m)}(\vec{X}, \{\vec{y}'\}) . \tag{2.18'}$$

The coefficient of $\partial^2/\partial \vec{X}^2$ has been denoted $\pi(\vec{y}';\vec{y})$ by analogy with the polarization operator $\pi(P^2)$ of an ordinary field-theory propagator. Similarly, the pieces in H^2 which refer only to the internal excitation configurations in ϕ have been denoted by M^2, the (mass)^2 operator. These analogies are quite useful, because, in fact, the naive continuum limit to reach (2.18') does *not* exist, and a process of renormalization is required.

Before discussing this further, let us briefly introduce another approach which leads to an equation of the form Eq. (2.18'). The approach which follows will *not* be used in this paper, so the reader may wish to skip ahead to the next section on a first reading. Within our field theory, there exist multilocal operators which couple a physical state of the system to the fully dressed vacuum. For example, with our previous notation, we define

$$\Phi^{(n,m)}(\vec{X}, t; \vec{p}) = \left\langle \Omega \middle| \prod_{\Gamma_n(\vec{x})} U(\theta \in \Gamma, t) \middle| \psi^{(m)}(\vec{p}) \right\rangle , \tag{2.21}$$

where $\vec{X}$ denotes a given reference point for a closed contour Γ_n, n links long.

The operator

$$\hat{\Gamma}_n(\vec{X}, t) = \prod_{\Gamma_n(\vec{x})} U(\theta \in \Gamma, t) \tag{2.22}$$

is a Heisenberg field, so it follows that

$$(E_\psi - E_\Omega)^2 \Phi = -\langle \Omega | \ddot{\hat{\Gamma}}_n(\vec{x}, t) | \psi^{(m)}(\vec{p}) \rangle . \tag{2.23}$$

Next, $\ddot{\hat{\Gamma}}_n$ can be computed using the Heisenberg equations of motion. *If*, within the planar, zero-width approximation, it is possible to rewrite

$$\langle \Omega | [H, [H, \hat{\Gamma}_n]] | \psi \rangle \approx \sum_{\vec{x}', \vec{n}} \langle \Omega | \hat{\Gamma}_{n'}(\vec{x}') | \psi \rangle \Delta_{n',n}(\vec{x}', \vec{x}) , \tag{2.24}$$

then we will have succeeded in obtaining an equation of the form (2.18'). All subsequent discussions regarding passage to the continuum limit and renormalization will be the same as for the earlier approach.

In the specific model studied in Sec. III, the operator method based on Eq. (2.22) runs into trouble within the zero-width and planar approximations. We reserve further comment until the conclusions, after the specific troubles that are encountered have been explained.

D. Continuum limit and renormalization

To validate the approach that is being advocated, it is necessary to replace the "suppose that" abstractions of the preceding subsection with a concrete, implementable framework for calculation.

This is no simple problem, and we do not pretend to be anywhere close to a solution. Even so, some things can be said which may serve to focus attention on the vital components of the problem.

First of all, even the zeroth-order term of the wave equation is ill-defined in the continuum

limit. Restoring the correct dimensions to the energy, one has $\epsilon_n \propto (g^2/a^2)P_n$, where P_n is the true, dimensional perimeter of the path n lattice units long. This is no different from the result in any other approach to lattice gauge theories. One would like to be able to replace a with a length parameter of relevance to hadronic physics, such as the Regge slope, $a^2 \propto \alpha'$.

Beyond the zeroth order, Eq. (2.18′) will contain, in general, other terms which are superficially divergent as $a \to 0$; finite terms, obtained when differences pass to derivatives, providing compensating a's in numerators; and terms which tend to zero as $a \to 0$, from very high derivatives.

Thus, as was mentioned earlier, the right-hand side of Eq. (2.18′) will look very much like the expansion for the inverse propagator of a cut off field theory. Unfortunately, the fact that some of the divergences are *quadratic* in the cutoff makes the analogy discouraging, as that is symptomatic of a nonrenormalizable field theory. What is worse, even if the coefficients of higher powers of $(\partial^2/\partial \vec{X}^2)$ vanish as the cutoff is removed, we see already at this stage that nothing ensures a relativistic spectrum at all mass levels because of the possible presence of $\pi(y, y')$.

The upshot of these remarks is that we do not really expect the conventional renormalization principles to be directly applicable to our wave equation. On the other hand, there may exist enough of a resemblance that some of the techniques of conventional renormalization theory can be brought to bear either directly, or in a slightly altered form.[15]

There is not much point in pursuing an abstract discussion of these issues much further than this. It will be much more profitable to bring the problems to life in the context of a specific example. The example to which we now pass simplifies much of the discussion by freezing out all the nontrivial degrees of freedom of the problem except one—the path length itself. Even when only this degree of freedom is allowed, $a \to 0$ infinities occur and must be dealt with.

III. A SIMPLIFIED MODEL AND ITS APPROXIMATE SOLUTION

In this section, we study a restricted class of motions of the quarkless hadrons present in the gauge theory. The restriction that is introduced enables us to focus on the dilational degree of freedom of these mesons. This degree of freedom can be characterized by a total length parameter σ, and the specific problem that is addressed is how the hadron wave function depends on this parameter.[16] In attacking this problem, we follow the general strategy outlined in Sec. II to obtain a wave equation.

A. Definition of the model

We continue to use the Hamiltonian of Sec. II, but restrict our attention to a one-dimensional array of plaquettes in the $\hat{x}$ direction as in Fig. 2. Thus, on the restricted lattice, the scaled Hamiltonian is

$$H = H_0 - \lambda V , \tag{3.1a}$$

$$H_0 = \sum_{\vec{n}, m} \pi^2(\vec{n}, m) , \tag{3.1b}$$

with $m = 1$ (2) corresponding to $\hat{x}$ ($\hat{y}$), respectively; $\lambda = 1/g^4$; and

$$V = \sum_x [B(x) + B^\dagger(x)] , \tag{3.1c}$$

with $B_{(x)}$ a box on a unit plaquette with lower left corner $(x\hat{x}, 0\hat{y})$, and circulation given by the right-hand rule.

In the space of the model, it is possible to introduce a reflection $x\hat{x} \to -x\hat{x}$. Under this operation, which we denote by P, one has

$$P: \; B(x) \to B^\dagger(-x-1) . \tag{3.2}$$

Thus, H is invariant under P, and states may be labeled by a P eigenvalue.

The states of interest in this model, subject to the approximations described in Sec. II, will be closed loops of U's, stretched along the x direction. Thus, an n-link gauge-invariant "right-handed" configuration can be labeled

$$\varphi^{(n)}(x) = \prod_{j=0}^{n-1} B(x+j) , \tag{3.3}$$

and a "left-handed" configuration is simply $\varphi^{\dagger(n)}(x)$. However, under P, one has

$$P: \; \varphi^{(n)}(x) \to \varphi^{\dagger(n)}(-x-n) . \tag{3.4}$$

Thus the linear combinations that transform into each other under P are

$$P: \; [\varphi^{(n)}(x) \pm \varphi^{\dagger(n)}(-x-n)] \to \pm[\varphi^{(n)}(x) \pm \varphi^{\dagger(n)}(-x-n)] . \tag{3.5}$$

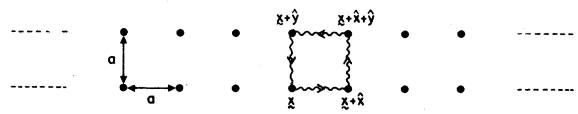

FIG. 2. A lattice one unit wide and infinitely long in the $\hat{x}$ direction. A plaquette $B(x)$ is indicated in accordance with the right-hand rule, and relabeled by $x = (x\hat{x}, 0\hat{y})$.

It will be enough for us to consider the $P+$ states of the system.

The spatial degeneracy of the problem can be taken into account by forming the combinations

$$\varphi^{(n)}(p)=\frac{1}{\sqrt{4\pi}}\sum_x[\varphi^{(n)}(x)e^{ipx}+\varphi^{\dagger(n)}(x)e^{-ip(x+n)}], \tag{3.6a}$$

$$|\varphi^{(n)}(p)\rangle=\varphi^{(n)}(p)|0\rangle . \tag{3.6b}$$

With the normalization conditions defined in (2.8), these states are orthonormal

$$\langle\varphi^{(m)}(q)|\varphi^{(n)}(p)\rangle=\delta_{n,m}\delta(p-q) . \tag{3.7}$$

Certain useful details of calculations such as that leading to Eq. (3.7) will be reserved to a later part of this section.

Within the zero-width planar approximation scheme described in Sec. II, the basic matrix element required for developing the model is

$$\langle\varphi^{(m)}(q)|V|\varphi^{(n)}(p)\rangle=\delta(p-q)(\delta_{m,n+1}a_p+\delta_{m,n-1}a_p^*) , \tag{3.8a}$$

where $a_p\equiv(1+e^{ip})$. The cases when n or m are equal to one require special treatment, because V can take those states to the bare vacuum state. Explicitly,

$$\langle\varphi^{(m)}(q)|V|0\rangle=\sqrt{4\pi}\,\delta_{m,1}\delta(q) , \tag{3.8b}$$

$$\langle\varphi^{(m>1)}(q)|V|\varphi^{(1)}(p)\rangle=\delta_{m,2}\delta(p-q)a_p . \tag{3.8c}$$

B. Ordinary perturbation theory

We record the results of the ordinary Rayleigh-Schrödinger perturbation theory to the second order in V for later reference. The states $|\phi^{(n)}(p)\rangle$ are eigenstates of H_0, with eigenvalues $\epsilon_n=2(n+1)$, as in Eq. (2.9). The perturbation will mix these states with $(n\pm1)$ configurations, and this has the effect of "broadening" the x extent of the gluon wave function. In addition, however, these transitions are responsible for displacing the bare gluon bound states in the $\pm x$ directions, and it is in this manner that the momentum dependence of the energy develops.[17]

It is straightforward to compute

$$|\Omega\rangle\approx|0\rangle+\tfrac{1}{4}\sqrt{4\pi}\,\lambda|\varphi^{(1)}(0)\rangle+\tfrac{1}{12}\sqrt{\pi}\,\lambda^2|\varphi^{(2)}(0)\rangle . \tag{3.9}$$

Similarly, retaining only the connected parts,

$$|G^{(n)}(p),P+\rangle\approx|\varphi^{(n)}(p)\rangle+\tfrac{1}{2}\lambda[a_p|\varphi^{(n+1)}(p)\rangle-a_p^*|\varphi^{(n-1)}(p)\rangle]+\tfrac{1}{8}\lambda^2[a_p^2|\varphi^{(n+2)}(p)\rangle+a_p^{*2}|\varphi^{(n-2)}(p)\rangle] . \tag{3.10}$$

C. The truncated Hamiltonian

1. A false start

It is evident that the operator $\phi(x)^{(m)}$ defined in Eq. (3.3) will couple the state $|G^{(n)}(p),\ P+\rangle\equiv|G\rangle$ to $|\Omega\rangle$ at some finite order of perturbation theory. In particular, if $m=n$, there will be a nonzero overlap at the zeroth order. Thus, it seems natural to consider the wave functions

$$\Phi^{(n)}(x,t)=\langle\Omega|\varphi^{(n)}(x,t)|G\rangle , \tag{3.11a}$$

$$(E_G-E_\Omega)^2\Phi^{(n)}(x,0)=\langle\Omega|[H,[H,\varphi^{(n)}(x,0)]]|G\rangle . \tag{3.11b}$$

The double commutator of H with $\phi(x)^{(n)}$ can be computed from the canonical relations (2.2), and gives a result of the form $\langle\Omega|\tilde{H}\phi(x)^{(n)}|G\rangle$, where $\tilde{H}$ is an operator containing both canonical coordinates and canonical momenta.

After lengthy computation, it is possible to rewrite

$$\langle\Omega|\tilde{H}\varphi^{(n)}(x,0)|G\rangle=\sum_{x',n'}\Delta^{(n,n')}(x,x')\Phi^{(n')}(x') , \tag{3.12}$$

with the "wave operator" $\Delta^{(n',n)}(x',x)$ expressed as a power series in λ.

Unfortunately, the procedure sketched above suffers from a subtle but fatal drawback within the zero-width, planar approximation. The problem is that these nonperturbative approximations violate unitarity. As a consequence, within the present model, the canonical commutators are violated, and the passage from $\langle\Omega|[H,[H,\phi(x)^{(n)}]]|G\rangle$ to $\langle\Omega|\tilde{H}\phi(x)^{(n)}|G\rangle$ is invalid. It is best to demonstrate this point with a specific example. Consider the operator

$$\hat{\delta}=\{V^2,\varphi^{(n)}(x)\}-2V\varphi^{(n)}(x)V , \tag{3.13}$$

which is zero by the canonical commutation relations, since V and ϕ each contain only field variables $B(x)$ and $B(x)^\dagger$. However, let us compute the matrix element of this operator between $\langle0|$ and $|\phi^{(n)}(p)\rangle$, by inserting intermediate states $|\phi^{(k)}(q)\rangle$:

$$1 \approx |0\rangle\langle 0| + \sum_{m=1}^{\infty} \int dq\, |\varphi^{(m)}(q)\rangle\langle\varphi^{(m)}(q)| \equiv |0\rangle\langle 0| + \sum_{m}{}' |m\rangle\langle m| , \tag{3.14a}$$

$$\langle 0|\hat{\delta}|\varphi^{(n)}(p)\rangle = \sum_{k,m}{}' [V_{0m}V_{mk}\langle k|\varphi^{(n)}_{(x)}|\varphi^{(n)}_{(p)}\rangle + \langle 0|\varphi^{(n)}(x)|m\rangle V_{mk}\langle k|V|\varphi^{(n)}(p)\rangle - 2\langle 0|V|m\rangle\langle m|\varphi^{(n)}(x)|k\rangle\langle k|V|\varphi^{(n)}(p)\rangle] . \tag{3.14b}$$

The right-hand side is, in general, different from zero, which illustrates that $[V, \phi(x)^{(n)}] \neq 0$ within our scheme of truncating the resolution of unity.

It can be shown by tedious calculation that if V and $\phi(x)^{(n)}$ are *not* commuted, but rather computed consistently in all their possible orderings, then the right-hand side of Eq. (3.11b) does reproduce the correct energy shift $(E_G - E_\Omega)^2$ to second order in λ. However, it soon becomes intractable to construct a nonperturbative wave equation for Φ within this framework. We turn to a slightly different approach.

2. *The Hamiltonian in the single-particle subspace*

The problem of commuting operators can be circumvented by working with expressions which do not involve commutators. Thus, we are led to consider the following second-order equation [cf. (2.21)]:

$$E_G{}^2\langle n|G\rangle = \sum_{m,k} H_{nm}H_{mk}\langle k|G\rangle . \tag{3.15}$$

In this equation, we are approximating 1 by $\sum_k |k\rangle\langle k|$, where the $|k\rangle$ are "single-particle states," including $|0\rangle$, as in Eq. (3.14a). Within this approximation, H_{nm} has a very simple form, which follows from (3.8) for n and m unequal to zero:

$$\langle\varphi^{(n)}(q)|H|\varphi^{(m)}(p)\rangle = \delta(p-q)[\epsilon_n\delta_{n,m} - \lambda(a_p\delta_{n,m+1} + a_p^*\delta_{n,m-1})] . \tag{3.16}$$

3. *The wave equation (long loops)*

In this subsection, we shall write the wave operator for the cases when $n \gg 1$. This simplifies several expressions, because overlaps with $|0\rangle$ can be neglected. To be precise, abbreviating $1 \cong |0\rangle\langle 0| + |m\rangle\langle m|$, since $\epsilon_0 = 0$ and $\langle 0|V|0\rangle = 0$, we have

$$H^2 = [|m\rangle H_{mk}H_{kn}\langle n|] - \lambda[|m\rangle H_{mk}V_{k0}\langle 0|] - \lambda[|0\rangle V_{0k}H_{kn}\langle n|] + \lambda^2[|m\rangle V_{m0}V_{0n}\langle n|] + \lambda^2[|0\rangle|V_{0k}|^2\langle 0|] . \tag{3.17}$$

(In this equation, the states $|m\rangle$ do *not* include $|0\rangle$.) For example, $\langle n|V|0\rangle = 0$ unless $n = 1$. Thus, $H_{mk}V_{k0}$ will only couple in to the states $|m=2\rangle$ and $|m=1\rangle$. We have, then, for large n

$$\langle n|H^2|G^{(n)}\rangle \approx H_{nk}H_{km}\langle m|G^{(n)}\rangle . \tag{3.18}$$

Making use of (3.16), we have

$$E_G{}^2\langle\varphi^{(n)}(p)|G^{(n)}\rangle = \sum_{n'}\Delta_{nn'}(p)\langle\varphi^{(n')}(p)|G^{(n)}\rangle , \tag{3.19}$$

$$\Delta_{nn'}(p) = \epsilon_n{}^2\delta_{n,n'} - \lambda(\epsilon_n + \epsilon_{n'}')(a_p\delta_{n',n-1} + a_p^*\delta_{n',n+1}) + \lambda^2(2a_p^*a_p\delta_{n',n} + a_p{}^2\delta_{n',n-2} + a_p{}^{*2}\delta_{n',n+2}) . \tag{3.20}$$

A configuration-space version of this equation may be written using

$$\psi^{(n)}(x) \equiv \frac{1}{2\pi}\int_{-\pi}^{\pi} dp e^{ipx}\langle\varphi^{(n)}(p)|G^{(n)}\rangle . \tag{3.21}$$

We obtain

$$E_G{}^2\psi^{(n)}(x) = \sum_{n',g}\Delta_{n,n'}(x,y)\psi^{(n')}(y) , \tag{3.22a}$$

where

$$\Delta_{n,n'}(x,y) \equiv \epsilon_n{}^2\delta_{n,n'}\delta_{x,y} - \lambda(\epsilon_n + \epsilon_{n'})[\delta_{n',n-1}(\delta_{x,y} + \delta_{y,x+1}) + \delta_{n',n+1}(\delta_{x,y} + \delta_{y,x-1})] + \lambda^2[2\delta_{n,n'}(2\delta_{x,y} + \delta_{y,x+1} + \delta_{y,x-1}) + \delta_{n',n-2}(\delta_{x,y} + 2\delta_{y,x+1} + \delta_{y,x+2}) + \delta_{n',n+2}(\delta_{x,y} + \delta_{y,x-2} + 2\delta_{y,x-1})] . \tag{3.22b}$$

It will be useful in our later considerations to refer to the perturbation theory expansion of these wave functions

$$\psi^{(m)}(x-y)\approx \delta_{n,m}\delta_{x,y}-\tfrac{1}{2}\lambda[\delta_{m,n+1}(\delta_{x,y}+\delta_{y,x+1})-\delta_{m,n-1}(\delta_{x,y}+\delta_{y,x-1})]\,. \tag{3.23}$$

We are now ready to discuss the physical interpretation of our wave equation in the quasicontinuum limit. (In the next subsection, the rules of calculation used throughout this section are reviewed. Readers familiar with these rules may wish to pass directly to the next section.)

D. Calculational details

The matrix elements used in this section are all of the form

$$G_n=\langle 0|A_1A_2\cdots A_n|0\rangle\,, \tag{3.24}$$

where $|0\rangle$ is the ground state of H_0, and $A_1, A_2,\ldots$, etc., are gauge-invariant operators. In the coordinate representation, this reads

$$G_n=\int \mathfrak{D}\theta\psi_0^*(\theta)a_1(\theta)\cdots a_n(\theta)\psi_0(\theta)\,. \tag{3.25}$$

The normalized zeroth-order vacuum wave function is

$$\psi_0(\theta)=1\,, \tag{3.26a}$$

and the operators in (3.25) are of the form

$$a_j(\theta)=\sum_{x_i\in\Gamma_j}(B_{x_1}B_{x_2}\cdots B_{x_n}\pm B^\dagger_{x_1}\cdots B^\dagger_{x_n})\,, \tag{3.26b}$$

as can be seen from Eqs. (3.3), (3.6).

Thus, keeping in mind that

$$B_x(\theta)=\exp\{i[\theta_1(\vec{x})+\theta_2(\vec{x}+\hat{1})-\theta_1(\vec{x}+\hat{2})-\theta_2(\vec{x})]\} \tag{3.27}$$

with $\vec{x}=(x\hat{x},0\hat{y})$, and the definition (2.6d) for the measure $D\theta$, it is clear that G_n vanishes unless each B is contracted with a $B^\dagger$. The weight assigned to each such contraction is a Kronecker δ on the x indices, since $e^{i\theta}e^{-i\theta}=1$, and the measure $D\theta$ contains factors $(2\pi)^{-1}$ to fix the normalization:

$$\int \mathfrak{D}\theta B_x(\theta)B^\dagger_y(\theta)=\delta_{x,y}\,. \tag{3.28}$$

We remind the reader that the "planar approximation" allows no more than one B or $B^\dagger$ at any plaquette labeled by the corner site.

Consequently, a matrix element G_n may be calculated by assigning a "plug" to each B and a "socket" to each $B^\dagger$ that appears in the product $\Pi_{j=1}^n A_j$. One simply inserts the plugs into the sockets in all possible ways. If there are any loose plugs or sockets left over, G_n vanishes.

IV. THE PHYSICAL INTERPRETATION OF THE MODEL

A. Passing to the quasicontinuum limit

As mentioned in Sec. II, the nontrivial step involved in passing to the continuum limit is to provide a suitable physical interpretation for the discrete labels used in the wave operator and in the wave functions. In the simplified model under consideration, the real-space length of the plaquette $\phi(x)^{(n)}$ is

$$\sigma=2(n+1)a\,. \tag{4.1}$$

Thus, we can write at once

$$\epsilon_n=\sigma/a\,. \tag{4.2}$$

We shall call replacements such as (4.1) a "quasicontinuum" limit, because continuum variables have been introduced, but the lattice spacing has not been eliminated.

Furthermore, we can consider two different path lengths, labeled by n and n', and write

$$\sigma'-\sigma=2a(n'-n) \tag{4.3a}$$

or

$$\delta\sigma=2a\Delta n\,. \tag{4.3b}$$

Then a function defined on the integers n might have a variation

$$F(n+\Delta n)\to F(\sigma+\delta\sigma)\approx F(\sigma)+\delta\sigma\frac{\partial F}{\partial\sigma}\,, \tag{4.4a}$$

or

$$F(n\pm 1)\to F(\sigma)\pm 2a\frac{\partial F}{\partial\sigma}\,. \tag{4.4b}$$

Thus, our goal will be to express the quantum-number changes $n\to n+\Delta n$ as differential variations of the wave functions on the length of the path to which n refers. We will find, however, that (4.4) is too naive for our applications, as it does not separate relative motions from center-of-mass motions.

Part of the action of the Hamiltonian is to provide the standard rigid-body translations of plaquettes of fixed length in space. A specific term of superficial order λ^2 in Eq. (3.22) is of

this type:

$$E_G{}^2\psi^{(n)}(x) \supset 2\lambda^2[2\psi^{(n)}(x)+\psi^{(n)}(x+1)+\psi^{(n)}(x-1)] - 2\lambda^2\left[a^2\frac{\partial^2}{\partial x^2}\psi^{(n)}(x)+4\psi^{(n)}(x)\right]. \quad (4.5)$$

Therefore, the term above contains a contribution to the c.m. motion of the plaquette, and an additional piece which amounts to an energy correction.

Next, we examine the term of superficial order λ in Eq. (3.22). This term requires careful scrutiny:

$$E_G{}^2\psi^{(n)}(x) \supset (-\lambda)\{(\epsilon_n+\epsilon_{n-1})[\psi^{(n-1)}(x)+\psi^{(n-1)}(x+1)] +(\epsilon_n+\epsilon_{n+1})[\psi^{(n+1)}(x)+\psi^{(n+1)}(x-1)]\}. \quad (4.6)$$

The wave functions in this expression themselves lead as order λ in perturbation theory, so *within* perturbation theory, the right-hand side of (4.6) is

$$\mathrm{RHS}(4.6) \approx \lambda^2\epsilon_n[\psi^{(n)}(x+1)-\psi^{(n)}(x-1)]. \quad (4.7)$$

(In obtaining this equation, we have approximated $\epsilon_n+\epsilon_{n\pm1}\approx 2\epsilon_n$.) Inspection of Eq. (4.6) leads us to expect that the quasicontinuum limit of the right-hand side will involve mixed derivatives in x and in σ. The perturbative result, Eq. (4.7), suggests that only $\partial/\partial x$ should appear. What is going on?

To clarify the situation, let us refer to Fig. 3. The wave functions are, of course, defined on the configurations displayed. We would like to extract the dependencies of these functions on "center-of-mass" (X) and "relative" (l) variables, $\psi(x)^{(n)} \to \psi(X, l)$, where $l=n$. To this end, consider functions $F(X, l)$ defined on these configurations:

$$F_1 = F(x+\tfrac{1}{2}(n+1), n+1), \quad (4.7a)$$

$$F_2 = F(x+\tfrac{1}{2}(n-1), n+1). \quad (4.7b)$$

Evidently, as the path length l changes, the center of mass X shifts relative to $(x+\frac{1}{2}n)$. We have as a consequence of those shifts

$$F_1+F_2 = \left[2+\left(\frac{a}{2}\right)^2\frac{\partial^2}{\partial X^2}\right]F(X=x+\tfrac{1}{2}n, n+1). \quad (4.8)$$

We shall then write for the relative motion

$$F(X, l+\Delta l) \approx F(X, l)+a\Delta l\frac{\partial F}{\partial l}+\frac{a^2}{2}(\Delta l)^2\frac{\partial^2 F}{\partial l^2}, \quad (4.9)$$

and for the total motion,

$$F_1+F_2 \approx 2\left(1+2a\frac{\partial}{\partial\sigma}+2a^2\frac{\partial^2}{\partial\sigma^2}+\frac{a^2}{8}\frac{\partial^2}{\partial X^2}\right)F(X, \sigma). \quad (4.10)$$

All we have done is disentangle the c.m. motion from the relative motion.

It is important to note that the ansatz Eq. (4.9) is a significant input to the quasicontinuum calculational scheme. From perturbation theory alone, we have, as was noted earlier,

$$\psi^{(n+1)}(x) = -\tfrac{1}{2}\lambda[\psi^{(n)}(x)+\psi^{(n)}(x+1)]+O(\lambda^2) \quad (4.11a)$$

This equation can be cast in the form

$$a\left(\frac{\partial\psi}{\partial n}+\frac{\lambda}{4}\frac{\partial\psi}{\partial x}\right) = -(1+\lambda)\psi, \quad (4.11b)$$

so that as $\lambda\to 0$, the solution is $\psi(x, n)\approx e^{-n/a}\psi(x, 0)$. The wave functions that satisfy our continuum wave equation may or may not be in qualitative agreement with perturbation theory results such as the above.

In any case, we are now in a position to write the right-hand side of Eq. (4.6) in the quasicontinuum limit, for large $\epsilon_n\to\sigma/a$:

$$\mathrm{RHS}(4.6) \to (-8\lambda)\left[\frac{\sigma}{a}\left(1+2a^2\frac{\partial^2}{\partial\sigma^2}+\frac{a^2}{8}\frac{\partial^2}{\partial X^2}\right) - 2a\frac{\partial}{\partial\sigma}\right]\psi(X, \sigma).$$

Two terms in Eq. (3.22) proportional to $\lambda^2\delta_{n, n'+2}$ and $\lambda^2\delta_{n, n'-2}$ remain to be discussed. The shift in the c.m. is extracted in a manner analogous to Eq. (4.8), and then Eq. (4.9) is applied to give the perimeter change. Collecting these terms, and all those already discussed, we obtain the quasicontinuum wave equation

$$E_G{}^2\psi(X, \sigma) = \left[\hat{m}^2(\sigma)-f(\sigma)\frac{\partial^2}{\partial X^2}\right]\psi(X, \sigma), \quad (4.12a)$$

$$\hat{m}^2(\sigma) = -\frac{\partial}{\partial\sigma}\left[\left(\frac{4\sigma}{a}-16\lambda\right)\frac{\partial}{\partial\sigma}\right]+\left[\frac{1}{4\lambda}\left(\frac{\sigma}{a^2}\right)^2-\frac{2\sigma}{a^3}+\frac{3\lambda}{a^2}\right], \quad (4.12b)$$

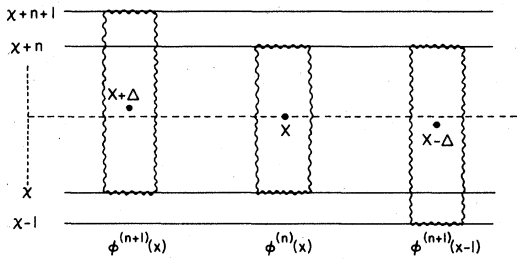

FIG. 3. We consider $\phi^{(m)}(\chi)=F(X=\chi+\frac{1}{2}m, l=m)$.

$$f(\sigma)=\left(\frac{\sigma}{4a}-\frac{3\lambda}{4}\right). \tag{4.12c}$$

In obtaining (4.12), we rescaled the time to dimensional units, i.e., $E_G^2 \to (2a/g^2)^2 E_G^2$. It is now necessary to discuss the normalization of this equation.

B. Naive dimensional transmutation

The first question of renormalization involves passing from the discrete labels to continuum variables in ψ:

$$(\psi,\psi)=\sum \psi^{*(m)}(x)\psi^{(m)}(x) \to \int_{-\infty}^{\infty} dx\psi^*(X,\sigma)\psi(X,\sigma)\,. \tag{4.13}$$

These renormalizations can be handled by $\psi(x) \overset{(n)}{\to} Z\psi(X,\sigma)$, and since the wave equation is linear in ψ, they will not alter its form. Let us pass directly to examining the equation itself.

Let us try a factorizing ansatz in Eq. (4.12):

$$\psi(X,\sigma)=e^{iPX}\tilde{\psi}(\sigma,P^2)\,, \tag{4.14a}$$

$$[\hat{m}^2(\sigma)+P^2 f(\sigma)]\tilde{\psi}_n(\sigma,P^2)=\epsilon_n^2(P^2)\tilde{\psi}_n(\sigma,P^2)\,. \tag{4.14b}$$

For sufficiently small λ, it is known from Sturm-Liouville theory that Eq. (4.14b) has an ascending spectrum of discrete eigenvalues ϵ_n^2, which depend parametrically on the coupling λ, the spacing a, and the c.m. momentum P^2. Let us discuss this point more fully.

We can rewrite Eq. (4.14) in the form

$$\left\{-\frac{\partial}{\partial\sigma}\left[\left(\frac{4\sigma}{a}-16\lambda\right)\frac{\partial}{\partial\sigma}\right]+V(\sigma,P^2)\right\}\tilde{\psi}=\epsilon^2\tilde{\psi}\,, \tag{4.15a}$$

$$V(\sigma,P^2)=\frac{1}{4\lambda}\left(\frac{\sigma}{a^2}\right)^2-\left(\frac{2}{a^2}-P^2\right)\frac{\sigma}{a}+3\lambda\left(\frac{1}{a^2}-\frac{P^2}{4}\right). \tag{4.15b}$$

Changing variables to $4\sigma/a-16\lambda=W^2$, we obtain

$$\left[-\frac{4}{a^2}\frac{1}{W}\frac{\partial}{\partial W}\left(W\frac{\partial}{\partial W}\right)+V(W,P^2)\right]\tilde{\psi}=\epsilon^2\tilde{\psi}\,. \tag{4.15c}$$

This equation has the form of a radial wave equation in cylindrical coordinates, for a wave function independent of the azimuthal or z coordinates, in a potential $V(W,P^2)$.

Our change of variables gives an imaginary W if $(\sigma/a<4\lambda)$. For small coupling λ, this occurs for very short paths. However, we do not trust our wave equation for small paths, and this complication is an artifact of our approximation.

One way to proceed is to restrict the range of σ so that W is always real. (We choose the positive branch of the square root, of course.) Alternatively, we can pass to a new equation which we trust in the large-σ, small-λ limit, but which is mathematically consistent down to $\sigma=0$. All that is required is that we modify the internal kinetic energy piece,

$$m^2(\sigma)=\hat{m}^2(\sigma)-16\lambda\frac{\partial^2}{\partial\sigma^2}\,. \tag{4.12'}$$

We shall proceed using this substitution for simplicity.

Next, observe that there is a range of P^2 over which, as the limit $P^2\to 0$ is taken, the eigenvalues (– functions) $\epsilon_n^2(\tilde{\psi}_n)$ go smoothly over into the eigenvalues (-functions) $E_n^2(\psi_n)$ of the equation

$$m^2(\sigma)\psi_n(\sigma)=E_n^2\psi_n(\sigma)\,. \tag{4.16}$$

(Crudely, one requires that the potential should not change sign. Also, however, the potential should not develop any new critical points, nor change inflection points into maxima or minima, as a function of P^2, for the desired continuity of the eigenfunctions and eigenvalues.)

It may be worthwhile to think of $2\pi/a\equiv\Lambda$ as a momentum cutoff. Then $P\ll\Lambda$ *defines* small P, and the coefficients $(2/a^2-P^2)$, $(1/a^2-P^2/4)$ in $V(P^2,\sigma)$ remain close to their $P^2=0$ values. So we can just treat $P^2 f$ as a perturbation onto $V(0,\sigma)$.

Rescaling $W=2\lambda^{1/6}\rho$, we have

$$\left[-\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)+\frac{\rho^4}{16}+\frac{a^2P^2}{4}(\lambda^{2/3}\rho^2+4\lambda^{4/3})\right]\tilde{\psi}_n=N_n\tilde{\psi}_n\,, \tag{4.17a}$$

$$N_n\equiv\lambda^{1/3}(a^2\epsilon_n^2+\lambda)\,, \tag{4.17b}$$

or

$$\epsilon_n^2(P^2)\approx E_n^2+P^2\bar{f}_n+\cdots\,, \tag{4.18a}$$

where

$$\eta_n\equiv\lambda^{1/3}(a^2E_n^2+\lambda) \tag{4.18b}$$

is a dimensionless increasing function of integers n determined from the eigenvalue equation (4.17a) for $P^2=0$, and

$$\bar{f}_n=\tfrac{1}{4}\lambda^{2/3}(\bar{\rho}_n^2+4\lambda^{2/3})\,. \tag{4.18c}$$

In this equation, $\bar{\rho}_n^2$ is the expected value of ρ^2 in the nth discrete eigenstate.

Evidently, the coefficient $\bar{f}_n$ can be interpreted as being proportional to the speed of light in the formula (4.18). Inasmuch as in the Hamiltonian lattice theory, the units of measurement of time and space have been defined incommensurate with $c=1$, we have the freedom to rescale the units

of spatial momentum to yield $c=1$ at the end of our calculation.

However, since the rest-mass scale is *also* determined by the lattice spacing "a," we should consistently exercise this remaining freedom by introducing a "renormalized" energy equation at a level of our choice labeled by $n=N$:

$$(\epsilon_N{}^2)_R=\epsilon_N{}^2\Big/\left(\frac{\partial\epsilon_N{}^2}{\partial P^2}\right)_{P^2=0}$$

$$\equiv m_N{}^2+P^2+\cdots . \tag{4.19}$$

The "physically observable" mass m_N is expressed in terms of the parameters of the bare theory by

$$m_N{}^2=\frac{\eta(N)\lambda^{-1/3}-\lambda}{a^2\bar{f}_N}\equiv\frac{G_N(\lambda)}{a^2} . \tag{4.20}$$

Thus, $m_N{}^2$ is a function of two variables, λ and "a." We shall, however, attempt to implement a very naive form of cutoff independence of the physical parameter m_N by the following argument.

Choose an "a" arbitrarily, and determine a λ such that Eq. (4.20) is satisfied, given $m_N{}^2$. Then choose a different "a" and ask how λ must change so that (4.20) continues to be satisfied. Evidently, if $a\to\mu a$, we must have $G_N(\lambda)\to\mu^2 G_N(\lambda)$, so

$$\frac{G_N(\lambda(a))}{G_N(\lambda(\mu a))}=\frac{1}{\mu^2} , \tag{4.21}$$

or

$$G_N(\lambda(\mu))=G_N(\lambda(1))\mu^2 .$$

Making use of our explicit formula for G_N in terms of $\eta(N)$ and $\bar{f}_N$, we find [with $G_N(\lambda(1))\equiv G_0$]

$$\lambda^{2/3}=\frac{-(\frac{1}{16}\mu^2G_0\bar{\rho}_N)\pm[(\frac{1}{16}\mu^2G_0\bar{\rho}_N)^2+4\eta_N(1+\frac{1}{4}\mu^2G_0)]^{1/2}}{2(1+\frac{1}{4}\mu^2G_0)} , \tag{4.22}$$

i.e.,

$$\lim_{\mu\to 0}\lambda=\eta_N{}^{3/4} .$$

That is, if λ approaches a fixed number as the chosen lattice spacing becomes infinitesimally small, the limit $a\to 0$ exists in Eq. (4.20). In this sense, we have achieved dimensional transmutation, and a relativistic dispersion law at level N. It is easy to verify that the terms superficially of order $a^{2(n-1)}P^{2n}$ not exhibited in (4.19) have finite coefficients as $a\to 0$ because $\lambda\to$ const, so these higher-order terms in P^2 vanish as the cutoff is removed.

However, our discussion has been carried out at a given level N. What happens to the other levels? Evidently

$$m_n{}^2=\frac{E_n{}^2}{\bar{f}_N}\to\frac{\eta_n-\eta_N}{\eta_N-\eta_N}m_N{}^2+\cdots+O(\mu) . \tag{4.23}$$

So all the other levels are pushed off to become tachyons if $n<N$, or of ∞ mass if $n>N$. Our naive procedure is seen to be much too naive.

C. Problems with covariance

The difficulty encountered in the preceding subsection arose from the observation that each level of excitation carries its own "speed of light." The spectrum is not at all relativistic, and a simple renormalization cannot cure this outcome of our approximation to the full theory. What is worse, it is not clear that the unrelativistic form of the spectrum is a result of the approximation scheme. Nonetheless, we can try to find physical reasons for our failure to achieve covariance.

One possible flaw in the zero-width approximation is that such an approximation may not be covariant even if the underlying dynamics is manifestly covariant. For example, in electrodynamics a boost may introduce an admixture of pairs into a state which is single particle at rest in the usual perturbative description. So it may be necessary to take into account processes in which long loops emit short loops and reabsorb them.

However, a much more naive shortcoming of our present formalism suggests itself immediately from the interpretation of $\bar{f}_n$ as having to do with $c\neq 1$. It is that if an object extended in the X direction has length L at rest, in the observer's rest frame the object has length L/γ if it has momentum P_X. We have not taken the Lorentz contraction into account in interpreting the parameter "σ" of our wave equation.

A completely consistent accounting for this effect is extremely complicated to implement exactly. In this paper, we shall settle for a heuristic approach to the problem. Let us consider the lattice as an arbitarily chosen reference grid for the motions of our hadrons in a continuum. Then the Lorentz-contracted length could be envisioned as a counting of the same n steps of the rest-frame lattice on a new lattice whose lattice

spacing was contracted by the factor γ. Since $\gamma=E/m$, we have a different γ at each mass level, so we can write [with $E_n^2=a^{-2}F_n(\lambda)$],

$$\epsilon_n^2=\frac{F_n+a^2P^2\bar{f}_n}{a_n^2}+\cdots, \tag{4.24}$$

where we assume

$$a_n^2=a^2+P^2b_n^2+\cdots. \tag{4.25}$$

Thus,

$$\epsilon_n^2\approx\frac{F_n}{a^2}+P^2\left(\bar{f}_n-\frac{F_nb_n^2}{a^4}\right)+\cdots. \tag{4.26}$$

We now attempt to set

$$\begin{aligned}&\text{(i)}\ \ \bar{f}_n-\frac{F_nb_n^2}{a^4}=1,\quad \forall n,\\ &\text{(ii)}\ \ [\text{cf. (4.19)}]\ m_n^2=\frac{F_n}{a^2}\left(\bar{f}_n-\frac{F_nb_n^2}{a^4}\right)^{-1},\end{aligned} \tag{4.27}$$

that is,

$$\begin{aligned}&\text{(i)}\ \ b_n^2\approx a^4(\bar{f}_n-1)/F_n,\\ &\text{(ii)}\ \ b_n^2\approx -a^2/m_n^2,\quad a\to 0,\end{aligned}$$

so that

$$\begin{aligned}&\text{(i)}\ \ a_n^2\approx a^2\left[1+\frac{a^2(\bar{f}_n-1)}{F_n}P^2+\cdots\right],\\ &\text{(ii)}\ \ a_n^2\approx a^2\left[1-\frac{P^2}{m_n^2}+\cdots\right].\end{aligned}$$

The correct Lorentz contraction would give

$$a_n^2=a^2\gamma_n^{-2}\approx a^2\left[1-\frac{P^2}{m_n^2}+\cdots\right].$$

So if $m_n^2=F_n/a^2$, we must have for case (i)

$$\bar{f}_n=0,\quad \forall n. \tag{4.28}$$

But from (4.18c), we see that this implies that λ must be a function of n, $\lambda=\lambda(n)$. This behavior characterizes a nonrenormalizable theory. An infinite number of counterterms must be added onto our *effective* Hamiltonian. This can reflect *either* the necessity of an infinite number of terms in the latticized original Hamiltonian to obtain relativistic covariance in the hadronic sector—*or* the necessity of including an infinite number of new (possible nonzero width or nonplanar) states in the resolution of the identity.

Case (ii) appears consistent with the Lorentz contraction formula to this order, but one might remain uneasy with introducing both a Lorentz contraction factor *and* "renormalizing" ϵ_N^2 as in (4.19). In any event, the discussion has been intended as a heuristic case for the possibilities which may be open, and the result is certainly not conclusive.

V. SUMMARY AND CONCLUSIONS

The principal aim of this investigation has been to use the lattice formulation of gauge theory to extract hadronic wave equations from which the spectrum of hadronic states may be computed. Two methods of formulating wave equations were discussed in Sec. II. These methods are quite general,[17] and may be used to construct wave equations for the lattice Hamiltonian theory independently of the zero-width and planar approximations, albeit with corresponding complications.

One of the methods discussed in Sec. II was illustrated by considering a simple but nontrivial model in Sec. III. On the lattice, the Schrödinger equation for the stationary states is a difference equation for a single-component wave function in the approximation scheme we have employed. (Coupling to other types of states, excluded within our approximation, would give rise to matrix equations with indices corresponding, e.g., to single-particle, double-particle, etc., states.) We attempted to construct solutions appropriate for the "long" gluon-bound-state sector by examining the continuum limit of the difference equation in the most straightforward manner possible.

This method is very different from the perturbative solution of the Schrödinger equation in powers of the inverse gauge field coupling constant.[18] At zero center-of-mass momentum, one obtains a well-defined eigenvalue equation for the mass spectrum, in which the internal fluctuation momenta and the potential appear in a physically plausible manner.

In principle, one can solve this equation for the largest part of the potential in closed form, and thereby avoid potentially misleading expansions, e.g., of square-integrable functions in terms of a small number of plane waves. At $P=0$, the equation we obtained implies the existence of an infinite spectrum of internal excitation, in accordance with our expectation from strong-coupling perturbation theory.

Unfortunately, once we consider any finite P, we find that the dispersion law is not of relativistic form. It cannot be stressed too often that no symmetry of the operator form of the theory ensures relativistic covariance of the spectrum. We cannot be sure that even the exact solution of the theory would exhibit covariance in the hadronic sector.

To force covariance on the theory, it appears necessary to go beyond the operator Hamiltonian formalism, as in the attempt to introduce a Lorentz contraction factor by hand in Sec. IV. Of course, the possibility remains that the trun-

cation scheme we employed in approximating the Hamiltonian operator is directly responsible for our breakdown of covariance. (In two-dimensional quantum chromodynamics, the $1/N$ expansion rules out two-meson intermediate states in leading order, yet the one-meson spectrum is relativistic. However, it is possible that the boost structure allows this in the light-cone quantization but not in the equal-time quantization.) If opening decay channels is required for covariance, the approach advocated in this paper is of very limited practical utility for the pure gluon sector. But, again, we cannot be certain that inclusion of more complicated states in the completeness sum would solve our problem.[19]

The most promising way to proceed to add structure to the formalism is probably to demand a precise operator method under which covariance of the theory in the $a \to 0$ limit could be checked at all stages of the calculation. This would allow one to test whether perturbation theory, or a certain scheme of truncation, or any other method of approximation could be expected to preserve covariance.

It is important to keep in mind that Wilson's four-dimensional latticization of gauge field theories is not necessarily immune to the problems of covariance encountered in the investigation reported here. The Euclidean lattice theories do not treat time asymmetrically. Nonetheless, an inverse propagator in that version of the theory might in general be expected to contain expressions of the form

$$-\Delta^{-1}(p^2) = p^2\pi(p^2) + m^2(p^2)\,.$$

As in the present work, we have no assurance that the analytic continuation of Δ^{-1} has poles at a sequence of (masses)^2, with $\Delta^{-1} \approx p^2 - m_n^{\,2}$ near each mass shell.

In spite of these difficulties we have encountered with Lorentz invariance, the possibility remains that a wave equation abstracted from the lattice formulation of the theory might be accurate in the case where the quark mass sets a natural scale for the problem in a nonrelativistic setting, as in charmonium.[20] The methods described in Sec. II can be extended to deal with this system quantitatively, without removing the dynamical nature of the string degrees of freedom. We hope to report on an investigation of the motions of this system in the near future.

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