PHYSICAL REVIEW D VOLUME 20, NUMBER 2 15 JULY 1979

Static sources in classical Yang-Mills theory

P. Sikivie
Stanford Linear Accelerator Center, Stanford University, Stanford, California 94305

Nathan Weiss
Department of Physics, University of Illinois, Urbana-Champaign, Illinois 61801
(Received 7 March 1979)

We examine initial-time Yang-Mills field configurations which satisfy Gauss's law in the presence of static external sources. We show that if such a configuration is an extremum of the energy, then it is a static solution of the Yang-Mills equations. Next we consider Yang-Mills field configurations which satisfy Gauss's law for one external point source. We show that there exists a large class of such configurations which have lower energy than the Coulomb solution.

I. INTRODUCTION

In a previous paper,[1] we gave a detailed analysis of the Yang-Mills field equations in the presence of static external sources. The main features and results of our discussion were the following (for further details, see Ref. 1):

(1) A static external source was defined as one which has vanishing space components $j^a_\mu(x) = \delta_{\mu 0} q^a(x)$, $a = 1, \ldots, n$ where n is the order of the gauge group. It was shown that in this case all the group invariants built out of $q^a(x)$ [such as $C(\vec{x}) = \sum_{a=1}^n q^a(x) q^a(x)$, for example] are time-independent.

(2) We agreed to specify the external sources by giving at each space point $\vec{x}$ the values of the r-independent group invariants $C_1(\vec{x}), \ldots, C_r(\vec{x})$ that can be built out of $q^a(x)$; r is the rank of the gauge group. It is our purpose to try to solve the Yang-Mills equations

$$(D_\mu F^{\mu\nu})^a = \delta^{\nu 0} q^a(x) \tag{1.1}$$

for a given external source $C_1(\vec{x}), \ldots, C_r(\vec{x})$. In the Abelian case, for a given static external source, the most general solution is the Coulomb field plus an arbitrary number of plane waves, and there is only one static solution—the Coulomb field itself. It turns out that in the non-Abelian case, the class of distinct, i.e., non-gauge-equivalent, solutions for a given external source $C_1(\vec{x}), \ldots, C_r(\vec{x})$ is definitely richer than it is in the non-Abelian case. We agreed to characterize these various solutions by gauge-invariant quantities such as their total energy, and their total isospin,[2] and such quantities as $F^a_{\mu\nu}(x) F^a_{\lambda\sigma}(x)$.

(3) We found that a set of static external point charges always admits a static Coulomb solution; but, whereas the Coulomb solution is unique in the Abelian case, in the non-Abelian case there are in general several distinct static Coulomb solutions (for the same spatial distribution of the external point sources) corresponding to the various ways one can "diagonalize" the external point sources in isospin space. One can characterize these distinct solutions by their distinct values of the energy and the total isospin.

(4) We found[3,1] that a continuous extended but localized external charge distribution always admits solutions of energy as low as one wishes. We called these "total screening" solutions because the field strength tensor $F^a_{\mu\nu}(x)$ vanishes outside the region where the external source is localized. These solutions are Coulombic in that their magnetic field vanishes everywhere; their existence is due to the fact that a continuous external charge distribution, which acts like an infinity of point charges localized in a small region of space, will tend, in the non-Abelian case, to go into the energetically favored state where the local isospins at nearby points cancel one another by being oriented in opposite directions. This is not possible in the Abelian case where the external charge distribution is gauge invariant and where the Coulomb solution is unique.

(5) We found[3,1] that continuous extended but localized charge distributions also admit static solutions of a new type, unrelated to the various Coulomb solutions mentioned so far. We called these "magnetic dipole" solutions because they have long-range spherically asymmetric magnetic fields whereas their electric fields are of short range. They have lower energy than the Coulomb solution when $gq/4\pi$, where g is the gauge coupling constant and q the total external charge, exceeds a certain critical value which depends on the shape of the external source. These solutions thus appear to be related to the instability Mandula[4] found in his analysis of the small oscillations around a Coulomb field.

(6) Finally, we formulated the initial-value problem of Yang-Mills fields in the presence of

external sources in the $A_0=0$ gauge. To any initial configuration $A_i^a(\vec{x}, t_0)$, $E_i^a(\vec{x}, t_0)=F_{0i}^a(\vec{x}, t_0)$ which satisfies Gauss's law, i.e., Eq. (1.1) for $\nu=0$, there corresponds a solution to the Yang-Mills equations, which in general will be time dependent. The time dependence is provided by the definition of E_i^a in the $A_0=0$ gauge: $dA_i^a/dt=E_i^a$, and the Yang-Mills Eq. (1.1) for $\nu=1, 2, 3$.

To the above, we would now like to add two new results. In Sec. II we show that an initial-time configuration $A_i^a(\vec{x}, t_0)$, $E_i^a(\vec{x}, t_0)$ which minimizes the energy under the constraint of Gauss's law is in fact a static solution. By definition, a static solution is one in which all gauge-invariant quantities are time independent. In Sec. III, we show that a point source always admits configurations of lower energy than the Coulomb solution. In these initial-time configurations, the electric field is screened at the expense of the appearance of a magnetic field.

II. EXTREMA OF THE ENERGY ARE STATIC SOLUTIONS

In Ref. 1 it was shown that the initial-value problem for classical Yang-Mills fields in the presence of static external sources has the following straightforward formulation in the $A_0=0$ gauge. An initial configuration $A_i^a(\vec{x}, t_0)$, $E_i^a(\vec{x}, t_0)$ propagates in time according to

$$\frac{dE_i^a}{dt}=(D_jF_{ji})^a, \quad \frac{dA_i^a}{dt}=E_i^a \tag{2.1}$$

with

$$F_{ij}^a=\partial_iA_j^a-\partial_jA_i^a+gc^{abc}A_i^bA_j^c, \tag{2.2a}$$

$$(D_jF_{ji})^a=\partial_jF_{ji}^a+gc^{abc}A_j^bF_{ji}^c. \tag{2.2b}$$

If the initial configuration $A_i^a(\vec{x}, t_0)$, $E_i^a(\vec{x}, t_0)$ satisfies the constraint of Gauss's law,

$$(D_iE_i)^a=\partial_iE_i^a+gc^{abc}A_i^bE_i^c=q^a(\vec{x}), \tag{2.3}$$

and if the time development of $A_i^a(\vec{x}, t)$ and $E_i^a(\vec{x}, t)$ is the one specified by Eq. (2.1), then this constraint will also be satisfied at all later times.

Consequently, to each initial configuration $A_i^a(\vec{x}, t_0)$, $E_i^a(\vec{x}, t_0)$ that satisfies Gauss's law, there corresponds a time-dependent solution of the Yang-Mills field equations. These solutions can be characterized by their energy and total isospin which are both gauge invariant and conserved, and which can thus be calculated from the initial values:

$$I^a=\int d^3x\,\partial_iE_i^a(\vec{x}, t_0), \tag{2.4a}$$

$$H=\tfrac{1}{2}\int d^3x\,[E_i^a(\vec{x}, t_0)E_i^a(\vec{x}, t_0)+B_i^a(\vec{x}, t_0)B_i^a(\vec{x}, t_0)], \tag{2.4b}$$

where

$$B_i^a(\vec{x}, t)=\tfrac{1}{2}\epsilon_{ikl}F_{kl}^a(\vec{x}, t). \tag{2.5}$$

Intuitively one might expect that a configuration which minimizes the energy under the constraint of Gauss's law must be static. We shall show that this is indeed so. Let us thus, at some given time t_0, minimize H with respect to $A_i^a(\vec{x}, t_0)$ and $E_i^a(\vec{x}, t_0)$ under the constraint of Gauss's law (2.3). For each constraint, we need to introduce a Lagrange multiplier $\phi^a(\vec{x})$. H will be an extremum when

$$\frac{\delta H}{\delta\psi(\vec{x})}-\partial_j\frac{\delta H}{\delta(\partial_j\psi(\vec{x}))} - \int d^3y\,\phi^a(\vec{y})\left[\frac{\delta(D_iE_i^a(\vec{y})-q^a(\vec{y}))}{\delta\psi(\vec{x})} -\partial_j\frac{\delta(D_iE_i^a(\vec{y})-q^a(\vec{y}))}{\delta(\partial_j\psi(\vec{x}))}\right]=0 \tag{2.6}$$

for both $\psi=A_i^a(\vec{x}, t_0)$ and $\psi=E_i^a(\vec{x}, t_0)$. We obtain

$$E_i^a-gc^{abc}\phi^bA_i^c+\partial_i\phi^a=0, \tag{2.7a}$$

$$-(D_jF_{ji})^a-gc^{bac}\phi^bE_i^c=0. \tag{2.7b}$$

To determine an extremum of the energy under the constraint of Gauss's law, we must solve Eq. (2.7) along with Eq. (2.3). We can combine Eq. (2.7) with Eq. (2.1) to determine the time development of such an extremum:

$$\frac{dA_i^a}{dt}=E_i^a=+g(\phi\times A_i)^a-\partial_i\phi^a, \tag{2.8a}$$

$$\frac{dE_i^a}{dt}=(D_jF_{ji})^a=+g(\phi\times E_i)^a, \tag{2.8b}$$

where $(\alpha\times\beta)^a=c^{abc}\alpha^b\beta^c$. Note that the time development of A_i^a and E_i^a is simply an infinitesimal gauge transformation. It is easy to show that if A_i^a and E_i^a satisfy Eq. (2.7) at a time t, then they also satisfy Eq. (2.7) at time $t+dt$. Thus, the time development of any extremum of the energy subject to the constraint of Gauss's law is simply a gauge transformation

$$A_i(\vec{x}, t)=\mathcal{U}A_i(\vec{x}, t_0)\mathcal{U}^\dagger-\frac{1}{g}(\partial_i\mathcal{U})\mathcal{U}^\dagger, \tag{2.9a}$$

$$E_i(\vec{x}, t)=\mathcal{U}E_i(\vec{x}, t_0)\mathcal{U}^\dagger \tag{2.9b}$$

with

$$\mathcal{U}(\vec{x}, t-t_0)=\exp[g\phi(\vec{x})(t-t_0)], \tag{2.10}$$

and where we have used the matrix notation $\phi(\vec{x})=(-i)T^a\phi^a(\vec{x})$ and similarly for E_i and A_i, the T^a being the representation matrices for the generators of the group. By applying the inverse gauge transformation, we obtain E_i^a and A_i^a fields which are time independent. In this new gauge

$$A_0(\vec{x}, t)=g\phi(\vec{x}), \tag{2.11}$$

which is also time independent. Also, notice that in this gauge Eq. (2.7a) is simply the definition of E^a_i, while Eqs. (2.7b) and (2.3) are the Yang-Mills field equations.

In conclusion, we have shown that for every extremum of the energy under the constraint of Gauss's law, there exists a gauge in which all A^a_μ are time- independent. These A^a_μ are a time-independent solution of the Yang-Mills field equations. This of course implies that gauge-invariant quantities built out of the A^a_μ will be time independent in any gauge. Such solutions are called static. Conversely, any static solution to the Yang-Mills equations is an extremum of the energy under the constraint of Gauss's law.

III. THE FIELD OF A POINT SOURCE

Consider the Yang-Mills equations in the presence of a single external point source

$$(D_\mu F^{\mu\nu})^a = \delta^{a3}\delta^{\nu 0} q\,\delta^3(\vec{x})\,. \tag{3.1}$$

In Ref. 1, we discussed the Coulomb solution for this point source which in the $A_0 = 0$ gauge is

$$E^a_i(\vec{x}) = \delta^{a3}\frac{q}{4\pi}\frac{x_i}{|\vec{x}|^3}\,, \tag{3.2a}$$

$$A^a_i(\vec{x}) = tE^a_i(\vec{x})\,. \tag{3.2b}$$

We shall now exhibit a field configuration $A^a_i(\vec{x}, t_0)$, $E^a_i(\vec{x}, t_0)$ at an initial time t_0 which satisfies Gauss's law, i.e., Eq. (3.1) for $\nu = 0$, and which has lower energy than the Coulomb solution.

Let us consider a configuration of the general form

$$\vec{E}^a = \frac{q}{4\pi}\frac{1}{r^2}[\delta^{a3}\hat{r}F(r,\theta) + \delta^{a2}\hat{\phi}G(r,\theta)]\,, \tag{3.3a}$$

$$\vec{A}^a = \delta^{a1}\hat{\phi}A(r,\theta)\,, \tag{3.3b}$$

where r, θ, ϕ are the usual spherical coordinates and $\hat{r}, \hat{\theta}, \hat{\phi}$ their associated unit vectors. We will require $G(r,0) = G(r,\pi) = A(r,0) = A(r,\pi) = 0$ so that E and A will not be singular at $\theta = 0$ and $\theta = \pi$. The configuration (3.3) solves Gauss's law provided

$$\frac{q}{4\pi}\left[\delta^3(\vec{x})4\pi F(0,\theta) + \frac{1}{r^2}\frac{\partial F}{\partial r}(r,\theta) + \frac{g}{r^2}G(r,\theta)A(r,\theta)\right] = q\,\delta^3(\vec{x})\,. \tag{3.4}$$

We thus require

$$F(0,\theta) = 1 \tag{3.5a}$$

and

$$\frac{\partial F}{\partial r}(r,\theta) = -gG(r,\theta)A(r,\theta)\,. \tag{3.5b}$$

The energy of the configuration (3.4) is given by

$$H = \tfrac{1}{2}\int d^3x\left[\left(\frac{q}{4\pi}\right)^2\frac{1}{r^4}(F^2+G^2) + (\vec{\nabla}\times(\hat{\phi}A))^2\right]. \tag{3.6}$$

There are many choices of F, G, and A for which H is lower than the energy of the Coulomb solution (3.2). One such choice is

$$F(r,\theta) = 1 - \beta\left(\frac{\mu r}{1+\mu r}\right)^p \sin^2\theta\,, \tag{3.7a}$$

$$G(r,\theta) = \beta\frac{p}{a}\frac{(\mu r)^{p/2}}{(1+\mu r)^{p+1}}\sin\theta\,, \tag{3.7b}$$

$$A(r,\theta) = \frac{a}{g}\frac{(\mu r)^{p/2}}{r}\sin\theta\,, \tag{3.7c}$$

where we have introduced the dimensionless parameters β, $p > 0$, and a, and the parameter $\mu > 0$ with dimension of mass. This configuration is such that the "chromoelectric" field $\vec{E}^3$ is a Coulombic very close to the origin, but becomes modified at distances large relative to μ^{-1}. The total charge, i.e., the charge viewed at distances large relative to μ^{-1}, is

$$I^a = \int d^3x(\vec{\nabla}\cdot\vec{E})^a = \int_{S_\infty} d^2S(\hat{n}\cdot\vec{E}^a) = \delta^{a3}q(1-\tfrac{2}{3}\beta)\,. \tag{3.8}$$

For $\beta > 0$, the external charge is screened whereas it is increased for $\beta < 0$. This is done at the expense of the appearance of a "chromomagnetic" field:

$$\vec{B}^a = \delta^{a1}(\vec{\nabla}\times(\hat{\phi}A)) = \delta^{a1}\frac{a}{g}\frac{(\mu r)^{p/2}}{r^2}(2\cos\theta\hat{r} - \tfrac{1}{2}p\sin\theta\hat{\theta})\,. \tag{3.9}$$

To calculate the energy, we introduce a short-distance cutoff δ. We find for $\delta \to 0$ and for $p < 1$ (otherwise our configuration has an infrared divergence in its magnetic energy)

$$H = \pi\int_0^\pi \sin\theta\,d\theta\int_0^\infty r^2dr\left[\left(\frac{q}{4\pi}\right)^2\frac{1}{r^4}(F^2+G^2) + (\vec{B})^2\right] = \frac{q^2}{8\pi}\frac{1}{\delta} + \frac{q^2}{12\pi}\frac{1}{1-p}\frac{(\mu\delta)^p}{\delta}\left[-2\beta + \frac{p^2\beta^2}{a^2} + \frac{4\pi^2a^2}{(gq)^2}(8+p^2)\right] + O\left(\frac{(\mu\delta)^{2p}}{\delta}\right). \tag{3.10}$$

The minimum of H with respect to a occurs when

$$a^2 = \frac{p|\beta|gq}{2\pi(8+p^2)^{1/2}} = a_{\min}{}^2\,. \tag{3.11}$$

Thus the minimum energy for a given p, μ, and β is

$$H\Big|_{a=a_{\min}}=\frac{q^2}{8\pi}\frac{1}{\delta}+\frac{q^2}{12\pi}\frac{1}{1-p}\frac{(\mu\delta)^p}{\delta}\times\left[-2\beta+\frac{4\pi}{gq}\,|\beta|\,p(8+p^2)^{1/2}\right]+O\left(\frac{(\mu\delta)^{2p}}{\delta}\right). \tag{3.12}$$

For $\beta<0$, i.e., when the total charge is larger than the external charge q, the energy is always greater than the Coulomb energy [the first term in (3.12)]. For $\beta<0$, i.e., when the external charge gets screened, H can always be made smaller than the Coulomb energy, since we can always choose p to satisfy

$$\frac{gq}{4\pi}>\tfrac{1}{2}p(8+p^2)^{1/2},\quad 0<p<1 \tag{3.13}$$

for any values of the external charge q and the gauge coupling constant g.

We have exhibited initial-time configurations $A_i^a(\vec{x},t_0)$ and $E_i^a(\vec{x},t_0)$ which satisfy Gauss's law for a point source and which have lower energy than the Coulomb solution. We have done so for any value of $gq/4\pi$. Since the energy is positive-definite, this implies that there exists a minimum of the energy under the constraint of Gauss's law, which lies below the Coulomb solution. From the previous section we know that this minimum is a static solution of the Yang-Mills equations in the presence of the external point source, albeit probably a very singular solution. One might presume, although this has by no means been proved, that the time development of our initial configuration (3.7) is such that, by emission of radiation at spatial infinity, it finally reaches this state of lowest energy.

We wish to make the following further comments:

(1) The Yang-Mills equations in the presence of one point source are scale invariant. Therefore, a configuration that solves Gauss's law and that depends on a scale μ will solve Gauss's law for any value of the scale μ. Such is precisely the case for (3.7). Similarly, a solution to the Yang-Mills equations that depends on a scale μ will be a solution for any value of μ.

(2) The energy (3.12) is infinitely lower than the Coulomb energy in the limit $\delta\to 0$, provided Eq. (3.13) is satisfied, which is always possible for an appropriate choice of p.

(3) There obviously is a very wide choice of configurations that solve Gauss's law for a point source and that have lower energy than the Coulomb solution. In particular, their behavior at large distances is quite arbitrary.

(4) With the exception of energy and isospin, it is unclear which properties if any of our initial configuration (3.7) persist till later times.

(5) It is interesting to note that when p approaches one from below, the critical value of $gq/4\pi$ given by Eq. (3.13) approaches $\frac{3}{2}$, which is precisely the critical value of $gq/4\pi$ found by Mandula[4] in his stability analysis of the Coulomb potential in Yang-Mills theory.

ACKNOWLEDGMENTS

We would like to thank our colleagues at SLAC, especially Larry McLerran, and at the University of Illinois at Urbana-Champaign for many useful and illuminating discussions. One of us (P. S.) is grateful for the hospitality of the Aspen Center for Physics, where part of this work was accomplished. This work was supported by the Department of Energy under Contract No. EY-76-C-03-0515.

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