Duffin-Kemmer formulation of gauge theories

Susumu Okubo and Yasunari Tosa
Department of Physics and Astronomy, University of Rochester, Rochester, New York 14627
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Gauge theories, including the Yang-Mills theory as well as Einstein's general relativity, are reformulated in first-order differential forms. In this generalized Duffin-Kemmer formalism, gauge theories take very simple forms with only cubic interactions. Moreover, every local gauge transformation, e.g., that of Yang and Mills or Einstein, etc., has an essentially similar form. Other examples comprise a gauge theory akin to the Sugawara theory of currents and the nonlinear realization of chiral symmetry. The octonion algebra is found possibly relevant to the discussion of the Yang-Mills theory.

I. INTRODUCTION

It is well known that equations of motion for a free field of arbitrary spin can be written as a first-order differential equation

$$(\beta^\mu \partial_\mu - M)\psi(x) = 0\,, \tag{1.1}$$

where β^μ and M are constant matrices. The corresponding Lagrangian may be given by

$$L(x) = \tfrac{1}{2}\bar{\psi}\,\beta^\mu \partial_\mu \psi - \tfrac{1}{2}\bar{\psi} M \psi\,, \tag{1.2}$$

where $\bar{\psi}(x)$ is related to $\psi(x)$ by

$$\bar{\psi}(x) = \psi^T(x) C\,, \tag{1.3}$$

for another constant matrix C. The superscript T denotes the transpose. It is easy to find the explicit forms of β^μ and M, using the fact that the complexification of SO(3,1) Lie algebra is just sl(2)$\oplus$sl(2). Then all possible irreducible finite-dimensional representations of the Lorentz group are specified by $D^{(u,v)}$, where both $2u$ and $2v$ are non-negative integers. Note that the derivative ∂_μ is an element of the $D^{(1/2,1/2)}$ representation. Decomposing the field $\psi(x)$ into a direct sum of irreducible components $\psi^{(u,v)}$, we can show that $\partial_\mu \psi^{(u,v)}$ transforms as

$$D^{(1/2,1/2)} \otimes D^{(u,v)} = D^{(u+1/2,v+1/2)} \oplus D^{(u+1/2,v-1/2)} \oplus D^{(u-1/2,v+1/2)} \oplus D^{(u-1/2,v-1/2)}\,. \tag{1.4}$$

Comparing these with Eq. (1.1), we are, in principle, able to determine the matrix forms of β^μ and M. For example, the Dirac field corresponds to the well-known four-dimensional decomposition $D^{(1/2,0)} \oplus D^{(0,1/2)}$. For a spin-zero field, it is the five-dimensional space $D^{(1/2,1/2)} \oplus D^{(0,0)}$, corresponding to $(\partial_\mu \phi, \phi)$, while for a vector field the representation content is the ten-dimensional space $D^{(1,0)} \oplus D^{(0,1)} \oplus D^{(1/2,1/2)}$. The latter two cases are known as the Duffin-Kemmer formulation.[1]

Hereafter, any first-order equation of the Eq. (1.1) type will be referred to as the (free) Duffin-Kemmer equation.

Although many articles and books[2] have been written on those relativistic equations and the results are now well known, its extension for cases involving interactions is less straightforward in a small number of works.[3] In this paper we are primarily interested in this aspect of the Duffin-Kemmer formulation, especially in connection with Yang-Mills gauge theories.[4] In Sec. II we will show that the Lagrangian for a pure Yang-Mills field can be rewritten as

$$L(x) = \tfrac{1}{2}\bar{\phi}(x)\beta^\mu \partial_\mu \phi(x) - \tfrac{1}{2}\bar{\phi}(x) M \phi(x) - g \sum_{i,j,k=1}^{N} a_{ijk}\phi_i(x)\phi_j(x)\phi_k(x)\,, \tag{1.5}$$

where $\phi(x) = (\phi_1(x), \phi_2(x), \ldots, \phi_N(x))^T$ is now an N-dimensional field and a_{jkl} $(j,k,l = 1,2,3,\ldots,N)$ are numerical constants which may have some relevance to the octonion algebra. The term $\bar{\phi}(x) M \phi(x)$, for example, implies

$$\sum_{i,j=1}^{N} \bar{\phi}_i(x) M_{ij} \phi_j(x)\,. \tag{1.6}$$

The generalization of Eq. (1.5) which includes the matter field is discussed in Sec. III. We may still write the whole Lagrangian in the form of Eq. (1.5), where $\phi(x)$ now stands for a large column vector involving both gauge and matter fields. The usual local gauge transformation for matter and gauge fields can be summarized as the transformation of the field $\phi(x)$:

$$\phi_j(x) \to \phi_j'(x) = \sum_{k=1}^{N} U_{jk}(x)\phi_k(x) + V_j(x)\,, \tag{1.7}$$

where $U_{jk}(x)$ and $V_j(x)$ are functions of the coordinates but independent of the field variables $\phi_j(x)$ themselves. In this respect the usual distinction between the gauge fields and the matter fields formally disappears. If the matter fields include fermions, then the supersymmetric theories[5] may also be considered as an example of Eqs. (1.5) and (1.7).

A similar thing happens in Einstein's general relativity. The Lagrangian density

$$L(x)=\sqrt{-g}R\,, \tag{1.8}$$

where R is the scalar curvature, is well known to have a very complicated nonlinear functional form of the metric tensor $g_{\mu\nu}(x)$. However, if we choose $\phi(x)$ as a 50-component vector consisting of $\sqrt{-g}\,g^{\mu\nu}$ and $\Gamma^\lambda_{\mu\nu}$ $(\lambda,\mu,\nu=0,1,2,3)$, then the nonlinear Lagrangian is remarkably reduced to the much simpler form of Eq. (1.5). Furthermore, the local coordinate transformation can be written in the form of Eq. (1.7). These facts suggest that all local gauge theories may be written and characterized by Eqs. (1.5) and (1.7). Of course, any attempt in this direction is very intricate and beyond the scope of the present article.

As another example of our approach we obtain the Lagrangian which leads to the same equation of motion as does the Sugawara theory of currents.[6] A minor modification of our Lagrangian, which is no longer local gauge-invariant, yields the Cremmer-Scherk-Kalb-Ramond Lagrangian for dual models.[7]

Section IV is devoted to the chiral-invariant Lagrangian in the Duffin-Kemmer formalism. We will consider both the nonlinear σ model[8] and the Weinberg model.[9] In the Sec. V the gauge theory in the five-dimensional space is considered. We have noticed that the group SO(4,1), which contains the Lorentz group SO(3,1) as its subgroup, appears somehow related to the Yang-Mills gauge theory. This fact has tempted us to investigate the gauge theory in the five-dimensional space.

II. YANG-MILLS GAUGE THEORY

The Yang-Mills Lagrangian[4] is given by

$$L(x)=\tfrac{1}{4}F^a_{\mu\nu}(x)F^{a\mu\nu}(x)+\tfrac{1}{2}m^2A^a_\mu(x)A^{a\mu}(x) -\tfrac{1}{2}F^a_{\mu\nu}(x)[\partial^\mu A^{a\nu}(x)-\partial^\nu A^{a\mu}x)+gf^{abc}A^{b\mu}(x)A^{c\nu}(x)], \tag{2.1}$$

where the notation is standard[10] except for an inclusion of the mass term $m^2A^a_\mu(x)A^{a\mu}(x)$. For the pure Yang-Mills theory, we have to set $m^2=0$, of course. The Lorentz metric is taken as

$$\eta_{00}=-\eta_{11}=-\eta_{22}=-\eta_{33}=1\,,\quad \eta_{\mu\nu}=0\ (\mu\neq\nu)\,. \tag{2.2}$$

The Latin indices in Eq. (2.1) refer to the internal symmetry with the Lie algebra

$$[t^a,t^b]=if^{abc}t^c\,, \tag{2.3}$$

for $a,b,c=1,2,\ldots,n$, where f^{abc} is the totally antisymmetric structure coefficient. Note that when we take variations with respect to both $F^a_{\mu\nu}$ and A^a_μ as independent variables, the Lagrangian (2.1) reproduces the standard equation of motion:

$$F^a_{\mu\nu}(x)=\partial_\mu A^a_\nu(x)-\partial_\nu A^a_\mu(x)+gf^{abc}A^b_\mu(x)A^c_\nu(x)\,, \tag{2.4a}$$

$$\partial^\nu F^a_{\mu\nu}(x)=m^2A^a_\mu(x)+gf^{abc}F^b_{\mu\nu}(x)A^{c\nu}(x)\,. \tag{2.4b}$$

In order to rewrite the Lagrangian (2.1) in the form of Eq. (1.5), we introduce $N(=10n)$ component column vector

$$\phi^a_j(x)\quad (j=1,2,\ldots,10,\ a=1,2,\ldots,n)$$

by (see Appendix A)

$$\phi^a(x)=(F^a_{23}(x),F^a_{31}(x),F^a_{12}(x),F^a_{01}(x),F^a_{02}(x),F^a_{03}(x),A^a_1(x),A^a_2(x),A^a_3(x),A^a_0(x))^T. \tag{2.5}$$

Then the Lagrangian (2.1) becomes

$$L(x)=\tfrac{1}{2}\bar\phi^a(x)\beta_\mu\partial^\mu\phi^a(x)-\tfrac{1}{2}\bar\phi^a(x)M\phi^a(x)-\frac{g}{3!}f^{abc}\Gamma_{ijk}\phi^a_i(x)\phi^b_j(x)\phi^c_k(x)\,, \tag{2.6}$$

with the ambiguity of total-divergence terms which do not affect the equation of motion. The 10×10 *real* matrices, β_μ and M, are given by

$$\beta_0=\left[\begin{array}{ccc|ccc|ccc|c} &&&&&&&&& \\ &&&&&&&&& \\ &&&&&&&&& \\ \hline &&&&&&1&0&0& \\ &&&&&&0&1&0& \\ &&&&&&0&0&1& \\ \hline &&&-1&0&0&&&& \\ &&&0&-1&0&&&& \\ &&&0&0&-1&&&& \\ \hline &&&&&&&&& \end{array}\right],\qquad \beta_1=\left[\begin{array}{ccc|ccc|ccc|c} &&&&&&0&0&0& \\ &&&&&&0&0&1& \\ &&&&&&0&-1&0& \\ \hline &&&&&&&&&1 \\ &&&&&&&&&0 \\ &&&&&&&&&0 \\ \hline 0&0&0&&&&&&& \\ 0&0&-1&&&&&&& \\ 0&1&0&&&&&&& \\ \hline &&&1&0&0&&&& \end{array}\right],$$

$$\beta_2=\left[\begin{array}{ccc|ccc|ccc|c} &&&&&&0&0&-1& \\ &&&&&&0&0&0& \\ &&&&&&1&0&0& \\ \hline &&&&&&&&&0 \\ &&&&&&&&&1 \\ &&&&&&&&&0 \\ \hline 0&0&1&&&&&&& \\ 0&0&0&&&&&&& \\ -1&0&0&&&&&&& \\ \hline &&&0&1&0&&&& \end{array}\right],\qquad \beta_3=\left[\begin{array}{ccc|ccc|ccc|c} &&&&&&0&1&0& \\ &&&&&&-1&0&0& \\ &&&&&&0&0&0& \\ \hline &&&&&&&&&0 \\ &&&&&&&&&0 \\ &&&&&&&&&1 \\ \hline 0&-1&0&&&&&&& \\ 1&0&0&&&&&&& \\ 0&0&0&&&&&&& \\ \hline &&&0&0&1&&&& \end{array}\right], \tag{2.7}$$

$$M=\left[\begin{array}{ccc|ccc|ccc|c} 1&0&0&&&&&&& \\ 0&1&0&&&&&&& \\ 0&0&1&&&&&&& \\ \hline &&&1&0&0&&&& \\ &&&0&1&0&&&& \\ &&&0&0&1&&&& \\ \hline &&&&&&m^2&0&0& \\ &&&&&&0&m^2&0& \\ &&&&&&0&0&m^2& \\ \hline &&&&&&&&&m^2 \end{array}\right].$$

The adjoint $\overline{\phi}^a(x)$ is defined by

$$\overline{\phi}^a(x)=(\phi^a(x))^T C\,, \tag{2.8}$$

where the 10×10 matrix C has the following diagonal form:

$$C=\left[\begin{array}{ccc|ccc|ccc|c} -1 & 0 & 0 & & & & & & & \\ 0 & -1 & 0 & & & & & & & \\ 0 & 0 & -1 & & & & & & & \\ \hline & & & 1 & 0 & 0 & & & & \\ & & & 0 & 1 & 0 & & & & \\ & & & 0 & 0 & 1 & & & & \\ \hline & & & & & & 1 & 0 & 0 & \\ & & & & & & 0 & 1 & 0 & \\ & & & & & & 0 & 0 & 1 & \\ \hline & & & & & & & & & -1 \end{array}\right]. \qquad (2.9)$$

We can verify that the matrices β_μ satisfy the Duffin-Kemmer relation[1]

$$\beta_\mu\beta_\lambda\beta_\nu+\beta_\nu\beta_\lambda\beta_\mu=-\eta_{\mu\lambda}\beta_\nu-\eta_{\nu\lambda}\beta_\mu . \qquad (2.10)$$

Furthermore, the standard relations for the transposed matrices β_μ^T and C^T are satisfied as follows:

$$\beta_\mu^T=-C\beta_\mu C^{-1}=-\beta^\mu(=-\eta^{\mu\nu}\beta_\nu)\,, \qquad (2.11)$$

$$C^T=C=C^{-1}=-1-2(\beta_0)^2 . \qquad (2.12)$$

The triple linear coefficient Γ_{ijk} in Eq. (2.6) has some complicated structure: First, it is completely antisymmetric in three indices i, j, and k, which is obvious from the construction. Second, it assumes only three values $+1$, 0, and -1. The value $+1$ is taken only for the combinations $(i,j,k)=(8,3,7)$, $(7,2,9)$, $(9,1,8)$, $(8,10,5)$, $(9,10,6)$, and $(7,10,4)$. Parallel to the octonion case, as done by Freudenthal,[11] we depict our case in Fig. 1. Actually, we can also use the diagram shown in Fig. 2 for the description of Γ_{ijk}. However, for the reason which will be clear soon, we will prefer to use Fig. 1 from now on.

Because the coefficient Γ_{ijk} is completely antisymmetric in i,j,k, it is naturally related to a noncommutative Jordan algebra.[12] Following the method of Osborn and Falkner,[13] we introduce a

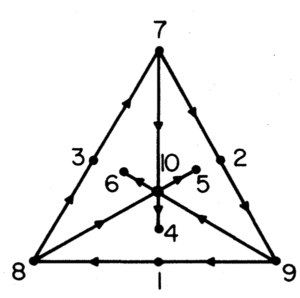

FIG. 1. The structure diagram of the coefficients Γ_{ijk} for the Yang-Mills theory.

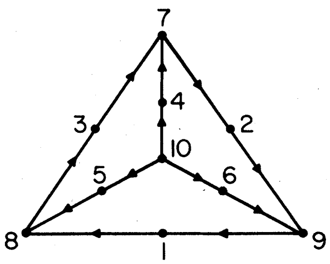

FIG. 2. Another representation of the coefficients Γ_{ijk}.

unit element e_0 together with 10 elements e_j. Consider a multiplication table defined as follows:

$$e_je_k=\mp\delta_{jk}e_0+\sum_{m=1}^{10}\Gamma_{jkm}e_m \quad (j,k\neq 0)\,, \qquad (2.13a)$$

$$e_je_0=e_0e_j=e_j\,,\quad e_0e_0=e_0 . \qquad (2.13b)$$

Then the algebra among e_j becomes a nonassociative 11-dimensional algebra, which is, however, a noncommutative Jordan algebra, since it is obviously Jordan-admissible.[14] Although both signs in Eq. (2.13a) are equally acceptable, we choose the upper sign -1 hereafter for a reason which will become clear.

Introducing the bilinear form (x,y) defined by

$$(e_j,e_k)=\delta_{jk} \quad (j,k=0,1,\ldots,10)\,, \qquad (2.14)$$

we can express the coefficient Γ_{jkm} as

$$\Gamma_{jkm}=(e_je_k,e_m)=(e_j,e_ke_m) \quad (j,k,m\neq 0) . \qquad (2.15)$$

We note that the algebra defined by Eqs. (2.13a) and (2.13b) is *not* Lie-admissible,[15] and therefore Γ_{ijk} cannot be a structure constant of any Lie algebra.

There is a relation between β_μ and Γ_{ijk}. Defining the 10×10 matrices Γ_j $(j=1,2,\ldots,10)$ by

$$(\Gamma_i)_{jk}=\Gamma_{ijk}\,, \qquad (2.16)$$

so that

$$\Gamma_i^T=-\Gamma_i\,, \qquad (2.17)$$

we have

$$C\beta_1=\Gamma_7,\quad C\beta_2=\Gamma_8,\quad C\beta_3=\Gamma_9,\quad C\beta_0=\Gamma_{10} . \qquad (2.18)$$

Our formalism is intimately related to the SO(3, 2) symmetry. Let us set

$$J_{\mu\nu}=-J_{\nu\mu}=\beta_\mu\beta_\nu-\beta_\nu\beta_\mu . \qquad (2.19)$$

Then in view of Eq. (2.10), we find

$$[J_{\mu\nu},\beta_\lambda]=\eta_{\lambda\mu}\beta_\nu-\eta_{\lambda\nu}\beta_\mu\,, \qquad (2.20)$$

as well as

$$[J_{\mu\nu},J_{\lambda\tau}]=\eta_{\nu\tau}J_{\mu\lambda}+\eta_{\mu\lambda}J_{\nu\tau}-\eta_{\mu\tau}J_{\nu\lambda}-\eta_{\nu\lambda}J_{\mu\tau}\,, \qquad (2.21)$$

for $\mu, \nu, \lambda, \tau = 0, 1, 2, 3$. Note that Eq. (2.21) defines the Lie algebra of Lorentz group SO(3, 1). The Lorentz invariance of the interaction terms of the Lagrangian (2.6) implies

$$(J_{\mu\nu})_{i'i}\Gamma_{i'jk}+(J_{\mu\nu})_{j'j}\Gamma_{ij'k}+(J_{\mu\nu})_{k'k}\Gamma_{ijk'}=0 . \tag{2.22}$$

If we introduce the fifth coordinate by setting

$$\eta_{55}=1, \quad \eta_{5\mu}=0 \quad \text{for } \mu \neq 5 , \tag{2.23}$$

and

$$\begin{aligned} &J_{\lambda 5} = -J_{5\lambda} = \beta_\lambda \quad \text{for } \lambda \neq 5 , \\ &J_{55} = 0 \quad \text{for } \lambda = 5 , \end{aligned} \tag{2.24}$$

we find that Eq. (2.21) is valid for $\mu, \nu, \lambda, \tau = 0, 1, 2, 3$, *and 5*. In other words, $J_{\mu\nu}$ ($\mu, \nu = 0, 1, 2, 3, 5$) now describes the infinitesimal generators of the SO(3, 2) group. Our reducible ten-dimensional representation space corresponding to $D^{(1,0)} \oplus D^{(0,1)} \oplus D^{(1/2,1/2)}$ of the SO(3, 1) group is now specified by a single ten-dimensional irreducible representation of the larger group SO(3, 2). Indeed, analogous to Eq. (2.24), if we define $F^a_{\lambda 5}$ by

$$\begin{aligned} &F^a_{\lambda 5} = -F^a_{5\lambda} = A^a_\lambda \quad \text{for } \lambda \neq 5 , \\ &F^a_{55} = 0 \quad \text{for } \lambda = 5 , \end{aligned} \tag{2.25}$$

then the antisymmetric tensor $F^a_{\mu\nu}$ ($\mu, \nu = 0, 1, 2, 3, 5$) specifies our irreducible tensor in the SO(3, 2) group. Of course, these results are well known if we neglect the interaction terms. As a matter of fact, the usual Duffin-Kemmer theory corresponds to a special case of a more general SO(5) theory by Bhabha.[16] Using the five-dimensional terminology, we see that both the mass terms and the interaction terms in the Lagrangian (2.6) behave as a $T_5^{\ 5}$ component of a tensor T^ν_μ with nonvanishing trace ($\mu, \nu = 0, 1, 2, 3, 5$), while the kinetic term behaves as a fifth component of a vector V_μ.

An interesting fact is that we can make the cubic interactions invariant under SO(3, 2) by adding the term

$$L' = -\frac{1}{3!}\, g f^{abc} F^a_{\mu\nu}(x) F^b_{\alpha\beta}(x) F^c_{\gamma\lambda}(x) \eta^{\nu\alpha}\eta^{\beta\gamma}\eta^{\mu\lambda} \tag{2.26}$$

to the old Lagrangian (2.1), where the summation over the indices runs only from 0 to 3. The resultant interactions in this new Lagrangian take the same form as Eq. (2.26), where the only difference is that the summation goes to the fifth component. Note that the local gauge invariance of the Yang-Mills type is *not violated* by the terms in Eq. (2.26). As before, we rewrite the new Lagrangian in the Duffin-Kemmer form. Then the coefficients Γ_{ijk} become more complicated than before, as shown in Figs. 3(a) and 3(b). Note that

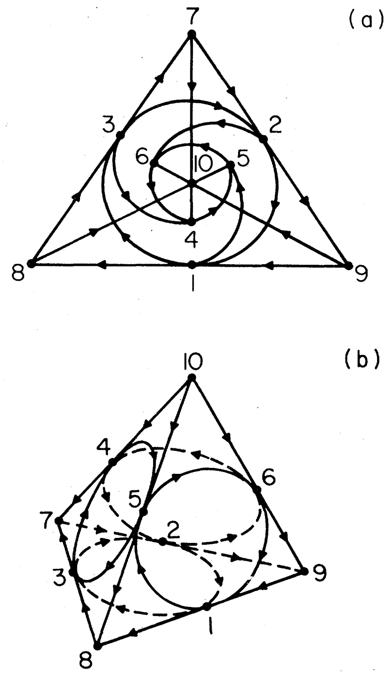

FIG. 3. (a) The structure diagram of the coefficients where the cubic interactions are invariant under the group SO(3, 2). (b) Another representation of (a).

this time they correspond to the structure constants of the Lie algebra of SO(3, 2).

The whole theory can be simplified if we discuss it in Euclidean space-time instead of Minkowski space-time. The metric tensor is now

$$\eta_{\mu\nu} = \delta_{\mu\nu} . \tag{2.27}$$

In this space it is allowed to have the fields (pseudoparticles or instantons) which satisfy the self-conjugate property of the tensor $F_{\mu\nu}$:[17]

$$F^a_{\mu\nu}(x) = {}^*F^a_{\mu\nu}(x) , \tag{2.28}$$

where the dual tensor ${}^*F^a_{\mu\nu}(x)$ is defined as usual by

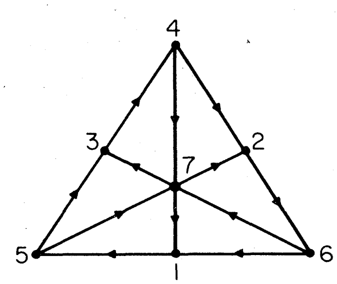

FIG. 4. The structure diagram of Γ_{ijk} when the self-conjugate properties are imposed on $F_{\mu\nu}$

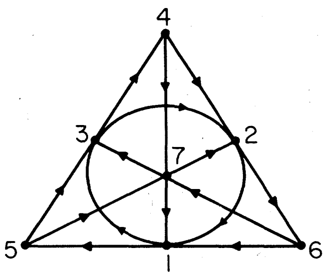

FIG. 5. The structure diagram of Γ_{ijk} for the self-conjugate $F_{\mu\nu}$ case when the additional interactions are present. This diagram exactly corresponds to that of the octonion algebra.

$$*F^a_{\mu\nu}(x)=\tfrac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^{a\alpha\beta}(x)\quad(\mu,\nu,\alpha,\beta=0,1,2,3)\,. \tag{2.29}$$

Consequently, as far as these fields are concerned, we can replace the ten-dimensional column vector $\phi^a_j(x)$ by the seven-dimensional vector:

$$\phi^a_j(x)=(F^a_{23}(x),F^a_{31}(x),F^a_{12}(x),A^a_1(x),\\ A^a_2(x),A^a_3(x),A^a_0(x))^T. \tag{2.30}$$

Accordingly, the interaction structure diagrams, Figs. 1 and 3(a), now become Figs. 4 and 5, respectively.

Note that Fig. 5 corresponds to the structure diagram of the octonion algebra[11] if we choose the upper sign in the right-hand side of Eq. (2.13a). Hence the cubic interaction terms

$$f^{abc}\Gamma_{ijk}\phi^a_i(x)\phi^b_j(x)\phi^c_k(x)$$

are invariant under the exceptional Lie group G_2, where $\phi^a_j(x)$ transforms as the fundamental seven-dimensional representation of G_2. Unfortunately, we do not know whether the invariance under the group G_2 has any physical meaning or not. This will be left to future investigations. In Appendix B we will present one of the transparent realizations of the octonion algebra.

III. FURTHER GENERALIZATION

First we show that our method can be applied to the cases where the matter fields are present.

Let $q(x)$ be fermion fields interacting with gauge fields $A^a_\mu(x)$ via the minimal coupling scheme:

$$\hat{L}=i\bar{q}(x)\gamma^\mu[\partial_\mu-igA^a_\mu(x)t^a]q(x)-\bar{q}(x)m'q(x)\,, \tag{3.1}$$

where t^a is the p-dimensional matrix of the underlying gauge symmetry for a fermion multiplet. Defining an $N(=8p+10n)$-component vector $\psi_j(x)$ as

$$\psi_j(x)=(q(x),\bar{q}(x),\phi^1_1(x),\dots,\phi^n_{10}(x))\,, \tag{3.2}$$

we can write the whole Lagrangian easily in the form of Eq. (2.6):

$$L_{\text{total}}=\tfrac{1}{2}\bar{\psi}(x)(\beta^\mu\partial_\mu-M)\psi(x)\\ +g\,\mathfrak{A}_{ijk}\psi_i(x)\psi_j(x)\psi_k(x)\,, \tag{3.3}$$

where the summation over the indices i,j,k runs from 1 through N. The numerical constant coefficients $\mathfrak{A}_{ijk}$ in the interactions no longer have any simple symmetry, since $\psi_j(x)$ now contains both fermions and bosons. Nevertheless, the local gauge transformation in our notation can be summarized as

$$\psi_j(x)\rightarrow\psi'_j(x)=\sum_{k=1}^{N}U_{jk}(x)\psi_k(x)+V_j(x)\,, \tag{3.4}$$

for some functions $U_{jk}(x)$ and $V_j(x)$ which do *not* depend upon the field ψ_j itself. Obviously any supersymmetric gauge theory can be written in the form of Eq. (3.3) with its gauge transformation in the form of Eq. (3.4).[5]

Next let us consider Einstein's general relativity. The Lagrangian density is given by

$$L=\sqrt{-g}\,R\,, \tag{3.5}$$

where the scalar curvature R is defined by

$$R=g^{\mu\nu}R_{\mu\nu}\,, \tag{3.6a}$$

$$R_{\mu\nu}=\partial_\alpha\Gamma^\alpha_{\mu\nu}-\tfrac{1}{2}(\partial_\nu\Gamma^\alpha_{\mu\alpha}+\partial_\mu\Gamma^\alpha_{\nu\alpha})\\ +\Gamma^\alpha_{\nu\mu}\Gamma^\beta_{\alpha\beta}-\Gamma^\alpha_{\beta\mu}\Gamma^\beta_{\alpha\nu}\,. \tag{3.6b}$$

It is well known[18] that the variational principle applied to this Lagrangian

$$\delta\int d^4x\,L=0 \tag{3.7}$$

yields Einstein's equations

$$R_{\mu\nu}=0\,, \tag{3.8a}$$

$$\Gamma^\lambda_{\mu\nu}=\tfrac{1}{2}g^{\lambda\alpha}(\partial_\mu g_{\alpha\nu}+\partial_\nu g_{\alpha\mu}-\partial_\alpha g_{\mu\nu})\,, \tag{3.8b}$$

when we choose both $g_{\mu\nu}$ and $\Gamma^\lambda_{\mu\nu}$ to be independent variational variables. It is easy to see that the complicated nonlinear Lagrangian (3.5) can be recast in the form of Eq. (3.3), if we define the 50-dimensional vector $\psi_j(x)$ by

$$\psi_j(x)=(\sqrt{-g}\,g^{\mu\nu},\Gamma^\lambda_{\mu\nu})\,, \tag{3.9}$$

and discard some of the total divergence terms in the Lagrangian. We simply remark here that the 50×50 matrices β_μ now obey complicated fifth-order polynomial identities, instead of the third-order Duffin-Kemmer algebra of Eq. (2.10). The detailed analysis on these properties of β_μ will be given elsewhere, and we will not pursue this case. At any rate, it is clear in our form that even general relativity may be regarded as the ordinary Minkowski-space field theory with cubic self-interaction terms where the gauge transformation is given by Eq. (3.4).

These results suggest that a general study of Eqs. (3.3) and (3.4) may be worthwhile. However, in this paper we will confine ourselves to the generalized cases of Yang-Mills theories because of the complexity.

We only consider a Lagrangian of the form described by Eq. (2.6):

$$L(x)=\tfrac{1}{2}\,\overline{\phi}^a(x)\beta^\mu\partial_\mu\phi^a(x)-\tfrac{1}{2}\,\overline{\phi}^a(x)M\,\phi^a(x)-\frac{g}{3!}\,f^{abc}\Gamma_{ijk}\phi_i^a(x)\phi_j^b(x)\phi_k^c(x)\,, \tag{3.10}$$

where we set

$$\overline{\phi}^a(x)\equiv\phi^{a^T}(x)C,\quad \hat{\beta}_\mu\equiv C\beta_\mu,\quad \hat{M}\equiv CM\,. \tag{3.11}$$

However, we do not necessarily assume the explicit forms of β_μ, M, and Γ_{ijk} as given in Sec. II, except the property that the matrices β_μ are diagonal in the internal-symmetry space. The general local gauge transformation is given by the same form as Eq. (3.4):

$$\phi_j^a(x)\to\phi_j'^a(x)=\sum_{k,\,b}U_{jk}^{ab}\phi_k^b(x)+V_j^a(x)\,. \tag{3.12}$$

Without loss of generality, we can assume the following properties of β_μ and M:

$$\hat{\beta}_\mu^T=-\hat{\beta}_\mu,\quad \hat{M}^T=\hat{M}\,, \tag{3.13}$$

since we are dealing only with bosons.

Now the invariance of the Lagrangian (3.10) under the local gauge transformation (3.12) demands the validities of the following equations:

$$U_{ik}^{ab}\,\hat{\beta}^\mu_{ij}U_{jl}^{ac}=\hat{\beta}^\mu_{kl}\delta^{bc}\,, \tag{3.14a}$$

$$U_{ik}^{ab}\hat{M}_{ij}U_{jl}^{ac}=\hat{M}_{kl}\delta^{bc}\,, \tag{3.14b}$$

$$f^{abc}U_{ii'}^{aa'}U_{jj'}^{bb'}U_{kk'}^{cc'}\Gamma_{ijk}=f^{a'b'c'}\Gamma_{i'j'k'}\,, \tag{3.14c}$$

as well as other equations involving $V_j(x)$ which are not reproduced here. Since we restrict ourselves to Hermitian boson fields, it is reasonable to choose U_{ij}^{ab} as a real unitary matrix in the sense that

$$(U^{ac})^TU^{bc}=E\delta^{ab}\,, \tag{3.15}$$

where we regard U_{ij}^{ab} as the (i,j) element of the matrix U^{ab} and E denotes the unit matrix. Then Eq. (3.14a) can be written as

$$[\hat{\beta}_\mu,U^{ab}]=0 \tag{3.16}$$

in the matrix notation. Hence, if the matrices $\hat{\beta}_\mu$ $(\mu=0,1,2,3)$ *either* form an irreducible matrix algebra *or* do not contain the same equivalent irreducible components more than once in their decomposition into irreducible spaces, Schur's lemma tells us that U^{ab} is diagonal in the i-j space, or

$$(U^{ab})_{ij}=\delta_{ij}u^{ab}\,, \tag{3.17}$$

in each irreducible space. This implies that we cannot mix the space-time indices with the internal-symmetry labels under the conditions described above.

It is easy to see that these conditions are actually satisfied by the ten-dimensional Duffin-Kemmer theory studied in Sec. II. However, this does not imply that the solutions for M and Γ_{ijk} are unique. Indeed, we have seen that we could have the solutions for Γ_{ijk} specified by Fig. 3, not by Fig. 1.

We can find another peculiar solution as follows. Let us choose the *mass* matrix $M=0$ in the Lagrangian (3.10). Then we obtain the following Lagrangian in an ordinary language:

$$L_{M=0}=-\tfrac{1}{2}F^a_{\mu\nu}(x)[\partial^\mu A^{a\nu}(x)-\partial^\nu A^{a\mu}(x)+gf^{abc}A^{b\mu}(x)A^{c\nu}(x)]\,. \tag{3.18}$$

This Lagrangian gives two equations of motion:

$$\partial_\mu A^a_\nu(x)-\partial_\nu A^a_\mu(x)+gf^{abc}A^b_\mu(x)A^c_\nu(x)=0\,, \tag{3.19a}$$

$$\partial^\mu f^a_{\mu\nu}(x)+gf^{abc}A^{b\mu}(x)F^c_{\mu\nu}(x)=0\,. \tag{3.19b}$$

Equation (3.19b) is compatible with Eq. (3.19a), since the integrability condition for $F_{\mu\nu}$

$$\partial^\mu\partial^\nu F^a_{\mu\nu}(x)=\partial^\nu\partial^\mu F^a_{\mu\nu}(x)=0\,, \tag{3.20}$$

does not contradict with Eqs. (3.19a) and (3.19b).

We recognize that Eq. (3.19a) is the same equation of motion which the Sugawara theory of currents[6] gives. In the latter, the fields $A^a_\mu(x)$ satisfy the usual algebra of currents at equal times, while in the former we should use, at least in principle, canonical commutation relations among $A^a_\mu(x)$. Thus our theory may differ in some aspects from the Sugawara theory. The quantization of our Lagrangian will be discussed elsewhere. The Lagrangian (3.18) is invariant under the usual local gauge transformation of Yang-Mills.

In closing this section, we would like to make the following remark. If we choose the *mass* matrix M as

$$M=\left[\begin{array}{ccc|ccc|ccc|c} 0 & & & & & & & & & \\ & 0 & & & & & & & & \\ & & 0 & & & & & & & \\ \hline & & & 0 & & & & & & \\ & & & & 0 & & & & & \\ & & & & & 0 & & & & \\ \hline & & & & & & 1 & & & \\ & & & & & & & 1 & & \\ & & & & & & & & 1 & \\ \hline & & & & & & & & & 1 \end{array}\right] \tag{3.21}$$

and consider the Abelian gauge group U(1), then our Lagrangian takes the form

$$L=-\tfrac{1}{2}(\partial_\mu A_\nu-\partial_\nu A_\mu)F^{\mu\nu}+\tfrac{1}{2}A_\mu A^\mu\,, \tag{3.22}$$

where we suppressed the index of the trivial one-dimensional U(1) symmetry. Although this is no longer invariant under the local gauge transformation, it is of some interest by the following reason.

Introducing new fields by

$$G_{\mu\nu\lambda}=-\epsilon_{\mu\nu\lambda\tau}A^\tau\,, \tag{3.23}$$

$$\phi_{\mu\nu}=\tfrac{1}{2}\epsilon_{\mu\nu\lambda\tau}F^{\lambda\tau}\equiv{}^*F_{\mu\nu}\,, \tag{3.24}$$

we obtain the following form from Eq. (3.22):

$$L=-\tfrac{1}{12}G_{\mu\nu\lambda}G^{\mu\nu\lambda}+\tfrac{1}{6}G_{\mu\nu\lambda}(\partial^\mu\phi^{\nu\lambda}+\partial^\nu\phi^{\lambda\mu}+\partial^\lambda\phi^{\mu\nu})\,, \tag{3.25}$$

which is the famous Cremmer-Scherk-Kalb-Ramond Lagrangian for dual models.[7] Note that this Lagrangian now has the local gauge symmetry of the type

$$\phi_{\mu\nu}\to\phi'_{\mu\nu}=\phi_{\mu\nu}+\partial_\mu\Lambda_\nu-\partial_\nu\Lambda_\mu\,, \tag{3.26a}$$

$$G_{\mu\nu\lambda}\to G'_{\mu\nu\lambda}=G_{\mu\nu\lambda}\,, \tag{3.26b}$$

where Λ_μ satisfies

$$\partial_\mu(\partial^\nu\Lambda^\mu-\partial^\mu\Lambda^\nu)=0\,. \tag{3.27}$$

The quantization of this model has been discussed by Kaul and Hagen.[19]

IV. CHIRAL-INVARIANT LAGRANGIANS

The nonlinear realizations[8,9,20] of the chiral group $SU_L(n)\otimes SU_R(n)$ can also be regarded as examples of our method. Here we consider the simplest case where $n=2$.

The first example we consider is the nonlinear σ model[8] where the Lagrangian is given by

$$L=\tfrac{1}{2}[\partial^\mu\vec\pi\cdot\partial_\mu\vec\pi+(\partial^\mu\sigma)(\partial_\mu\sigma)]\,, \tag{4.1}$$

with the nonlinear constraint

$$\vec\pi^2(x)+\sigma^2(x)=C^2=\text{const}\,. \tag{4.2}$$

The field $\vec\pi$ is an isospin vector while σ is an isoscalar. Because of the constraint Eq. (4.2) the Lagrangian (4.1) is nonlinear. However, we can write the equivalent Lagrangian to Eqs. (4.1) and (4.2) by introducing the auxiliary fields $\vec\pi_\mu(x)$, $\sigma_\mu(x)$, and $\phi(x)$:

$$\hat L=\vec\pi_\mu(x)\partial^\mu\vec\pi(x)+\sigma_\mu(x)\partial^\mu\sigma(x)-\tfrac{1}{2}(\vec\pi_\mu\vec\pi^\mu+\sigma_\mu\sigma^\mu)-\phi(x)(\vec\pi^2+\sigma^2-C^2)\,. \tag{4.3}$$

Note that $\phi(x)$ is the Lagrange-multiplier field. Choosing $\vec\pi_\mu$, $\vec\pi$, σ_μ, σ, and ϕ to be independent variational fields, we obtain

$$\vec\pi_\mu(x)=\partial_\mu\vec\pi(x)\,, \tag{4.4}$$

$$\sigma_\mu(x)=\partial_\mu\sigma(x)\,, \tag{4.5}$$

$$\vec\pi^2(x)+\sigma^2(x)=C^2\,. \tag{4.6}$$

Therefore, the new Lagrangian produces the same equations of motion as the old one with the constraint (4.2) gives. Now it is easy to rewrite the Lagrangian (4.3) in our form (1.5). Introducing the 21-component vector $\psi_j(x)$ $(j=1,\ldots,21)$ by

$$\psi_j(x)=(\vec\pi_\mu(x),\vec\pi(x),\sigma_\mu(x),\sigma(x),\phi(x))\,, \tag{4.7}$$

we obtain

$$\hat L=\tfrac{1}{2}\bar\psi(x)\beta^\mu\partial_\mu\psi(x)-\tfrac{1}{2}\bar\psi(x)M\psi(x)+C_j\psi_j(x)+\frac{1}{3!}\Gamma_{ijk}\psi_i(x)\psi_j(x)\psi_k(x)\,. \tag{4.8}$$

Note that we now have a linear term in $\psi_j(x)$. The chiral transformation is linear in this model. Thus we can write it as

$$\psi_j(x)\to\psi'_j(x)=U_{jk}\psi_k(x)\,. \tag{4.9}$$

The second example consists of the canonical realization by Weinberg.[9] The chiral-invariant Lagrangian is

$$L=\alpha D_\mu\vec\pi(x)\cdot D^\mu\vec\pi(x)\,, \tag{4.10}$$

where D_μ denotes the covariant derivative. The most general form of covariant derivative is given by

$$D_\mu\pi^a(x)=\left[\frac{1}{(\vec\pi^2+f^2)^{1/2}}\delta^{ab}+\frac{1}{\vec\pi^2+f^2}(2f'-v)\pi^a\pi^b\right]\partial_\mu\pi^b(x)\,, \tag{4.11}$$

where v satisfies the equation

$$v=-\frac{1}{f+(\vec\pi^2+f^2)^{1/2}}\,. \tag{4.12}$$

If we choose f equal to

$$f=\frac{1}{2\lambda}[1-\lambda^2\vec\pi^2(x)]\,, \tag{4.13}$$

for some constant λ, the Lagrangian (4.9) becomes

$$L=\frac{1}{[1+\lambda^2\vec\pi^2(x)]^2}\partial_\mu\vec\pi(x)\partial^\mu\vec\pi(x)\,, \tag{4.14}$$

where we have chosen the scalar factor α as

$$\alpha=1/4\lambda^2\,. \tag{4.15}$$

In order to rewrite the Lagrangian (4.14) in the form (1.5), we introduce the auxiliary field $\psi^a_\mu(x)$ by

$$\psi^a_\mu(x)=\frac{1}{[1+\lambda^2\vec\pi^2(x)]}\partial_\mu\pi^a(x)\,. \tag{4.16}$$

Then the Lagrangian (4.14) is equivalent to

$$L=\tfrac{1}{2}\bar{\phi}^a(x)\beta^\mu\partial_\mu\phi^a(x)$$
$$-\tfrac{1}{2}\bar{\phi}^a(x)M\phi^a(x)[1+\lambda^2\bar{\phi}^b(x)\Gamma\phi^b(x)]^2\,, \qquad (4.17)$$

where we have set

$$\phi^a(x)=(\psi_\mu^a(x),\ \pi^a(x))^T\,, \qquad (4.18a)$$

$$\bar{\phi}^a(x)=(\phi^a(x))^T C\,. \qquad (4.18b)$$

Unfortunately the chiral transformation in this model is nonlinear:

$$\delta_A\pi^a(x)=(1/\lambda)\theta_A^b[\tfrac{1}{2}\delta^{ba}(1-\lambda^2\vec{\pi}^2)+\lambda^2\pi^b\pi^a]\,, \qquad (4.19)$$

where θ_A^b is a small constant parameter of the chiral transformation. Therefore it is impossible to write the transformation as Eq. (3.12).

The explicit forms of 5×5 matrices β_μ, C, M, and Γ are given as follows:

$$\beta_0=\left(\begin{array}{c|ccc|c} & & & & 1\\ \hline & & & & \\ & & & & \\ & & & & \\ \hline -1 & & & & \end{array}\right),\quad \beta_1=\left(\begin{array}{c|ccc|c} & & & & \\ \hline & & & & -1\\ & & & & 0\\ & & & & 0\\ \hline & -1 & 0 & 0 & \end{array}\right),\quad M=\left(\begin{array}{c|ccc|c} 1 & & & & \\ \hline & 1 & 0 & 0 & \\ & 0 & 1 & 0 & \\ & 0 & 0 & 1 & \\ \hline & & & & 1\end{array}\right),$$

$$\beta_2=\left(\begin{array}{c|ccc|c} & & & & \\ \hline & & & & 0\\ & & & & -1\\ & & & & 0\\ \hline & 0 & -1 & 0 & \end{array}\right),\quad \beta_3=\left(\begin{array}{c|ccc|c} & & & & \\ \hline & & & & 0\\ & & & & 0\\ & & & & -1\\ \hline & 0 & 0 & -1 & \end{array}\right),\quad C=\left(\begin{array}{c|ccc|c} 1 & & & & \\ \hline & -1 & 0 & 0 & \\ & 0 & -1 & 0 & \\ & 0 & 0 & -1 & \\ \hline & & & & 1\end{array}\right), \qquad (4.20)$$

$$\Gamma=\left(\begin{array}{c|c|c} & & \\ \hline & & \\ & & \\ \hline & & 1\end{array}\right).$$

Note that the 5×5 matrices β_μ satisfy the same Duffin-Kemmer relations,

$$\beta_\mu\beta_\lambda\beta_\nu+\beta_\nu\beta_\lambda\beta_\mu=-\eta_{\lambda\mu}\beta_\nu-\eta_{\lambda\nu}\beta_\mu\,, \qquad (4.21a)$$

$$\beta_\mu^T=-C\beta_\mu C^{-1}=-\beta^\mu\,,\quad C^T=C=C^{-1}=-1-2(\beta_0)^2\,, \qquad (4.21b)$$

as those for the vector field case, Eqs. (2.10), (2.11), and (2.12).

It is regrettable that the new Lagrangian (4.17) is more complicated, since it contains a sixth-order polynomial term in $\phi_j^a(x)$. In this respect, only the σ-model realization (4.8) has resemblance to the Yang-Mills case.

V. GAUGE THEORIES IN THE FIVE-DIMENSIONAL SPACE

In Sec. II, we have seen that the group SO(3, 2) may have some relevance. However, as shown in Sec. II, the whole Lagrangian is *not* invariant under this group, since it contains both T_5^5 component of a tensor T_ν^μ and V_5 component of a vector V_μ. In this section we will construct a simple example of the gauge theories which are invariant under the de Sitter group SO(4, 1).[21]

We introduce a fifth coordinate x_5, formally, although we assume that somehow the gauge fields are independent of x_5, or at least their dependence is practically negligible. The departure of the present scheme from that given in Sec. II is that we do *not* regard $F_{\mu\nu}^a(x)$ as independent variables, but those given in terms of $A_\mu^a(x)$:

$$F_{\mu\nu}^a(x)=\partial_\mu A_\nu^a(x)-\partial_\nu A_\mu^a(x)+gf^{abc}A_\mu^b(x)A_\nu^c(x)\,, \qquad (5.1)$$

for $\mu,\nu=0,1,2,3,5$. Hence the Lagrange density is now simply written as

$$L=-\tfrac{1}{4}F_{\mu\nu}^a(x)F^{a\mu\nu}(x)\,, \qquad (5.2)$$

where the summation over the "Lorentz" indices runs from 0 to 5.

From our assumption on the dependence of the fifth coordinate,

$$\frac{\partial}{\partial x_5} A^a_\mu(x) \cong 0 . \qquad (5.3)$$

Defining the scalar field $\phi^a(x)$ by

$$\phi^a(x) = A^a_5(x) , \qquad (5.4)$$

with the metric $\eta_{55} = -1$, instead of Eq. (2.23), we can rewrite the Lagrangian (5.2) as follows:

$$L = L_0 + L_1 , \qquad (5.5)$$

$$L_0 = -\tfrac{1}{4} \sum_{\mu,\nu=0}^{3} F^a_{\mu\nu}(x) F^{a\mu\nu}(x) , \qquad (5.6)$$

$$L_1 = \tfrac{1}{2} \sum_{\mu=0}^{3} (D_\mu \phi^a(x))(D^\mu \phi^a(x)) , \qquad (5.7)$$

where we have set

$$D_\mu \phi^a(x) \equiv F^a_{\mu 5} = \partial_\mu \phi^a(x) + g f^{abc} A^b_\mu(x) \phi^c(x) . \qquad (5.8)$$

The resultant Lagrangian (5.5) is actually equivalent to the usual four-dimensional theory where the scalar field $\phi^a(x)$ is coupled to the gauge field $A^a_\mu(x)$ via the minimal way. Note that the scalar field $\phi^a(x)$ belongs to the adjoint representation of the group.

It is interesting to note that in the grand unified gauge theories[22] at least one of the Higgs scalars which break the unified symmetry to $SU(2) \otimes U(1) \otimes SU_{color}(3)$ transforms like the adjoint representation. In particular, in the SU(5) case, you need only one Higgs scalar, which belongs to the adjoint representation, for that job. Thus the present theory may provide a clue to the grand unification scheme. However, we will not pursue this point further in this paper.

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Note added. After completion of this paper we were notified of the following: First, even supergravity has the interaction which can be written as a single cubic term by a suitable redefinition of the fields. We would like to thank Professor S. Deser for this comment. Second, there are series of papers on the interactions among Duffin-Kemmer-Petiau fields [see B. G. Kenny, D. C. Peaslee, and Michael Martin Nieto, Phys. Rev. D 13, 757 (1976) and references therein]. The general Bhabha SO(5) formalism for a free arbitrary-spin field has been dealt by R. A. Krajcik and Michael Martin Nieto, Phys. Rev. D 15, 445 (1977) and references therein. The supersymmetry of arbitrary spin has been developed by Jarmo Hietarinta, Phys. Rev. D 13, 838 (1976). We are grateful to Professor Michael Martin Nieto for this information.

APPENDIX A

In the text we neglect the dimensional consideration of the fields, since we are only interested in the transformation properties of them. For example, in the Yang-Mills case the vector $\phi'^a(x)$ which has the correct dimension is given by

$$\phi'^a(x) = \left(\frac{1}{m_0} F^a_{\mu\nu}(x),\ A^a_\mu(x) \right),$$

where m_0 has the dimension of mass. This parameter m_0 is arbitrary and need not be identified with the mass of the gauge fields. The choice $m_0 = 1$ corresponds to the one used in the text.

The Lagrangian still takes the same form as Eq. (2.26), except for the following changes in the definition of the matrices β_μ and M, and the coupling constant g:

$$\beta_\mu \to \beta'_\mu = m_0 \beta_\mu ,$$

$$M \to M' = \left[\begin{array}{ccc|ccc|ccc|c} m_0^2 & & & & & & & & & \\ & m_0^2 & & & & & & & & \\ & & m_0^2 & & & & & & & \\ \hline & & & m_0^2 & & & & & & \\ & & & & m_0^2 & & & & & \\ & & & & & m_0^2 & & & & \\ \hline & & & & & & m^2 & & & \\ & & & & & & & m^2 & & \\ & & & & & & & & m^2 & \\ \hline & & & & & & & & & m^2 \end{array}\right] ,$$

$$g \to g' = m_0 g .$$

Note that the parameter m_0 is spurious in the sense that it does not change the theory at all.

However, as we will do later, the introduction of the additional cubic interactions, Eq. (2.26), makes m_0 have the physical meaning, since the added terms in the correct mass dimension are the following:

$$L' = -\frac{1}{m_0^2}\frac{g}{3!} f^{abc} F^a_{\mu\nu} F^b_{\alpha\beta} F^c_{\gamma\lambda} \eta^{\nu\alpha}\eta^{\beta\gamma}\eta^{\mu\lambda} .$$

Thus the solutions with these additional interactions have the inherent size $1/m_0$. At any rate, the form of the Lagrangian and its symmetries in the Duffin-Kemmer formulation remain the same.

APPENDIX B

We consider the following realization of the octonion algebra. Although this realization may not be entirely new,[23] we present it in view of its relevance to the discussion in Sec. II. Hereafter all greek indices refer to the Euclidean space-time with $x_4 = ix_0$.

Let $f_{\mu\nu}$ be a self-conjugate antisymmetric tensor,[17] i.e.,

$$^*f_{\mu\nu} \equiv \tfrac{1}{2}\epsilon_{\mu\nu\alpha\beta} f_{\alpha\beta} = f_{\mu\nu} . \tag{B1}$$

Let a_μ be a vector. Furthermore, we introduce the unit element e_0. Then the set consisting of $f_{\mu\nu}$, a_μ, and e_0 provide the bases for an eight-dimensional space. Now we impose the following multiplication table

$$a_\mu a_\nu = f_{\mu\nu} - \delta_{\mu\nu} e_0 , \tag{B2}$$

$$f_{\mu\nu} a_\lambda = -a_\lambda f_{\mu\nu} = \delta_{\mu\lambda} a_\nu - \delta_{\nu\lambda} a_\mu + \epsilon_{\mu\nu\lambda\tau} a_\tau , \tag{B3}$$

$$f_{\mu\nu} f_{\alpha\beta} = \delta_{\nu\alpha} f_{\mu\beta} - \delta_{\mu\alpha} f_{\nu\beta} - \delta_{\nu\beta} f_{\mu\alpha} + \delta_{\mu\beta} f_{\nu\alpha} - (\delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\mu\beta}\delta_{\nu\alpha} + \epsilon_{\mu\nu\alpha\beta}) e_0 . \tag{B4}$$

The consistency among Eq. (B1)–Eq. (B4) can be checked easily.

If we set

$$e_1 = f_{14} = f_{23}, \quad e_2 = f_{24} = f_{31}, \quad a_3 = f_{34} = f_{12} , \quad e_4 = a_1, \quad e_5 = a_2, \quad e_6 = a_3, \quad e_7 = a_4 , \tag{B5}$$

we can rewrite Eqs. (B1)–Eq. (B4) as follows:

$$e_j e_k = -\delta_{jk} e_0 + \sum_{m=1}^{7} \Gamma_{jkm} e_m \quad (j, k \neq 0) , \tag{B6}$$

where the totally antisymmetric coefficients Γ_{jkm} assume the values $+1$, 0, -1, only, and behave like those in Fig. 5. Therefore, the eight objects e_j $(j = 0, 1, \ldots, 7)$ now define the desired octonion algebra. The fact that the cubic terms $\Gamma_{jkm}\phi_j^{(1)}\phi_k^{(2)}\phi_m^{(3)}$ for the three seven-dimensional objects $\phi_j^{(a)}$ $(a = 1, 2, 3)$ are invariant under the group G_2 requires some further algebraic discussion. This is left to the reader.

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[23]Professor Gürsey informed us of the following reason why we require both a self-conjugate tensor $f_{\mu\nu}$ and a vector a_μ: When the group G_2 is reduced into its subgroup $SU(2) \otimes SU(2)$, the seven-dimensional representation of G_2 splits into a direct sum of the four-dimensional representation $D^{(1/2,1/2)}$ and the self-contragradient three-dimensional one $D^{(1,0)}$ (see Günaydin and Gürsey, Ref. 11). He also informed us that a similar realization of the octonion algebra has been considered by Günaydin in his Ph.D. thesis. We would like to thank Professor Gürsey for these remarks.