Static Yang-Mills fields with sources

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Yang-Mills theory with a static, c-number external charge density is studied. Time-independent, non-Coulombic solutions are found for a wide variety of sources with arbitrary strength. Their energy as a function of source strength is lower than the Coulomb energy; moreover, the multiplicity of solutions increases when the source strength exceeds a critical magnitude. Intrinsically nonperturbative configurations that satisfy the field equations are presented. The quantal significance of these results is briefly discussed.

I. INTRODUCTION

In the last few years we have seen successful analyses of various quantum field-theoretic models through the semiclassical method. This approach begins by taking the field equations to be classical (c-number) partial differential equations, with solutions which carry nonperturbative information about the corresponding quantum theory. Results for Yang-Mills theories are of course the most interesting, owing to their occurrence in models of strong and electro-weak interactions. Correspondingly, the monopoles and pseudoparticles (instantons) are well-known examples of unexpected solutions.

Very recently attention has been paid to solutions of Yang-Mills equations with static, c-number sources.[1] While the physical, quantal relevance of these remains obscure (some observations on this topic comprise Sec. V of our paper), they are sufficiently novel and different from their Abelian counterpart to warrant study. The present investigation is a further contribution to this, mainly mathematical, subject. We show that a wide variety of static sources, for example arbitrary spherically symmetric charge distributions, support static solutions with energy lower than that of the corresponding Coulomb solution. While this is true even for infinitesimally small sources, whenever the source is sufficiently strong the number of such solutions increases.

This paper is organized as follows: After additional introductory remarks, we begin in Sec. II with a discussion of the energy relationships in Yang-Mills theory and recall the Coulomb solution. In Sec. III our new solution is presented for weak sources, while strong sources are studied in Sec. IV and further solutions are found. The weak case can be completely solved analytically; numerical computation is used for strong sources. In the Conclusion, Sec. V, we mention some questions which remain open and we address the topic of how the quantum theory is illuminated by these results. We argue that the specification of the source should be supplemented by a gauge condition. Then the number of distinct solutions decreases; yet for sufficiently strong sources a multiplicity of solutions persists. Various technical computations are relegated to the Appendices.

The source ρ_a with which we concern ourselves is the time component of an external current 4-vector J_a^μ, whose spatial components are zero. The source is always chosen to be static, $\partial_t \rho_a = 0$. Hence the Yang-Mills equation reads

$$\mathfrak{D}_\mu F^{\mu\nu} = g\delta^{\nu 0}\rho \,, \tag{1.1}$$

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu + g[A^\mu, A^\nu] \,, \tag{1.2}$$

$$\mathfrak{D}_\mu = \partial_\mu + g[A_\mu, \;\;] \,. \tag{1.3}$$

[We study the SU(2) theory with coupling strength g and use interchangeably component notation: ρ_a, $a=1,2,3$, and matrix notation $\rho = \rho_a \sigma^a/2i$, σ^a = Pauli matrices.]

Consistency of the four equations in (1.1) requires the right-hand side to be covariantly conserved, which in the present circumstance simply means that A^0 commutes with ρ:

$$[A^0, \rho] = 0 \,. \tag{1.4a}$$

The commutation is not quantum mechanical; all our quantities are c numbers. Rather, the commutator is in the Pauli matrix space of the SU(2) theory, and (1.4a) is equivalent to the component expression

$$\epsilon_{abc} A_b^0 \rho_c = 0 \,. \tag{1.4b}$$

We shall be discussing solutions of (1.1) and of its consequence (1.4). These equations are not gauge invariant owing to the presence of the external charge density. They are, however, gauge covariant in the sense that if A^μ is a solution with source ρ, then another source ρ' which is gauge equivalent to ρ,

$$\rho' = U^{-1}\rho U\,, \tag{1.5a}$$

is solved by A'^μ, the gauge transform of A^μ:

$$A'^\mu = U^{-1}A^\mu U + \frac{1}{g} U^{-1}\partial^\mu U\,. \tag{1.5b}$$

[Here U is a $2\times 2\,\mathrm{SU}(2)$ matrix; we take it to be time independent so as to preserve the static nature of the source.] Use of this gauge covariance allows the alignment of the charge density into a preferred direction in isospace, albeit the gauge transformation may in general be singular, as is further discussed in connection with Eqs. (3.11) below. We shall frequently take the source to be of the form

$$\begin{aligned} \rho_a &= \delta_{a3} q\,, \\ \rho &= \frac{\sigma^3}{2i} q\,. \end{aligned} \tag{1.6}$$

A physical realization of the model that we are thus led to study is a system of photons (A_3^μ) and massless, charged vector mesons ($A_{1,2}^\mu$); also an external, static, electromagnetic charge density (q) is present. The photons interact with the vector mesons through the usual minimal coupling and through a nonminimal magnetic interaction; additionally the charged vector mesons interact among themselves with a quartic self-interaction. The various couplings are related by the underlying non-Abelian gauge principle.

Although a transparent physical analog exists when the problem is formulated in the above standard gauge frame (1.6), it may be that the mathematical equations are more tractable in some other frame. In other words, when solving (1.1), (1.4), and (1.6), it may be useful first to gauge-rotate into a frame where the source is ρ'. The (simpler) equations are solved in this frame, and the potentials A'^μ are determined; the potentials for the problem of interest A^μ are then gotten by gauge transforming A'^μ. Of course, gauge-invariant properties of the solutions, for example the energy, can be calculated in any gauge frame.

II. ENERGY RELATIONS FOR YANG-MILLS FIELDS

The energy for Yang-Mills fields can be taken to be the gauge-invariant Hamiltonian:

$$\begin{aligned} \mathcal{E} &= \tfrac{1}{2}\int d\vec{r}\,(\vec{E}_a^{\,2} + \vec{B}_a^{\,2})\,, \\ E_a^i &= F_a^{i0}, \quad B_a^i = -\tfrac{1}{2}\epsilon^{ijk}F_a^{jk}\,. \end{aligned} \tag{2.1}$$

This quantity remains conserved in the presence of external static sources; note that they do not occur explicitly in (2.1). We shall be concerned exclusively with gauge-field configurations of finite total energy—a requirement which imposes regularity constraints on the charge density that will not be spelled out explicitly.

In simple dynamical systems, it follows from Hamilton's equations of motion that stationary values of the energy are given by static (time-independent) solutions. Moreover, explicit properties of the Hamiltonian frequently ensure that the minimum-energy configuration is one of these static solutions. In a gauge theory, where there are constraints, the situation is more complicated. Let us first record the static Yang-Mills equations:

$$\vec{\nabla}\cdot\vec{E}_a - g\epsilon_{abc}\vec{A}_b\cdot\vec{E}_c = g\rho_a\,, \tag{2.2}$$

$$\vec{E}_a = -\vec{\nabla}A_a^0 + g\epsilon_{abc}\vec{A}_b A_c^0\,, \tag{2.3}$$

$$\vec{\nabla}\times\vec{B}_a - g\epsilon_{abc}\vec{A}_b\times\vec{B}_c = g\epsilon_{abc}A_b^0\vec{E}_c\,, \tag{2.4}$$

$$\vec{B}_a = \vec{\nabla}\times\vec{A}_a - \tfrac{1}{2}g\epsilon_{abc}\vec{A}_b\times\vec{A}_c\,. \tag{2.5}$$

A consistency condition on ρ_a follows from these:

$$\epsilon_{abc}A_b^0\rho_c = 0\,. \tag{2.6}$$

The form for the charge density is

$$\rho_a = \delta_{a3}q\,. \tag{2.7}$$

The above equations emerge not by demanding stationariness of the positive-definite energy (2.1), but rather of the related object

$$\overline{\mathcal{E}} = \mathcal{E} - \int d\vec{r}\,A_a^0(\vec{\nabla}\cdot\vec{E}_a - g\epsilon_{abc}\vec{A}_b\cdot\vec{E}_c - g\rho_a)\,, \tag{2.8}$$

where $\vec{B}_a$ is viewed as a function of $\vec{A}_a$, while $\vec{E}_a$, $\vec{A}_a$, and A_a^0 are varied independently. $\overline{\mathcal{E}}$ coincides with the energy only when Gauss's law (2.2) is satisfied; in other words, static equations follow by varying $\mathcal{E}$, as a functional of $\vec{E}_a$ and $\vec{A}_a$, subject to the constraint of Gauss's law. Since $\overline{\mathcal{E}}$ does not separate into positive-definite kinetic and static parts, it is not easy to see which configurations minimize the energy and satisfy (2.6).

Further insight is had by solving the constraint, expressing the energy (Hamiltonian) in terms of independent variables, and studying the properties of the resulting formula. In an Abelian theory one finds, as is well known,

$$\mathcal{E} = \frac{g^2}{8\pi}\int d\vec{r}\,d\vec{r}'\frac{q(\vec{r})q(\vec{r}')}{|\vec{r}-\vec{r}'|} + \tfrac{1}{2}\int d\vec{r}\,(\vec{E}_T^{\,2} + \vec{B}^2)\,. \tag{2.9}$$

The first term is the Coulomb energy; the second, involving the transverse electric field $\vec{E}_T$ and magnetic field $\vec{B}$, is positive-definite and depends only on independent variables. One sees that the energy is minimized (for fixed sources) by letting $\vec{E}_T$ and $\vec{B}$ vanish. In this way, one arrives at the static Coulomb solution as the minimum-energy configuration for an Abelian theory.

The Coulomb solution of the non-Abelian theory [with source in the third direction as in (2.7)] is easily obtained. Clearly the relevant equations are solved by

$$\vec{A}_a=0, \quad A^0_{1,2}=0, \quad A^0_3=-\frac{1}{\nabla^2}gq\ , \tag{2.10}$$

and the Coulomb energy is as in the Abelian case:

$$\mathcal{E}_C=\frac{g^2}{8\pi}\int d\vec{r}\,d\vec{r}\,'\frac{q(\vec{r})q(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\ . \tag{2.11}$$

But unlike in the Abelian case, we cannot argue now that this minimizes the energy. The construction analogous to (2.9) gives rise, in the Coulomb gauge, to a complicated formula:

$$\mathcal{E}=-\tfrac{1}{2}\int g\rho^{\text{tot}}\frac{1}{d}\nabla^2\frac{1}{d}g\rho^{\text{tot}}+\tfrac{1}{2}\int d\vec{r}\,(\vec{E}_{Ta}{}^2+\vec{B}_a{}^2)\ . \tag{2.12a}$$

Here ρ^{tot} is the total charge density

$$\rho_a^{\text{tot}}=\epsilon_{abc}\vec{A}_b\cdot\vec{E}_{Tc}+\rho_a\ , \tag{2.12b}$$

while d is the differential operator

$$d_{ac}=\nabla^2\delta_{ac}-g\epsilon_{abc}\vec{A}_b\cdot\vec{\nabla}\ . \tag{2.12c}$$

In the Coulomb solution (2.10), $\vec{A}$ is set equal to zero, as is the transverse electric field and the magnetic field. Consequently, $\rho^{\text{tot}}=\rho$ and $d=\nabla^2$. But this need not correspond to the minimum value of $\mathcal{E}$. We see that choosing nonvanishing $\vec{A}$ and $\vec{E}_T$ certainly increases the second contribution to (2.12a) over its vanishing Coulombic form; but in the first contribution both ρ^{tot} and d are altered in a complicated way and a net decrease of $\mathcal{E}$ could result. That this indeed does happen has been established for spherically symmetric charge densities $q(\vec{r})=q(r)$, where explicit time-dependent solutions have been presented with energy below the Coulomb energy; in fact the energy can be made as small as desired.[2]

We are here interested in static solutions, for which even less has been proved. It is known that sufficiently strong external sources render the Coulomb solution unstable.[3] Furthermore, a method has been given for constructing sources of appropriate strength to support static solutions with energy lower than the Coulomb energy.[4] We have found that a wide class of arbitrarily weak sources also gives rise to static solutions with energy below the Coulomb energy. Moreover, further non-Coulombic solutions appear as the strength of the source is increased.

We conclude this section by recording another formula for the energy of a static Yang-Mills solution, which follows from (2.1)–(2.6), whenever the electric and magnetic fields fall off faster than $r^{-3/2}$ at large distance:

$$\mathcal{E}=g\int d\vec{r}\,\rho_a\vec{r}\cdot\vec{E}_a\ . \tag{2.13}$$

This expression, derived in Appendix A, is especially convenient when the source is composed of δ functions since then the energy is given by a local expression. We shall use (2.13) in Sec. IV.

III. STATIC YANG-MILLS FIELDS WITH WEAK SOURCES

We solve Eqs. (2.2)–(2.7) perturbatively in the source. To keep track of orders of the perturbation, we take the charge density q to be of order Q, which is some convenient scale of q. (For example, Q can be an overall factor in q.) Observe that the Coulomb solution (2.10) and (2.11) produces scalar potentials which are $O(Q)$, and an energy which is $O(Q^2)$. To explore the possibility of other, non-Coulombic solutions, let us recall first that in the absence of sources the only finite-energy solution has zero energy, hence, vanishing electric and magnetic fields.[5]

Such a trivial configuration may be achieved by setting the Yang-Mills potentials equal to zero. Correspondingly, in the presence of sources, we take A^0 to be $O(Q)$ and $\vec{A}$ to vanish with Q. This leads to the Coulomb solution (2.10) and (2.11).

However, the trivial configuration in the absence of sources may also be realized by a pure gauge form for $\vec{A}$. Hence, an alternate *Ansatz* for a static solution can be given,

$$A^0=O(Q)\ , \tag{3.1a}$$

$$\vec{A}=-\frac{1}{g}\,U^{-1}\vec{\nabla}U+O(Q^2) \tag{3.1b}$$

with U being a static SU(2) matrix. In order to study this *Ansatz* further, it is convenient to transform away the pure gauge portion of $\vec{A}$. Thus we pass to a new gauge frame, where the potentials are

$$A'^0=UA^0U^{-1}=O(Q)\ , \tag{3.2a}$$

$$\vec{A}'=U\vec{A}U^{-1}-\frac{1}{g}\,U\vec{\nabla}U^{-1}=O(Q^2)\ . \tag{3.2b}$$

Correspondingly, the source becomes

$$\rho'=U\rho U^{-1}=U\frac{\sigma^3}{2i}\,U^{-1}q\ ,\qquad \rho_a'=\hat{\rho}_a(\vec{r})q\ , \tag{3.3}$$

where $\hat{\rho}_a$ is a unit isovector; it is the gauge rotation of δ_{a3}. In the new frame, the relevant equations are again (2.2)–(2.6), except all quantities are primed. We solve these perturbatively in Q. To lowest order in Q

$$\vec{E}'_a = -\vec{\nabla} A'^0_a = O(Q) \tag{3.4a}$$

and

$$\vec{B}'_a = \vec{\nabla}\times\vec{A}'_a = O(Q^2)\,. \tag{3.4b}$$

Gauss's law (2.2) becomes

$$\nabla^2 A'^0_a = -g\rho'_a\,, \tag{3.5}$$

and a consistent $O(Q)$ solution is obtained:

$$A'^0_a = -\frac{1}{\nabla^2} g\rho'_a\,, \tag{3.6a}$$

$$\vec{E}'_a = \vec{\nabla}\frac{1}{\nabla^2} g\rho'_a\,. \tag{3.6b}$$

Ampère's law (2.4) reduces to

$$\vec{\nabla}\times\vec{B}'_a = g\epsilon_{abc}A'^0_b\vec{E}'_c\,, \tag{3.7}$$

which is consistently $O(Q^2)$. When the energy is sought only to $O(Q^2)$, $\vec{B}'$ and $\vec{A}'$ need not be determined since they contribute $O(Q^4)$ terms. However, one should check that the right-hand side of (3.7) is divergenceless. That it indeed is follows from (3.4a) and (3.5), provided (2.6) is satisfied. Thus we further require the validity of the consistency condition

$$\epsilon_{abc}\rho'_b\frac{1}{\nabla^2}\rho'_c = 0\,. \tag{3.8}$$

[It is straightforward to solve for $\vec{A}'$ from (3.4b) and (3.7). Choosing a divergenceless form for $\vec{A}'$, we find

$$\vec{A}'_a = \frac{1}{\nabla^2} g\epsilon_{abc}\left(\frac{1}{\nabla^2} g\rho'_b\right)\vec{\nabla}\left(\frac{1}{\nabla^2} g\rho'_c\right). \tag{3.9}$$

This is indeed $O(Q^2)$.]

To summarize: Whenever we can find a gauge transformation U, such that the charge density $\rho_a = \delta_{a3}q$ gives rise upon rotation by U to a new charge density $\rho'_a = \hat{\rho}_a q$ which satisfies (3.8), we can construct perturbatively a static non-Coulombic Yang-Mills solution with energy

$$\mathcal{E}_I = \frac{g^2}{8\pi}\int d\vec{r}\, d\vec{r}'\frac{\rho'_a(\vec{r})\rho'_a(\vec{r}')}{|\vec{r}-\vec{r}'|} + O(Q^4)\,. \tag{3.10a}$$

The $O(Q^2)$ part of this, $\mathcal{E}_I^{(2)}$, can also be given in a form which facilitates comparison with the Coulomb energy $\mathcal{E}_C$ of Eq. (2.11):

$$\mathcal{E}_I^{(2)} = \frac{g^2}{8\pi}\int d\vec{r}\, d\vec{r}'\hat{\rho}_a(\vec{r})\hat{\rho}_a(\vec{r}')\frac{q(\vec{r})q(\vec{r}')}{|\vec{r}-\vec{r}'|}\,. \tag{3.10b}$$

Since $\hat{\rho}_a(\vec{r})\hat{\rho}_a(\vec{r}') \le 1$, $\mathcal{E}_I^{(2)}$ is obviously less than $\mathcal{E}_C$ when q is everywhere positive or negative. In this circumstance our new solution (3.6) and (3.9) has lower energy than the Coulombic one, for sufficiently weak sources where terms of lowest order in Q are dominant. It may happen that even for q's which change sign, $\mathcal{E}_I^{(2)}$ is less than $\mathcal{E}_C$. Indeed, this occurs for the explicit example discussed below.

It remains to determine under what circumstances the consistency condition (3.8) can be satisfied. We have not analyzed this question in general, but specific examples are easily given. Take a spherically symmetric source

$$\rho_a = \delta_{a3}q(r) \tag{3.11a}$$

and choose for ρ'_a a form which is radially oriented in SU(2) space and gauge equivalent to (3.11a):

$$\rho'_a = \hat{r}^a q(r)\,,\qquad \hat{\rho}_a(\vec{r}) = \hat{r}^a\,. \tag{3.11b}$$

[To avoid singularities, $q(0)$ should vanish.] Since $(1/\nabla^2)\rho'_a$ also points in the $\hat{r}^a$ direction, the condition (3.8) is met. This configuration has the further interesting property, proved in Appendix B, that $\mathcal{E}_I^{(2)}$ is less than $\mathcal{E}_C$ for all $q(r)$, even those which change sign. Observe also that the gauge transformation which takes (3.11a) to (3.11b) is singular, as it must be since ρ_a has a vanishing Kronecker index, while for ρ'_a that index is one.

Other examples are also available. Consider the potential

$$A'^0_a(\vec{r}) = \eta^a(\hat{r})\mathcal{Q}(r)\,, \tag{3.12}$$

where η^a is a spherical harmonic of total angular momentum l, depending only on angles, and $\mathcal{Q}(r)$ is spherically symmetric and independent of angles. When the Laplacian is applied to (3.12) the result remains proportional to $\eta^a(\hat{r})$:

$$\nabla^2 A'^0_a(\vec{r}) = \eta^a(\hat{r})\left[\nabla^2\mathcal{Q}(r) - \frac{l(l+1)}{r^2}\mathcal{Q}(r)\right]. \tag{3.13}$$

Therefore, if we call the right-hand side of (3.13) $-g\rho'_a$, this charge density satisfies (3.8) and, when ρ'_a is rotated into the $a=3$ direction,

$$-g\rho_a = \delta_{a3}[\eta^b(\hat{r})\eta^b(\hat{r})]^{1/2} \times\left[\nabla^2\mathcal{Q}(r) - \frac{l(l+1)}{r^2}\mathcal{Q}(r)\right], \tag{3.14}$$

the above spherically nonsymmetric charge density provides another instance of the phenomenon that we have discovered.

The final result then is that all radially symmetric charge distributions, and others as well, support non-Coulombic static solutions with energy lower than the Coulomb energy for sufficiently weak sources.

In conclusion, let us note an observation due to Goldstone[6]: The lowest order in Q theory can be elegantly summarized by the single variational statement that the quantity

$$\mathfrak{C}=\frac{g^2}{8\pi}\int d\vec{r}\,d\vec{r}\,'\frac{\rho_a'(\vec{r})\rho_a'(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} \tag{3.15}$$

must be invariant against gauge transformations of ρ_a'. Upon performing an infinitesimal gauge transformation, we see that this requirement demands

$$\epsilon_{abc}\left[\frac{g}{4\pi}\int d\vec{r}\,'\frac{\rho_b'(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\right]\rho_c'(\vec{r})=0\,, \tag{3.16}$$

which is equivalent to (3.5) and (3.8). Evaluated at its stationary value, the quantity $\mathfrak{C}$ is just the $O(Q^2)$ energy.

IV. STATIC YANG–MILLS FIELDS WITH ARBITRARY RADIAL SOURCES

To progress beyond the perturbative approximation, we need to simplify the highly nonlinear equations (2.2)–(2.6). We choose to work with the radial example of Eqs. (3.11) which, even in the perturbative regime, exhibits sufficiently interesting behavior to merit further discussion. To begin, it is easy to convince oneself that, for a source of the radial form (3.11b), iteration of the equations to all orders in Q produces a scalar potential A_a^0 proportional to $\hat{r}^a$ [see (3.6a)] and a vector potential A_a^i proportional to $\epsilon^{aij}\hat{r}^j$ [see (3.9)]. (Primes are suppressed; in this section we always remain in the radial frame.) We may therefore make the following *Ansatz* for the exact potentials:

$$\rho_a(\vec{r})=\frac{1}{g^2r_0^3}\,\hat{r}^a q(r/r_0)\,, \tag{4.1}$$

$$A_a^0(\vec{r})=\frac{1}{gr}\,\hat{r}^a f(r/r_0)\,, \tag{4.2a}$$

$$A_a^i(\vec{r})=\frac{1}{gr}\,\epsilon^{aij}\hat{r}^j[a(r/r_0)-1]\,, \tag{4.2b}$$

where r_0 is an arbitrary length scale.

Recall that radial symmetry in non-Abelian gauge theory can be realized in two ways: the Abelian fashion, where everything points in a fixed direction in isospace and is radially invariant; the non-Abelian fashion, where explicit radial symmetry is absent, but any rotational noninvariance can be compensated by a gauge transformation.[7] The above *Ansatz* belongs to the second category, but it is not the most general radial formula which, as is well known, involves four functions of r rather than two.[8] Nevertheless, we show in Appendix C that the greater generality adds nothing new; the most general radially (gauge) invariant solution is equivalent to (4.2).

With (4.1) and (4.2a) the consistency condition (2.6) is satisfied. The field strengths become

$$E_a^i=-\frac{\hat{r}^i\hat{r}^a}{gr_0^2}\left(\frac{f}{x}\right)'-\frac{1}{gr_0^2}(\delta^{ia}-\hat{r}^i\hat{r}^a)\frac{fa}{x^2}\,, \tag{4.3a}$$

$$B_a^i=-\frac{\hat{r}^i\hat{r}^a}{gr_0^2}\,\frac{a^2-1}{x^2}-\frac{1}{gr_0^2}(\delta^{ia}-\hat{r}^i\hat{r}^a)\frac{a'}{x}\,. \tag{4.3b}$$

All functions depend on $x=r/r_0$ and the prime indicates differentiation with respect to that variable. The differential equations for f and a emerging from (2.2)–(2.5) are[9]

$$-f''+\frac{2a^2}{x^2}\,f=xq\,, \tag{4.4a}$$

$$-a''+\frac{a^2-1-f_a^2}{x^2}\,a=0\,, \tag{4.4b}$$

while the energy (2.1) is

$$\mathcal{E}=\frac{4\pi}{g^2r_0}\int_0^\infty dx\left[(a')^2+\frac{1}{2x^2}\,(a^2-1)^2+\tfrac{1}{2}(f')^2+\frac{1}{x^2}\,f^2a^2\right]. \tag{4.5}$$

Equations (4.4) follow from varying the non-positive-definite quantity

$$\overline{\mathcal{E}}=\frac{4\pi}{g^2r_0}\int_0^\infty dx\left[(a')^2+\frac{1}{2x^2}\,(a^2-1)^2-\tfrac{1}{2}(f')^2-\frac{1}{x^2}f^2a^2+xfq\right], \tag{4.6}$$

which coincides with $\mathcal{E}$ when Gauss's law (4.4a) is satisfied. The alternate formula for $\mathcal{E}$, Eq. (2.13), becomes

$$\mathcal{E}=-\frac{4\pi}{g^2r_0}\int_0^\infty dx\,x^3q\left(\frac{f}{x}\right)'. \tag{4.7}$$

Finiteness of the energy (4.5) requires that a^2 go to 1 and that f vanish at the origin. When xq vanishes at the origin, f vanishes as x^2 and a^2 approaches 1 to $O(x^2)$. For sources vanishing faster than x^{-4} at large distances, f vanishes as x^{-1} and a^2 approaches 1 to $O(x^{-1})$. The *Ansätze* (4.1) and (4.2) are respected by gauge transformations involving rotations by $\pm\pi$ about the $\hat{r}$ axis, thus changing the sign of a, an evident symmetry of (4.4). This remaining gauge freedom may be eliminated by requiring $a(0)=1$.

With a source satisfying the above conditions at the origin and at infinity, we are led to *two* types of solutions of (4.4), which differ in their asymptotic behavior at large distances: type I with large x asymptotes

$$a(x)\approx 1+a_{-1}^1x^{-1}\,,\qquad f(x)\approx f_{-1}^1x^{-1}\,, \tag{4.8}$$

and type II with

$$a(x) \approx -1 + a_{-1}^{\rm II} x^{-1}, \quad f(x) \approx f_{-1}^{\rm II} x^{-1}. \tag{4.9}$$

The small x asymptote is the same for both types

$$a(x) \approx 1 + a_2 x^2, \quad f(x) \approx f_2 x^2. \tag{4.10}$$

The constants a_2, f_2, a_{-1}, and f_{-1} cannot be determined by local considerations alone. The first type (4.8) is the solution which we discussed perturbatively in Sec. III; $a=1$ and $f=0$ throughout all space corresponds to vanishing source, and $A^\mu = 0$. The second type (4.9) requires a lower bound on the magnitude of the source, otherwise there would be a nontrivial, sourceless static solution of finite energy which is known not to exist.[5] Note that the type-II solution interpolates, as r passes from zero to infinity, between a vanishing vector potential and a nontrivial pure gauge:

$$r \to 0, \quad A_a^i = 0, \qquad r \to \infty, \quad A_a^i = \frac{2}{g} \epsilon^{iaj} \frac{\hat{r}^j}{r}. \tag{4.11}$$

Further analysis has been accomplished by numerical rather than analytic methods. The conclusion is that the type-I solution persists and behaves much in the same way as in the small source regime. For the type-II solution we find that not only does it exist, but surprisingly there are two branches which begin with a bifurcation point. We conclude this section by presenting the detailed results obtained when (4.4) is solved numerically with a wide range of magnitudes for the source q.

The general strategy in the numerical solutions is to integrate the equations from $x=0$, with a definite source, assuming trial values of the constants a_2 and f_2. Each choice of a_2 and f_2 determines a solution of the equations which is by construction well behaved at the origin, but in general will not satisfy the boundary conditions at $x=\infty$. (As a matter of fact, because of the nonlinearity of the equations, the solution may become singular also at finite x.) The a_2 and f_2 are varied until the required behavior at $x=\infty$ is obtained, within a pre-established degree of accuracy.

With a slight modification of this method, we have actually chosen to integrate the equations from $x=0$ up to $x=1$ with given a_2 and f_2, and from $x=\infty$ down to $x=1$ with given a_{-1} and f_{-1}; the constants of integrations a_2, f_2, a_{-1}, and f_{-1} are then varied until a match is obtained between the values found for a, f, a', and f' on the two sides of $x=1$. This procedure is particularly advantageous if one deals with sources concentrated at $x=1$. Indeed, setting

$$q(x) = Q\delta(x-1), \tag{4.12}$$

the equations reduce to the free equations (i.e., with $q=0$) in the intervals $0 \leq x \leq 1$ and $1 \leq x \leq \infty$, together with the requirement that a, a', and f be continuous at $x=1$, whereas f' must have a discontinuity of magnitude $-Q$ there. Another ad-

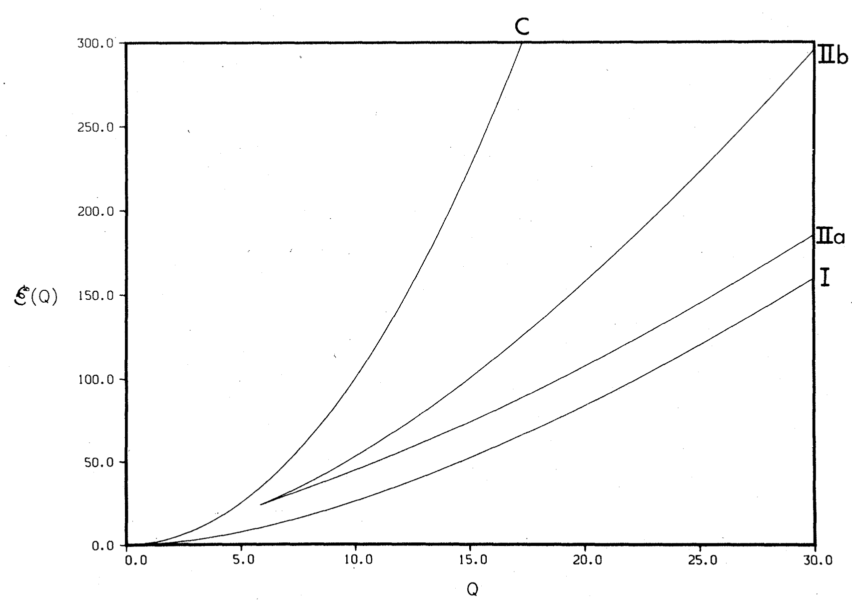

FIG. 1. Energy in units of $2\pi/g^2 r_0$ from (4.13) versus Q. Curve C is the Coulomb parabola. Curve I is the perturbatively attainable type-I solution. Curves IIa and IIb describe the two branches of the nonperturbative type-II solution.

vantage of (4.12) is that the energy (4.7) is readily evaluated:

$$\mathcal{E} = -\frac{4\pi}{g^2 r_0} Q\left(\frac{f}{x}\right)'\Big|_{x=1}. \qquad (4.13a)$$

Since f' is discontinuous at $x=1$, the above becomes

$$\mathcal{E} = \frac{4\pi}{g^2 r_0} Q[f(1) - \tfrac{1}{2}f'(1^+) - \tfrac{1}{2}f'(1^-)]. \qquad (4.13b)$$

Equation (4.13b) provides a direct local evaluation of the energy from the solutions of the differential equation.

The results which we present here have all been derived for a δ-function source. We have considered this type of source to abstract the effects which are associated with the strength of the source from the residual dependence on its actual shape. Thus the whole class of solutions depends on a single parameter, the magnitude Q of the source itself. With additional numerical computations we have checked that the qualitative features seen with a strictly localized source do indeed extend to sources spread over a range of values of x.

We have explored a wide range of initial data,

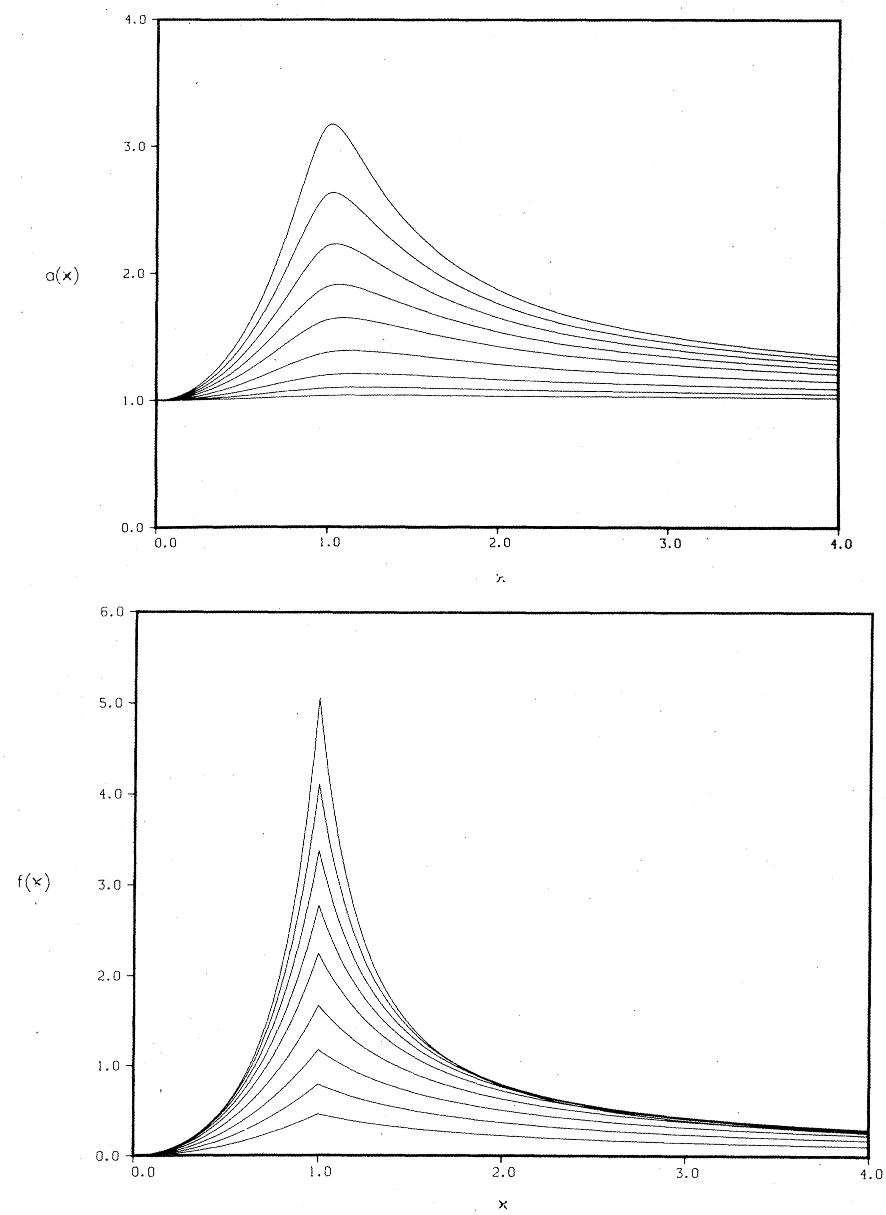

FIG. 2. Profiles of the functions a and f for the type-I solution. Starting from the lowest curves the values of Q are 1.41, 2.53, 4.04, 6.43, 10.05, 14.18, 19.93, 28.38, and 41.72.

using a computer to tabulate the values of a, a', and f on both sides of $x=1$ as functions of a_2, f_2, a_{-1}, and f_{-1}. The continuity requirement on a, a', and f at $x=1$ then imposes three constraints among four unknowns and determines one-parameter families of solutions to the equations, for which the boundary conditions at $x=0$, $x=\infty$ are met.

It is convenient to regard these families as curves in the space of solutions: Each point corresponds to a solution where a, a', and f are continuous, but no condition on the discontinuity of f' has yet been imposed. The discontinuity of f' specifies the value of the source: $Q=-\Delta f'=f'(1^-)-f'(1^+)$. $\Delta f'$ varies smoothly along the curves; each curve thus describes a family of solutions, related by continuity, with different values of Q.

Our numerical analysis has produced the following results. There is one family of type-I solutions [i.e., $a(0)=a(\infty)=1$], with values of Q extending all around the origin. For small Q these solu-

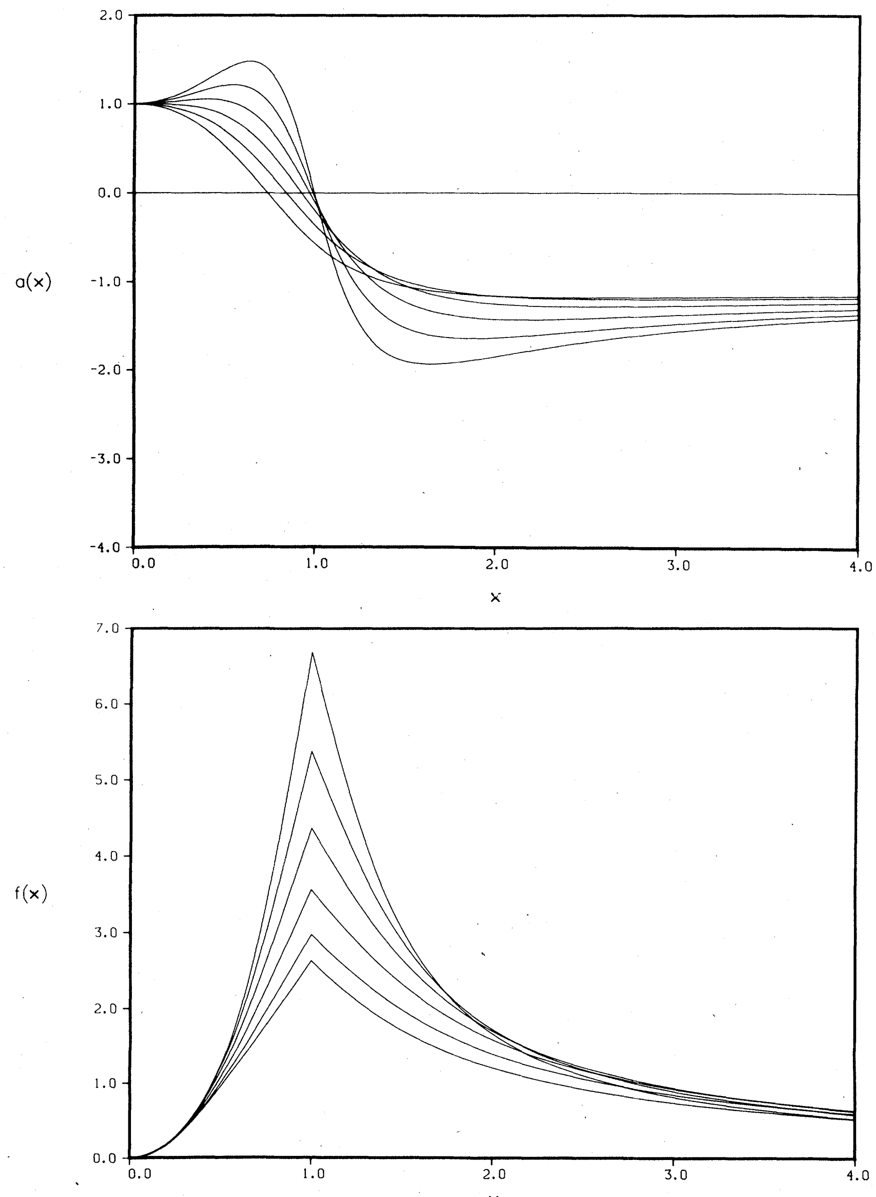

FIG. 3. Profiles of the functions a and f for the (a) (lower) branch of type-II solutions for $Q=5.86$, 6.44, 8.09, 11.05, 15.71, and 23.19. Correspondence between the individual curves and these values of Q is established by the fact that, as Q increases, so do $a''(0)$ and $f(1)$.

tions reduce to the perturbative solutions described previously. A graph of the energy versus source strength displays a curve (I in Fig. 1) reminiscent of the parabola that gives the Coulomb energy (C in Fig. 1) but consistently lower than the latter. This shows that the results inferred from the perturbative analysis actually extend to the nonperturbative domain.

We also find a family of type-II solutions[10] [i.e., $a(0)=-a(\infty)=1$]. The values taken by Q are bounded below by a minimum source strength $Q_0>0$. This means that as one moves continuously along the curve representing the solutions, Q first decreases to a minimum value Q_0 and then increases again. As a consequence, there are actually two distinct type-II solutions for each $Q>Q_0$. The existence of this second, nonperturbative family of solutions is, we believe, an interesting result.[11]

The curves of E versus Q for these solutions (IIa and IIb in Fig. 1) are entirely contained between the curves giving the energies of the Coulomb solution and of the type-I solution. Cusplike

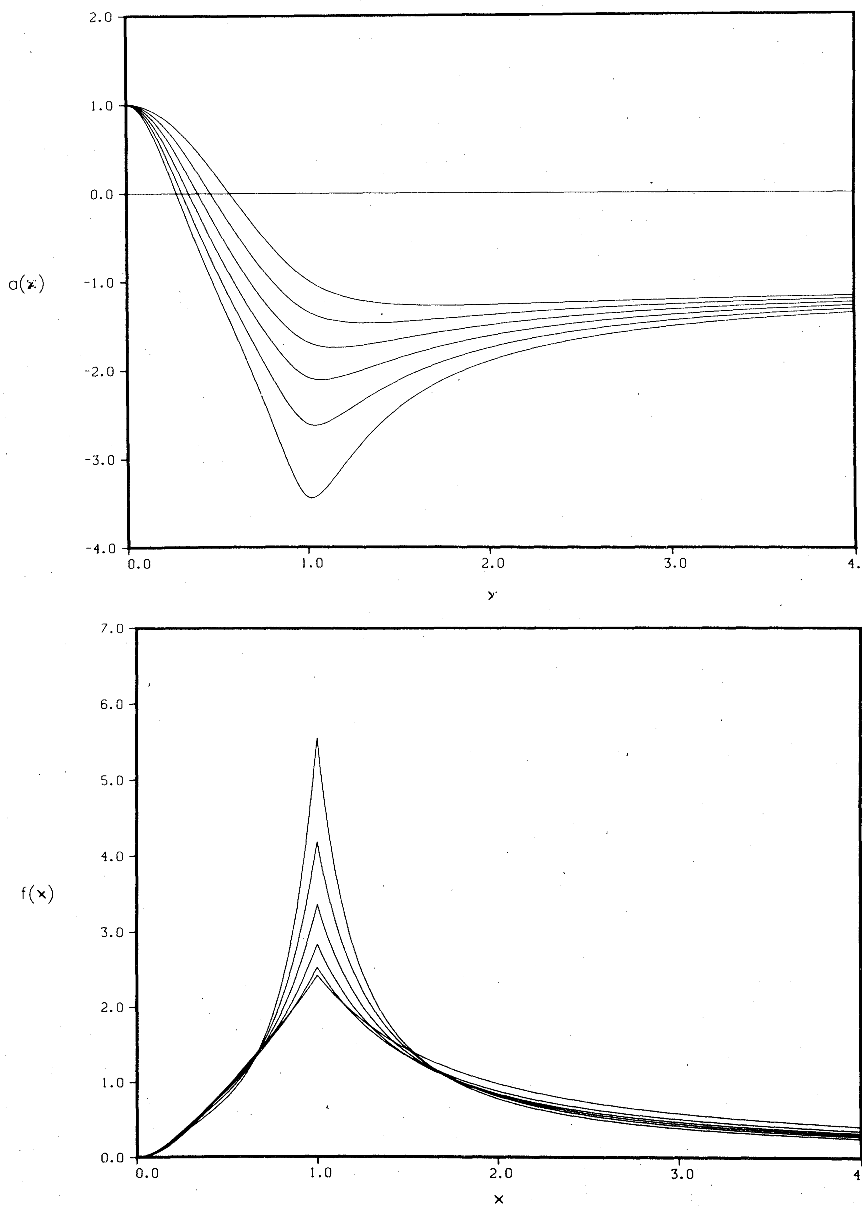

FIG. 4. Profiles of the functions a and f for the (b) (upper) branch of type-II solutions for $Q=6.53$, 8.61, 12.10, 17.85, 28.06, and 49.16. For these curves, as Q increases, $a''(0)$ decreases and $f(1)$ increases.

behavior is displayed, which is rather typical of phenomena with critical points. Finally, in Figs. 2–4 we exhibit the profiles of the functions a and f for a variety of solutions. The discontinuity in f', related to the δ-function source, is particularly evident.

V. CONCLUSION

We have demonstrated that static sources with arbitrary strength support multiple, finite-energy, static solutions to Yang-Mills equations, with the Coulomb solution typically carrying more energy than the new ones. Our results suggest, in a natural way, further mathematical questions. One wonders whether there is an *a priori* classification of the variety of solutions, which we see is at least twofold. Firstly, there is the variety found perturbatively for weak sources and confirmed analytically for a strong source, involving a change of gauge frame. Secondly, in the radial frame, we found for a sufficiently strong source two types of solutions, differing in their asymptotic behavior. Numerical computation exposed a further doubling: The type-II solution with nontrivial asymptotes has two branches.

There are tantalizing hints that a topological classification may be appropriate, but we have not found the relevant formulation. To be sure, the source in the two gauge frames (3.11a) and (3.11b) has a different Kronecker index, but the other, nonspherical examples that we have constructed in Eqs. (3.12)–(3.14) involve arbitrary spherical harmonics, to which a Kronecker index is not assigned.

The distinction between type-I and type-II solutions lies in their asymptotic forms, with the second tending to a nontrivial, pure gauge, rather than simply vanishing rapidly as does the first. But when we make a gauge transformation with a radial gauge function $U=\exp[i(\vec{\sigma}\cdot\hat{r}/2)\theta(r)]$, which leaves the source (4.1) unchanged, the form of the vector potentials becomes

$$A_a^i \to \frac{1}{gr}\,\epsilon^{aij}\hat{r}^j(a\cos\theta-1)+\frac{1}{gr}\,(\delta^{ai}-\hat{r}^a\hat{r}^i)(a\sin\theta) + \frac{1}{g}\,\hat{r}^a\hat{r}^i\theta' . \tag{5.1}$$

If θ is allowed to vary between 0 and π as r passes from 0 to ∞, it is hard to recognize in (5.1) the nontrivial properties of the type-II solution. Nevertheless, a distinction remains in comparison with the type-I solution, since the identical gauge transformation of that configuration exhibits a nontrivial large r asymptote. Moreover, if a gauge condition is imposed (see below) so that the vector potential is frozen into the form (4.2b), the topologically interesting behavior will remain. How to distinguish between the two branches of the type-II solution remains unclear.

A related question is whether or not there are other static solutions, and what is their energy in comparison to the ones we have found. For weak coupling, the variational formulation, Eqs. (3.15) and (3.16), shows that the Coulomb solution has the highest energy. We have exhibited other solutions with lower energy, but whether these exhaust the finite-energy static solutions is not known.

The stability of our solutions should be examined. As indicated above, we do not see any topological reason for stability, but nontopological stability remains a possibility.[12] In this connection we call attention to the fact that within our radial *Ansatz*, the energy (4.5) of the type-II solution is separated by an infinite barrier from that of type I.

Finally we turn to the question of physical relevance; what is the significance, for quantized Yang-Mills theory, of the multiplicity of static finite-energy solutions with c-number static sources? A c-number source can play one of two roles in a quantum theory. It may be a purely formal device inserted for purposes of computing a generating functional for Green's functions, which are obtained in the limit of vanishing sources after differentiation with respect to the source. Alternatively, the source may provide an approximate description of a system to which the Yang-Mills fields are coupled, and whose dynamics may be ignored. In the first case, one is interested in weak sources. In the second, strong sources are relevant, for it is only in that limit that the quantal noncommutativity of physical sources can be ignored.

However, once we contemplate the quantum theory another consideration comes into play which we have thus far not mentioned, and that is the necessity of choosing a gauge as a prerequisite to quantization. If a gauge condition is specified, some of the multiplicity of solutions disappears.

Let us consider the weak-source case, with the source aligned in the third direction, as in (2.7). If the gauge is fixed completely, let us say by demanding transversality of $\vec{A}_a$ (supplemented by appropriate boundary conditions[13,14]), then (3.1) is no longer an acceptable solution, since that vector potential is not transverse. The only solution is the Coulombic one of (2.10). It must be stressed that we are not saying that (3.1) is a gauge transform of (2.10); it is not. Rather, our point of view is that (3.1) is not an admissible solution once the gauge is fixed to be transverse. It is true that the gauge transform of (3.1) given by (3.2), (3.3), (3.6a), and (3.9) is in the correct transverse gauge, but this configuration is a solution with the source (3.3) which differs from (2.7). In short, if both the gauge and the source are specified, then the multiplicity here exhibited for weak sources is absent and quantization proceeds in a conventional manner.[15] (The same is true for the other, time-dependent solutions that have been discussed in the literature.[2])

When strong sources are considered, then a multiplicity remains. Consider the radial configuration (3.11b) or (4.1) and a transversality condition on $\vec{A}$. The ordinary Coulomb solution (2.10), after a gauge transformation which takes (2.7) into a radial form, is not acceptable since the vector potential is not transverse. However, there still remain three transverse solutions: the type-I solution with lowest energy and the two type-II solutions.[16] The type I is a Coulomb type and represents conventional physics for a radial source. The two type-II solutions are unexpected. For the quantum theory, where the sources approximate an assembly of many heavy quarks, the type-II solutions hint at the existence of higher excited states. Since we are ignorant about the classical stability question, we cannot say whether these states are stable, quasistable, or unstable. Also we do not have a description of the dynamics that would give rise to the requisite sources. Nevertheless, we believe that owing to the fact that the multiple classical solution exists for essentially arbitrary sources of sufficient strength, the states that we have found are, in general, interesting and new phenomena in Yang-Mills theory.

ACKNOWLEDGMENTS

This investigation, initiated with J. Goldstone, was originally directed toward a different question. Although he did not take part in much of the subsequent development, he aided us through frequent conversations and suggestion. These are gratefully acknowledged. Also, we benefited from comments by R. Giles. Our work was supported in part through funds provided by the U.S. Department of Energy (DOE) under Contracts Nos. EY-76-C-02-3069 and EY-76-C-02-0016.

APPENDIX A

We show that the energy formula (2.13) follows from (2.1) for static solutions with decrease at infinity. Consider the energy-momentum tensor

$$\theta^{\mu\nu} = -F_a^{\mu\alpha}F^{\nu}_{a\alpha} + \tfrac{1}{4}g^{\mu\nu}F_a^{\alpha\beta}F_{a\alpha\beta}\,. \tag{A1}$$

This quantity is traceless, but not conserved owing to the presence of external charges; rather from (1.1) and (1.2) we have

$$\partial_\mu \theta^{\mu\nu} = g\rho_a F_a^{0\nu}\,. \tag{A2}$$

We now begin with Eq. (2.1) for the energy and by a series of transformations arrive at (2.13)

$$\begin{aligned}\mathcal{E} &= \tfrac{1}{2}\int d\vec{r}\,(\vec{E}_a^{\,2}+\vec{B}_a^{\,2}) = \int d\vec{r}\,\theta^{00} \\ &= \int d\vec{r}\,\theta^{ii} = \int d\vec{r}(\partial_k x^i)\theta^{ki} = -\int d\vec{r}\,x^i\partial_k\theta^{ki} \\ &= -\int d\vec{r}\,x^i\partial_\mu\theta^{\mu i} = -g\int d\vec{r}\,x^i\rho_a F_a^{0i} \\ &= g\int d\vec{r}\rho_a\vec{r}\cdot\vec{E}_a\,.\end{aligned} \tag{A3}$$

The third equality follows from the second since $\theta^{\mu\nu}$ is traceless. The integration by parts needed to pass from the fourth equality to the fifth produces no surface terms, provided θ^{ki} decreases more rapidly than r^{-3} at large distances; this is ensured when the fields fall away faster than $r^{-3/2}$. In the sixth equaltiy we use the time independence of the fields and (A2) gives the next step, with which we arrive at the desired result.

APPENDIX B

We prove that

$$\mathcal{E}_{\rm II}^{(2)} = \frac{g^2}{8\pi}\int d\vec{r}\,d\vec{r}\,'\hat{r}\cdot\hat{r}'\frac{q(r)q(r')}{|\vec{r}-\vec{r}\,'|} \tag{B1}$$

is always less than the Coulomb energy:

$$\mathcal{E}_C = \frac{g^2}{8\pi}\int d\vec{r}\,d\vec{r}\,'\frac{q(r)q(r')}{|\vec{r}-\vec{r}\,'|}\,. \tag{B2}$$

We begin by evaluating the angular integrations in (B1):

$$\mathcal{E}_{\rm II}^{(2)} = g^2\frac{4\pi}{3}\int_0^\infty dr' q(r')\int_0^{r'} dr\,r^3 q(r)\,, \tag{B3}$$

Next the charge densities are expressed in terms of their Fourier transforms:

$$F(k) = \int d\vec{r}\,e^{i\vec{k}\cdot\vec{r}}q(r)\,, \tag{B4a}$$

$$q(r) = \frac{1}{2\pi^2 r}\int_0^\infty dk\,kF(k)\sin kr\,. \tag{B4b}$$

Spherical symmetry of q is used to arrive at (B4b). In terms of F, $\mathcal{E}_C$ is given by

$$\mathcal{E}_C = \frac{g^2}{4\pi^2}\int_0^\infty dk\,F^2(k)\,. \tag{B5}$$

Substitute (B4b) in (B3) and evaluate the r and r' integrals to get

$$\mathcal{E}_{\rm II}^{(2)} = -\frac{g^2}{6\pi^2}\int_0^\infty dk'\,k'F(k')\int_0^\infty dk\,kF(k) \times\left[\frac{\delta'(k-k')}{k'} - \frac{\delta(k-k')}{k^2} + \frac{2}{k^3}\theta(k-k')\right]. \tag{B6}$$

The first term in the brackets evaluates to

$$-\frac{g^2}{6\pi^2}\int_0^\infty dk\, kF(k)F'(k) = -\frac{g^2}{12\pi^2}\int_0^\infty dk\, k[F^2(k)]' = \frac{g^2}{12\pi^2}\int_0^\infty dk F^2(k)\,.$$

The second gives $(g^2/6\pi^2)\int_0^\infty dk F^2(k)$. Hence these two combine into $(g^2/4\pi^2)\int_0^\infty dk F^2(k) = \mathcal{E}_C$. Thus we find

$$\mathcal{E}_{II}^{(2)} = \mathcal{E}_C - \Delta\,, \quad \Delta = \frac{g^2}{3\pi^2}\int_0^\infty dk' \frac{F(k')}{(k')^2}\int_0^{k'} dk\, k^3 \frac{F(k)}{(k)^2}\,. \tag{B7}$$

The argument is completed by demonstrating that Δ is always positive. Comparison with (B1) and (B3) shows that

$$\Delta = \frac{g^2}{32\pi^4}\int d\vec{k}\, d\vec{k}' \frac{\hat{k}\cdot\hat{k}'}{|\vec{k}-\vec{k}'|}\frac{F(k)}{(k)^2}\frac{F(k')}{(k')^2}\,. \tag{B8a}$$

Next we introduce the Fourier transform

$$\vec{g}(\vec{r}) = \int \frac{d\vec{k}}{(2\pi)^3} e^{-i\vec{k}\cdot\vec{r}}\,\hat{k}\,\frac{F(k)}{k^2}\,, \qquad \hat{k}\frac{F(k)}{k^2} = \int d\vec{r}\, e^{i\vec{k}\cdot\vec{r}}\vec{g}(\vec{r})\,. \tag{B8b}$$

It follows that

$$\Delta = \frac{g^2}{32\pi^4}\int d\vec{k}\,d\vec{k}' \frac{1}{|\vec{k}-\vec{k}'|}\int d\vec{r}\,d\vec{r}' e^{i(\vec{k}\cdot\vec{r}+\vec{k}'\cdot\vec{r}')} \times \vec{g}(\vec{r})\cdot\vec{g}(\vec{r}')\,,$$
$$= g^2\int \frac{d\vec{r}}{r^2}\vec{g}(\vec{r})\cdot\vec{g}(-\vec{r})\,,$$
$$= g^2\int \frac{d\vec{r}}{r^2}\vec{g}(\vec{r})\cdot\vec{g}^*(\vec{r}) > 0\,. \tag{B8c}$$

This establishes the desired result.

APPENDIX C

Here it is demonstrated that taking the most general, radially (gauge) symmetric *Ansatz*[8] for the Yang-Mills potential

$$A_a^0 = \frac{1}{g}A^0\hat{r}^a\,, \qquad A_a^i = \frac{1}{g}\epsilon^{iaj}\frac{\hat{r}^j}{r}(1+\varphi_2) + \frac{1}{gr}(\delta^{ia}-\hat{r}^i\hat{r}^a)\varphi_1 + \frac{1}{g}\hat{x}^i\hat{x}^a A^1 \tag{C1}$$

does not lead to a more general theory. As is well known, the above choice of invariant functions produces a U(1) Higgs-gauge model in two-dimensional space-time with constant curvature. The equations are

$$\partial_\mu(r^2F^{\mu\nu}) + 2\varphi_i\epsilon_{ij}\mathfrak{D}_{jk}^\nu\varphi_k = r^2 q\,\delta^{\nu 0}\,, \tag{C2a}$$

$$\mathfrak{D}_{ij}^\mu\mathfrak{D}_{jk\mu}\varphi_k + \frac{1}{r^2}\varphi_i(\varphi^2-1) = 0\,. \tag{C2b}$$

Here the two coordinates x^μ are time ($\mu=0$) and radius ($\mu=1$); indices are raised and lowered with the metric tensor

$$g^{\mu\nu} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The gauge-covariant quantities are defined by

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu\,, \tag{C3a}$$

$$\mathfrak{D}_{ij}^\mu = \partial^\mu\delta_{ij} - A^\mu\epsilon_{ij}\,. \tag{C3b}$$

Abelian gauge invariance of this theory allows setting A^1 to zero. Then, for static fields, Eqs. (C2) become

$$\varphi_1\varphi_2' = \varphi_2\varphi_1'\,, \tag{C4a}$$

$$-(rA_0)'' + \frac{2\varphi^2}{r^2}(rA_0) = rq\,, \tag{C4b}$$

$$-\varphi_i'' + \frac{1}{r^2}(\varphi^2 - 1 - r^2A_0^{\,2})\varphi_i = 0\,. \tag{C4c}$$

One solution of (C4a) is $\phi_1 = 0$, and then (C4) reduce to (4.4) with the identification $\phi_2 = -a$, $A^0 = f/r$.

The other solution to (C4a) is $\phi_2 = c\phi_1$ with c an arbitrary constant. To see that this too reduces to our *Ansatz*, define a new field $-a = \phi_1(1+c^2)^{1/2}$. Then the vector potential becomes

$$A_a^i = \frac{1}{g}\epsilon^{iaj}\frac{\hat{r}^j}{r}\left(1 - \frac{c}{(1+c^2)^{1/2}}a\right) - \frac{1}{gr}(\delta^{ia}-\hat{r}^i\hat{r}^a)\frac{a}{(1+c^2)^{1/2}}\,, \tag{C5}$$

which can be brought to the form (4.2b) by the gauge transformation $U = \exp[i(\vec{\sigma}\cdot\hat{r}/2)\theta]$, $\cot\theta = c$.

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[1]J. Mandula, Phys. Rev. D 14, 3497 (1976); Phys. Lett. 67B, 175 (1977); 69B, 495 (1977); M. Magg, *ibid.* 74B, 246 (1977); 77B, 199 (1978); 78B, 481 (1978); P. Sikivie and N. Weiss, Phys. Rev. Lett. 40, 1411 (1978); Phys. Rev. D 18, 3809 (1978); K. Cahill, Phys. Rev. Lett. 41, 599 (1978); 41, 913(E) (1978); P. Pirilä and P. Prešnajder, Nucl. Phys. B142, 229 (1978).

[2]P. Sikivie and N. Weiss, Ref. 1. This is the authors'

"total screening" solution.

[3]J. Mandula, Ref. 1.

[4]P. Sikivie and N. Weiss, Ref. 1. This is the authors' "magnetic dipole" solution.

[5]S. Coleman, in *New Phenomena in Sub-Nuclear Physics*, edited by A. Zichichi (Plenum, New York, 1977); S. Deser, Phys. Lett. 64B, 463 (1976).

[6]J. Goldstone, private communication.

[7]J. Harnad, L. Vinet, and S. Shnider, Université de Montréal report (unpublished).

[8]E. Witten, Phys. Rev. Lett. 38, 121 (1977).

[9]These equations were first obtained with J. Goldstone, in the course of a quite different investigation of the Yang-Mills theory.

[10]More precisely, two families, one with positive values of Q, the other with negative values of Q, but this is of course a trivial consequence of the invariance of the equations under charge conjugation.

[11]Since we have explored a finite, although quite extended, range of initial data, we cannot rule out the existence of further solutions. We found, however, no hint in our results pointing to this possibility.

[12]R. Friedberg, T. D. Lee, and A. Sirlin, Phys. Rev. D 13, 2739 (1976). The cusp at the opening of the two type-II branches in the energy curve is reminiscent of the behavior found in nontopological solitons.

[13]V. Gribov, Lecture at 12th Winter School, Leningrad (unpublished); Nucl. Phys. B139, 1 (1978).

[14]R. Jackiw, I. Muzinich, and C. Rebbi, Phys. Rev. D 17, 1576 (1978); R. Jackiw, in *New Frontiers in High-Energy Physics*, edited by B. Kurşunoğlu, A. Perlmutter, and L. Scott (Plenum, New York, 1978); M. Ademollo, E. Napolitano, and S. Sciuto, Nucl. Phys. B134, 477 (1978).

[15]This argument was also urged upon us by R. Giles.

[16]The situation is somewhat reminiscent of the ambiguity in the Coulomb gauge discussed by Gribov, Ref. 13. However, there are important differences: Gribov presents gauge copies of the same gauge-field configuration; we find that a fixed source and the Coulomb gauge condition allow for multiple, gauge-nonequivalent solutions.