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Discretization problems of functional integrals in phase space

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We study functional integrals in phase space and show that the definition without limiting procedure by a formal series expansion has the same ambiguities as the definition through discretization, these last ones being related to ordering problems. We exhibit the way to obtain the prescription that makes the series expansion unambiguous and study the mechanism that makes the expansion independent of the chosen discretization.

I. INTRODUCTION

In previous works[1-3] we have studied discretization problems associated with functional integrals and we have shown in particular that perturbation expansions were independent of the discretization. We study here these problems in a systematic way, and we make precise the notion of discretization we have introduced before. This notion turns out to be basic to avoiding ambiguities in the definition of functional integrals as limits of multidimensional integrals. We also discuss the intimate connection between the concept of discretization and ordering of noncommuting operators. Among the previous works on this problem we must mention several interesting and clarifying papers by Dowker[4] and the work of Leschke and Schmutz.[5]

The definition of functional integrals without limiting procedure[6,7] is considered, and we show that this method has exactly the same ambiguities as the definition through discretization. In fact we prove that both types of ambiguities are in one-to-one correspondence and that the concept of discretization allows a complete treatment of these difficulties. We treat the case of one variable q, i.e., a phase space (p,q), since in relation to discretization problems there is no essential change for more variables and the generalization is straightforward.

In Sec. II we state the problem in general terms and then we treat it with a simple class of discretizations which allows us to introduce in a simple and self-contained way all the essential facts. The complete systematic treatment is given in Sec. III, relying heavily on the connection between operators and phase-space functions as exposed by Agarwal and Wolf.[8] The concept of discretization γ is defined here carefully and an equivalence relation between discretizations is introduced. It becomes clear from the content of this section that it is in fact the equivalence classes of discretizations that are relevant. In Sec. IV we study the dependence of a given functional integral on its discretization and we comment on recent work by Mizrahi.[9] Finally in Sec. V we give our conclusions.

We want to remark that we have not treated here the interesting problem of doing the best WKB approximation with corrections, a situation in which the notion of discretization also plays a role. For this to be done one needs first to determine the Lagrangian L_p (or the Hamiltonian H_p) whose Euler-Lagrange equations determine the most probable differentiable path, in order to know the best quadratic part one should split. This problem we have solved in Ref. 10. Then one can write a functional integral representation for the propagator with this Lagrangian L_p (or the corresponding H_p in phase space). This determines a discretization γ_p which we have calculated and called $\gamma_3(\frac{1}{2})$ in Ref. 2, and which one needs to know in order to compute corrections higher than the Gaussian ones. These developments will be published elsewhere.[17,18]

II. THE NOTION OF DISCRETIZATION: AN ILLUSTRATIVE EXAMPLE

A. Preliminaries

We consider the equation

$$\dot{P}(q,t)=\left[\frac{\partial}{\partial q}\left(A(q)+\frac{1}{2}\frac{\partial}{\partial q}D(q)\right)-V(q)\right]P(q,t)\,, \qquad (2.1)$$

which is the most general one of the form $\dot{P}=L(q,\partial/\partial q)P$ when L does not contain higher derivatives than the second. The Schrödinger equation is of the form (2.1) and when $V(q)=0$, Eq. (2.1) is the Fokker-Planck equation corresponding to the Markovian process $q(t)$ determined by the Langevin equation [$f(t)$ is a Gaussian white noise][1]:

$$\dot{q}(t)+A(q(t))=-\frac{1}{4}\frac{\partial D(q(t))}{\partial q(t)}+[D(q(t))]^{1/2}f(t). \quad (2.2)$$

We are interested in the fundamental solution or propagator $P(q,t;Q_0,t_0)$ of (2.1) such that

$$P(q,t_0;Q_0,t_0)=\delta(q-Q_0).$$

Let $\hat{q}$ and $\hat{p}$ be the usual quantum-mechanical operators, $[\hat{q},\hat{p}]=i$, and define the operator $\hat{H}$ by

$$\hat{H}(\hat{p},\hat{q})=-\tfrac{1}{2}i\hat{p}^2D(\hat{q})-\hat{p}A(\hat{q})-iV(\hat{q}). \quad (2.3)$$

Then (using the standard notation[1])

$$P(Q,t;Q_0,t_0)=\langle Q\,|\,U(t,t_0)\,|\,Q_0\rangle, \quad (2.4)$$

with

$$i\frac{\partial U(t,t')}{\partial t}=\hat{H}U(t,t'), \quad U(t',t')=1. \quad (2.5)$$

Setting $U(t)\equiv U(t,0)$ we introduce the operators

$$\hat{p}(t)=U^{-1}(t)\hat{p}U(t), \quad \hat{q}(t)=U^{-1}(t)\hat{q}U(t)$$

and the vectors

$$|q,t\rangle^R=U^{-1}(t)|q\rangle, \quad {}^L\langle q,t|=\langle q|U(t).$$

One has

$$P(Q,t;Q_0,t_0)={}^L\langle Q,t\,|\,Q_0,t_0\rangle^R.$$

We shall also consider the quantities

$$G^{(m,n)}(t_1',\ldots,t_m';t_1,\ldots,t_n)\equiv{}^L\langle Q,t\,|\,T\hat{p}(t_1')\cdots\hat{p}(t_m')\hat{q}(t_1)\cdots\hat{q}(t_n)\,|\,Q_0,t_0\rangle^R, \quad t\geq t_i',t_j\geq t_0, \quad (2.6)$$

where T is the usual chronological product.

In order to obtain a functional integral representation for $P(Q,t;Q_0,t_0)$ we write (2.4) as (in the following always $t_j=t_0+j\epsilon$, $t_{n+1}=t$, $q_0=Q_0$, and $q_{n+1}=Q$)

$$\langle Q\,|\,U(t,t_0)\,|\,Q_0\rangle=\int\prod_{i=1}^{n}dq_i\prod_{j=1}^{n+1}\langle q_j\,|\,U(t_j,t_{j-1})\,|\,q_{j-1}\rangle \quad (2.7)$$

(completeness relation or Markov property). Using (2.5) one has

$$\langle q_j\,|\,U(t_j,t_{j-1})\,|\,q_{j-1}\rangle=\langle q_j\,|\,1-i\epsilon\hat{H}(\hat{p},\hat{q})\,|\,q_{j-1}\rangle, \quad (2.8)$$

where we keep only terms up to $O(\epsilon)$ now and in what follows. It is clear that this is sufficient since we are finally interested in the limit $n\to\infty$, $\epsilon\to 0$. Nevertheless, an explicit proof can be given computing the other terms and showing that they are all zero in the limit. This we have done in Ref. 2.

B. A simple class of discretizations

Definition by limiting procedure

Using $[\hat{q},\hat{p}]=i$, the operator $\hat{H}$ can be written as (α is an arbitrary number and the primes denote derivatives with respect to q)

$$\hat{H}(\hat{p},\hat{q})=-\tfrac{1}{2}i[(1-\alpha)\hat{p}^2D(\hat{q})+\alpha D(\hat{q})\hat{p}^2]-\{(1-\alpha)\hat{p}[A(\hat{q})+\alpha D'(\hat{q})]+\alpha[A(\hat{q})+\alpha D'(\hat{q})]\hat{p}\}$$
$$-i[V(\hat{q})-\alpha A'(\hat{q})-\alpha(\alpha-\tfrac{1}{2})D''(\hat{q})]. \quad (2.9)$$

Then (2.8) can be written as

$$\langle q_j\,|\,U(t_j,t_{j-1})\,|\,q_{j-1}\rangle=\int\frac{dp}{2\pi}\exp[ip(q_j-q_{j-1})][1-i\epsilon h^{\alpha}(p,q_j,q_{j-1})] \quad (2.10)$$

with

$$h^{\alpha}(p,q_j,q_{j-1})=-\tfrac{1}{2}ip^2[(1-\alpha)D(q_{j-1})+\alpha D(q_j)]-p\{(1-\alpha)[A(q_{j-1})+\alpha D'(q_j)]+\alpha[A(q_j)+\alpha D(q_j)]\}$$
$$-i[V(q_{j-1})-\alpha A'(q_{j-1})-\alpha(\alpha-\tfrac{1}{2})D''(q_{j-1})]. \quad (2.11)$$

Replacing (2.10) [up to $O(\epsilon)$] in (2.7) one obtains ($\Delta_j=q_j-q_{j-1}$)

$$\langle Q\,|\,U(t,t_0)\,|\,Q_0\rangle=\int\left(\prod_{i=1}^{n}dq_i\prod_{j=1}^{n+1}\frac{dp_j}{2\pi}\right)\exp\left\{i\epsilon\sum_{j=1}^{n+1}\left[p_j\frac{\Delta_j}{\epsilon}-h^{\alpha}(p,q_j,q_{j-1})\right]\right\}. \quad (2.12)$$

Setting

$$H^{\alpha}(p,q)=h^{\alpha}(p,q',q)\,|_{q'\to q}$$
$$=-\tfrac{1}{2}ip^2D(q)-p[A(q)+\alpha D'(q)]-i[V(q)-\alpha A'(q)-\alpha(\alpha-\tfrac{1}{2})D''(q)], \quad (2.13)$$

one can, in the limit $n\to\infty$ ($\epsilon\to 0$, $q_j\to q_{j-1}$) by the usual formal replacements $p_j\to p(\tau)$, $q_j\to q(\tau)$, $\Delta_j/\epsilon\to\dot{q}(\tau)$, write (2.12) as a phase-space path integral

$$\langle Q\,|\,U(t,t_0)\,|\,Q_0\rangle=\int_{\gamma(\alpha)} DqDp\exp\left\{i\int_{t_0}^{t}d\tau[p(\tau)\dot q(\tau)-H^{\alpha}(p(\tau),q(\tau))]\right\}\delta(q(t_0)-Q_0)\delta(q(t)-Q)\,. \tag{2.14}$$

The subscript $\gamma(\alpha)$ in (2.14) stands for "discretization $\gamma(\alpha)$" and indicates that (2.14) is defined by the limit when $n\to\infty$ of the multidimensional integral (2.12), i.e., $\gamma(\alpha)$ is related to the knowledge of $h^{\alpha}(p,q_j,q_{j-1})$. It says that $H^{\alpha}(p,q)$ has to be discretized as

$$(1-\alpha)H^{\alpha}(p,q_{j-1})+\alpha H^{\alpha}(p,q_j)\,.$$

Since $\langle Q\,|\,U(t,t_0)\,|\,Q_0\rangle$ is obviously independent of α, the correct interpretation of the explicit α dependence of the integrand

$$\exp\left\{i\int d\tau[p\dot q-H^{\alpha}(p,q)]\right\}$$

is clear: It is just there to cancel the α dependence of the definition of the functional integral contained in the discretization.[1,2]

Definition by formal series expansion

Let us split from H^{α} a quadratic part writing, for instance, $D(q)=c+D_1(q)$, $A(q)=\mu q+A_1(q)$, $V(q)=\frac{1}{2}\omega^2q^2+V_1(q)$. Then $H^{\alpha}=H_0^{\alpha}+H_1^{\alpha}$ with

$$H_0^{\alpha}(p,q)=-\tfrac{1}{2}icp^2-\mu pq-\tfrac{1}{2}i\omega^2q^2+i\alpha\mu\,, \tag{2.15a}$$

$$H_1^{\alpha}(p,q)=-\tfrac{1}{2}ip^2D_1(q)-p[A_1(q)+\alpha D_1'(q)]-i[V_1(q)-\alpha A_1'(q)-\alpha(\alpha-\tfrac{1}{2})D_1''(q)]\,. \tag{2.15b}$$

We note that we could split also a more general quadratic part of the form $c_1(\tau)p^2+c_2(\tau)pq+c_3(\tau)q^2$ with $c_i(\tau)$ given functions, but this would not change our final conclusion.

The "free" generating functional is defined by

$$Z_0[j,j^*]=\int_{\gamma(\alpha)} DqDp\exp\left\{i\int_{t_0}^{t}d\tau[p\dot q+i\tfrac{1}{2}cp^2+\mu pq+\tfrac{1}{2}i\omega^2q^2-i\alpha\mu+j(\tau)q(\tau)+j^*(\tau)p(\tau)]\right\}\delta(q(t_0)-Q_0)\delta(q(t)-Q)\,. \tag{2.16}$$

It is a Gaussian integral and can be computed exactly (it corresponds to the Fourier transform of the Gaussian measure in the language of Refs. 6 and 7). Then (2.14) can be formally written as

$$\langle Q\,|\,U(t,t_0)\,|\,Q_0\rangle=\exp\left[-i\int_{t_0}^{t}d\tau H_1^{\alpha}\left(\frac{1}{i}\frac{\delta}{\delta j^*(\tau)},\frac{1}{i}\frac{\delta}{\delta j(\tau)}\right)\right]Z_0[j,j^*]\Big|_{j(\tau)=j^*(\tau)=0}\,, \tag{2.17}$$

and developing $\exp[-i\int H_1^{\alpha}]$, one obtains an explicit expression as an infinite formal series for the functional integral in (2.14). It is this formal series that is taken as the definition of the functional integral without limiting procedure in Refs. 6 and 9. This definition has in fact the same problems as the definition through discretization since, as we shall see now, each term in the series is ambiguous and the way to eliminate the ambiguity is precisely by using the discretization $\gamma(\alpha)$.

Elimination of ambiguities

The value of $Z_0[j,j^*]$ is computed in Appendix A. One finds

$$Z_0[j,j^*]=K\exp\left\{-\int_{t_0}^{t}dt'\int_{t_0}^{t}dt''[\tfrac{1}{2}j^*(t')\Delta^{11}(t',t'')j^*(t'')+j^*(t')\Delta^{12}(t',t'')j(t'')+\tfrac{1}{2}j(t')\Delta^{22}(t',t'')j(t'')]+\int_{t_0}^{t}dt'[S_1(t')j^*(t')+S_2(t')j(t')]\right\}, \tag{2.18}$$

where K does not depend on j or j^* and $\Delta^{ij}(t',t'')$ and $S_i(t')$ are known functions. The point to be remarked here is that $\Delta^{kk}(t,t-\epsilon)=\Delta^{kk}(t,t+\epsilon)$, $\epsilon\to0^+$, $k=1,2$, while $\Delta^{12}(t,t-\epsilon)=B(t)$, $\Delta^{12}(t,t+\epsilon)=B(t)+i$, i.e., $\Delta^{12}(t',t'')$ has a jump of value i at $t'=t''$. This was the essential point in our proof of the discretization independence of perturbation theory in Ref. 1 and it has also been remarked recently in Ref. 9. The reason (2.17) is not defined is now clear: In $H_1^{\alpha}(p(\tau),q(\tau))$ one has products $p(\tau)q(\tau)$ at the same time τ. These give, when

$$p(\tau)\to\frac{1}{i}\frac{\delta}{\delta j^*(\tau)}\,,\quad q(\tau)\to\frac{1}{i}\frac{\delta}{\delta j(\tau)}\,,$$

a term

$$-\frac{\delta}{\delta j^*(\tau)}\frac{\delta}{\delta j(\tau)}Z_0[j,j^*]$$

which contains [see (2.18)] a contribution $\Delta^{12}(\tau,\tau)$ (i.e., a tadpole) which is not defined. All the other terms in the expansion (2.17) are well defined.

The way to define $\Delta^{12}(t',t')$ is of course using the discretization $\gamma(\alpha)$ which tells us [see (2.11) to (2.15)] that the right-hand side of (2.17) should be interpreted as [setting $A_1^\alpha = A_1 + \alpha D_1'$; $V_1^\alpha = V_1 - \alpha A_1' - \alpha(\alpha - \frac{1}{2})D_1''$]

$$\exp\Big(\int_{t_0}^{t} d\tau\{-\tfrac{1}{2}p(\tau)^2[(1-\alpha)D_1(q(\tau-\epsilon))+\alpha D_1(q(\tau+\epsilon))] + ip(\tau)[(1-\alpha)A_1^\alpha(q(\tau-\epsilon))+\alpha A_1^\alpha(q(\tau+\epsilon))] - V_1^\alpha(q(\tau))\}\Big)\Big|_{\substack{p=\delta/i\delta j^* \\ p=\delta/i\delta j}} Z_0[j,j^*]\Big|_{\substack{j=j^*=0 \\ \epsilon\to 0^+}} . \qquad (2.19)$$

This formula gives an unambiguous value to all the terms $\Delta^{12}(\tau,\tau)$ arising in the expansion of (2.17). The simple prescription one can read from (2.19) is as follows: (a) When one has the product of two terms $\Delta^{12}(\tau,\tau)$, coming from $p(\tau)^2D_1(q(\tau))$ in H_1^α, the value is $(1-\alpha)B(\tau)^2+\alpha[B(\tau)+i]^2$, and (b) when one term $\Delta^{12}(\tau,\tau)$ comes either from $p(\tau)^2D_1(q(\tau))$ or $p(\tau)A_1^\alpha(q(\tau))$, the value is $(1-\alpha)B(\tau)+\alpha[B(\tau)+i]$. We see that these terms take α-dependent values, but one should recall that in $A_1^\alpha(q(\tau))$ and $V_1^\alpha(q(\tau))$ there is an explicit α dependence. These two types of α-dependent contributions must combine in such a way that the α dependence cancels completely in order to obtain the final result independent of α, $\langle Q\,|\,U(t,t_0)\,|\,Q_0\rangle$.

Cancellation mechanism

In order to see how this cancellation mechanism works we recall first briefly the standard graphical representation by Feynman graphs of the terms of the series (2.17), and then we treat a simple example. We introduce the graphs in Fig. 1 to denote the functions $\Delta^{ij}(t',t'')$, $S_i(t')$. We see from this figure and (2.18) that an undulated line is associated with $p(\tau)$ while a straight line is associated with $q(\tau)$. Then each term in the expansion of (2.17) is represented by graphs without external lines (except for the lines finishing in a cross) constructed from the graphs of Fig. 1 joined in interaction vertices determined by the function $H_1^\alpha(p,q)$, i.e., by the functions $D_1(q)$, $A_1(q)$, $V_1(q)$, and the value of α.

We consider now the example $D_1(q)=\frac{1}{2}\lambda q^2$, $A_1(q)=0$, $V_1(q)=0$, for which

$$H_1^\alpha(p,q) = -i\tfrac{1}{4}\lambda p(\tau)^2 q(\tau)^2 - \alpha\lambda p(\tau)q(\tau) + i\alpha(\alpha-\tfrac{1}{2})\lambda\,. \qquad (2.20)$$

The interaction vertices (that we denote by a square) are shown in Fig. 2. The contribution of first order in λ in the expansion of (2.17) can be read from

$$\langle Q\,|\,U(t,t_0)\,|\,Q_0\rangle = \Big(1 + i\int_{t_0}^{t} d\tau[i\tfrac{1}{4}\lambda p(\tau)^2q(\tau)^2 + \alpha\lambda p(\tau)q(\tau) - i\alpha(\alpha-\tfrac{1}{2})\lambda] + \cdots\Big)\Big|_{\substack{p(\tau)=\delta/i\delta j^*(\tau) \\ q(\tau)=\delta/i\delta j(\tau)}} Z_0[j,j^*]\Big|_{j=j^*=0} \qquad (2.21)$$

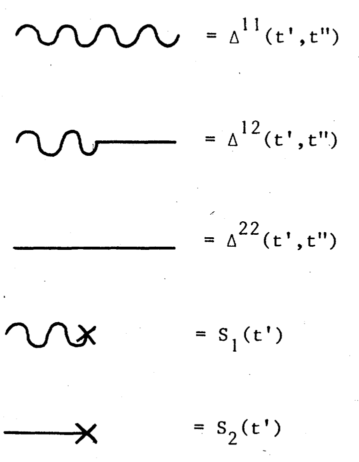

FIG. 1. Graphs representing the functions $\Delta^{ij}(t,t')$ and $S^i(t)$, $i=1,2$, appearing in $Z_0[j,j^*]$ given by (2.18).

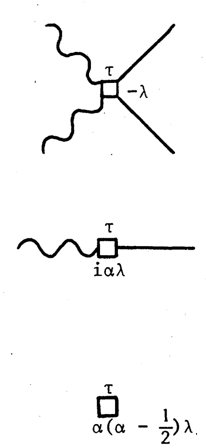

FIG. 2. Interaction vertices of the $D_1(q)=\frac{1}{2}\lambda q^2$, $A_1(q) = V_1(q)=0$ theory.

and is represented graphically in Fig. 3.

In Figs. 3(a), 3(b), and 3(c), one has the graphs coming from $-\frac{1}{4}\lambda p(\tau)^2 q(\tau)^2$, in Fig. 3(d) the graphs from $i\alpha\lambda pq$, and in Fig. 3(e) the graph from $\alpha(\alpha-\frac{1}{2})$. One should note that in first order in λ all closed loops are tadpoles as shown in Fig. 3; moreover, all graphs there are well defined except the ones containing the tadpole $\Delta^{12}(\tau,\tau)$, which are the second graph in 3(a), the last two in 3(b), and the first one in 3(d). The sum of these last graphs with the graphs depending explicitly on α [the second graph in 3(d) and 3(e)] must give an α-independent result, and in fact, an inspection of Fig. 3 shows that the α dependence should cancel independently in the graphs without crosses [second one of 3(a), first one of 3(d) and 3(e)] and in the ones with crosses [last two of 3(b) and second one of 3(d)]. By the prescriptions given in the preceding paragraph the former sum up to

$$-\tfrac{1}{2}\lambda\int_{t_0}^{t}d\tau\{(1-\alpha)B(\tau)^2+\alpha[B(\tau)+i]^2\}+i\alpha\lambda\int_{t_0}^{t}d\tau\{(1-\alpha)B(\tau)+\alpha[B(\tau)+i]\}+\alpha(\alpha-\tfrac{1}{2})\lambda\int_{t_0}^{t}d\tau\,, \tag{2.22}$$

and the latter to

$$-\lambda\int_{t_0}^{t}d\tau S_1(\tau)S_2(\tau)\{(1-\alpha)B(\tau)+\alpha[B(\tau)+i]\}+i\alpha\lambda\int_{t_0}^{t}d\tau S_1(\tau)S_2(\tau)\,. \tag{2.23}$$

In both cases one immediately checks the cancellation.

Conclusion

What we have done up to now shows then that the definition of functional integrals without limiting procedure has the same ambiguities as the definition through discretization, these last ones being related to ordering problems. We have also exhibited explicitly the way to obtain the prescription that makes all terms in the formal series expansion unambiguous, and we have seen that the mechanism that is responsible for the discretization independence of the series is a cancellation between discretization dependent vertices and contributions from tadpoles. This was our conclusion in Ref. 1 that has been corroborated since then, using operator methods involving a generalized Wick expansion theorem and a generalization of the chronological T product to equal times.[11]

The functions $G^{(m,n)}(t_1',\ldots,t_n)$

We close this section with some considerations with respect to the functions $G^{(m,n)}(t_1',\ldots,t_n)$ defined by (2.6). We define the complete generating functional $Z[j,j^*]$ by

$$Z[j,j^*]=\int_{\gamma(\alpha)}DqDp\exp\left\{i\int_{t_0}^{t}d\tau[p\dot q-H^\alpha(p(\tau),q(\tau))+j(\tau)q(\tau)+j^*(\tau)p(\tau)]\right\}\delta(q(t_0)-Q_0)\delta(q(t)-Q)\,, \tag{2.24}$$

and also $(t\geq t_i',t_j\geq t_0)$

$$\begin{aligned}\langle p(t_1')\cdots p(t_m')q(t_1)\cdots q(t_n)\rangle &\equiv\frac{1}{i^{m+n}}\frac{\delta^{m+n}Z[j,j^*]}{\delta j^*(t_1')\cdots\delta j^*(t_m')\delta j(t_1)\cdots\delta j(t_n)}\bigg|_{j=j^*=0}\\ &=\int_{\gamma(\alpha)}DqDp\,p(t_1')\cdots q(t_n)\exp\left\{i\int_{t_0}^{t}d\tau[p\dot q-H^\alpha(p,q)]\right\}\delta(q(t_0)-Q_0)\delta(q(t)-Q)\,.\end{aligned} \tag{2.25}$$

A simple calculation [in fact with minor changes the same one as done from (2.7) to (2.14)] shows that when $t_i'\neq t_j$ for all i's and j's one has

$$G^{(m,n)}(t_1',\ldots,t_m';t_1,\ldots,t_n)=\langle p(t_1')\cdots p(t_m')q(t_1)\cdots q(t_n)\rangle\,. \tag{2.26}$$

But now one can notice that the right-hand side of (2.26) is defined for all t_i''s and t_j's and in fact when $t_i'=t_j$, it takes an α-dependent value due to the discretization $\gamma(\alpha)$. This value is immediately determined using the definition of $\gamma(\alpha)$ as (Ref. 1)

$$\langle p(t_1')\cdots p(t_n')q(t_1)\cdots q(t_n)\rangle|_{t_1'=t_1}=\lim_{\epsilon\to 0}[(1-\alpha)\langle p(t_1')\cdots q(t_n)\rangle|_{t_1=t_1'-\epsilon}+\alpha\langle p(t_1')\cdots q(t_n)\rangle|_{t_1=t_1'+\epsilon}]\,. \tag{2.27}$$

We see then that quantities computed in the $\gamma(\alpha)$ discretization can take α-dependent values, and this happens when the corresponding object calculated in the operator formalism is not defined. This is the case in (2.26) since

$$G^{(n,m)}(t_1',\ldots,t_m',t_1,\ldots,t_n)$$

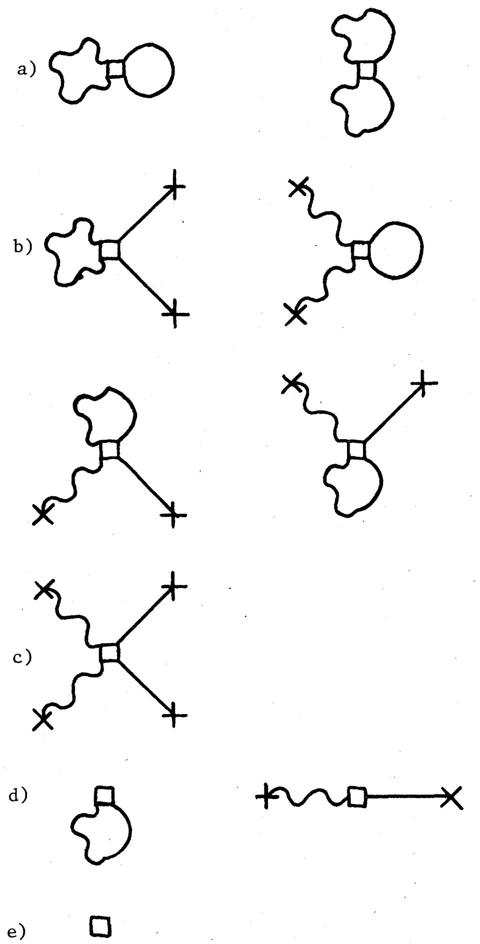

FIG. 3. Graphs representing the contribution of first order in λ in the expansion (2.17) of $\langle Q|U(t,t_0)|Q_0\rangle$ in the $D_1(q)=\frac{1}{2}\lambda q^2$, $A_1(q)=V_1(q)=0$ theory.

is not defined for $t_1'=t_1$ due to the chronological T product.

The functional $Z[j,j^*]$ can be expressed as

$$Z[j,j^*]=\exp\left[-\int_{t_0}^{t}d\tau H_1^{\alpha}\left(\frac{1}{i}\frac{\delta}{\delta j^*(\tau)},\frac{1}{i}\frac{\delta}{\delta j(\tau)}\right)\right]Z_0[j,j^*]\,, \tag{2.28}$$

and of course is to be calculated interpreting the exponential as in (2.19), i.e., as in the calculation of the propagator $P(Q,t;Q_0,t_0)$ to which $Z[j,j^*]$ reduces for $j=j^*=0$. All quantities (2.25) can then be obtained from (2.28). The interpretation of the quantities we have calculated here is clear in quantum mechanics. In the case of stochastic processes one can easily check that for $V(q)=0$ in (2.1) the quantities

$$\int dQ\langle p(t_1')\cdots p(t_m')q(t_1)\cdots q(t_n)\rangle$$
$$=\langle L|T\hat{p}(t_1')\cdots\hat{p}(t_m')\hat{q}(t_1)\cdots\hat{q}(t_n)|Q_0,t_0\rangle^R \tag{2.29}$$

with $\langle L|\equiv\int dQ\langle Q|$, are just the set of response and correlation functions of the Markovian process defined by (2.1) [for fixed initial condition $q(t_0)=Q_0$], and $P(Q,t;Q_0,t_0)$ is the conditional probability density.[1]

III. GENERAL DISCRETIZATIONS. SYSTEMATIC TREATMENT

A. Correspondence between operators and phase-space functions

The systematic treatment we present here uses results concerning phase-space functions associated with operators for which we refer to Ref. 8 (we shall use their notation). We recall that given an operator $\hat{B}(\hat{p},\hat{q})$ and a function $\Omega(u,v)$ satisfying certain conditions one can associate with $\hat{B}(\hat{p},\hat{q})$ a function $B^{\Omega}(p,q)$ by

$$\hat{B}(\hat{p},\hat{q})=\int dp\,dq\,B^{\Omega}(p,q)\Delta^{(\Omega)}(q-\hat{q},p-\hat{p}) \tag{3.1}$$

with

$$\Delta^{(\Omega)}(q-\hat{q},p-\hat{p})\equiv\frac{1}{(2\pi)^2}\int du\,dv\,\Omega(u,v)\times\exp\{-i[u(q-\hat{q})+v(p-\hat{p})]\}\,. \tag{3.2}$$

We introduce

$$\tilde{\Omega}(u,v)\equiv[\Omega(-u,-v)]^{-1}$$

and

$$\bar{\Omega}(u,v)\equiv\tilde{\Omega}(u,v)\exp(\tfrac{1}{2}iuv)\,.$$

Then (3.1) can be inverted as (see Appendix B for details and proofs)

$$B^{\Omega}(p,q)=\bar{\Omega}\left(i\frac{\partial}{\partial q},i\frac{\partial}{\partial p}\right)B_{AS}\,, \tag{3.3}$$

where $B_{AS}(p,q)$ is defined by the relation

$$\langle p|\hat{B}(\hat{p},\hat{q})|q\rangle=B_{AS}(p,q)\langle p|q\rangle=B_{AS}(p,q)\frac{\exp(-ipq)}{(2\pi)^{1/2}}\,. \tag{3.4}$$

For our purposes we restrict ourselves to functions $\Omega(u,v)$ such that $\Omega(0,v)=\Omega(u,0)=1$; this implies from (3.3) and (3.4) that if $\hat{B}(\hat{p},\hat{q})=\hat{p}^n$ then $B^{\Omega}(p,q)=p^n$ and if $\hat{B}(\hat{p},\hat{q})=\hat{q}^n$ then $B^{\Omega}(p,q)=q^n$. Using now $\hat{B}(\hat{p},\hat{q})$ written as in (3.1), one can introduce in a natural way the function $b^{\Omega}(p,q',q)$, computing $\langle q'|\hat{B}(\hat{p},\hat{q})|q\rangle$ and writing its value as[12]

$$\langle q'|\hat{B}(\hat{p},\hat{q})|q\rangle=\int\frac{dp}{2\pi}e^{ip(q'-q)}b^{\Omega}(p,q',q)\,. \tag{3.5}$$

Then $b^{\Omega}(p,q',q)$ can be taken as

$$b^{\Omega}(p,q',q)=\Omega\left(-i\frac{\partial}{\partial\bar{q}},-\Delta\right)B^{\Omega}(p,\bar{q})\,, \tag{3.6}$$

where

$$\bar{q} \equiv \frac{q+q'}{2}, \quad \Delta \equiv q' - q ,$$

and as $\Omega(u,0)=1$ we see that

$$b^{\Omega}(p,q',q)\,|_{q' \to q} = B^{\Omega}(p,q) . \tag{3.7}$$

The function $\Omega(u,v)$ is related to the different ways in which one can write the fixed operator $\hat{B}(\hat{p},\hat{q})$, i.e., the different orderings of the noncommuting operators $\hat{p}$ and $\hat{q}$. In fact what we have done in the previous section corresponds, as we shall see, to using

$$\Omega(u,v) = (1-\alpha)\exp(\tfrac{1}{2}iuv) + \alpha \exp(-\tfrac{1}{2}iuv) .$$

Let us apply this now to $\hat{H}(\hat{p},\hat{q})$ given by (2.3). We consider only functions $\Omega(u,v)$ of the product uv, that is, $\Omega(u,v)=\Omega(uv)$. This covers all usual orderings and, moreover, consideration of the more general case is not very illuminating for our purpose here. Then $\bar{\Omega}$ is also a function of uv and the primes in Ω and $\bar{\Omega}$ denote derivatives with respect to uv, and one has $\Omega(0)=\bar{\Omega}(0)=1$. One obtains

$$H^{\Omega}(p,q) = -\tfrac{1}{2}ip^2 D(q) - pA^{\Omega}(q) - iV^{\Omega}(q) , \tag{3.8}$$

with

$$A^{\Omega}(q) = A(q) - i\bar{\Omega}'(0)D'(q) , \tag{3.9a}$$

$$V^{\Omega}(q) = V(q) + i\bar{\Omega}'(0)A'(q) + \tfrac{1}{2}\bar{\Omega}''(0)D''(q) , \tag{3.9b}$$

and

$$h^{\Omega}(p,q',q) = -\tfrac{1}{2}ip^2 d^{\Omega}(q',q) - pa^{\Omega}(q',q) - iv^{\Omega}(q',q) , \tag{3.10}$$

with

$$d^{\Omega}(q',q) = D(\bar{q}) + i\Omega'(0)\Delta D'(\bar{q}) - \tfrac{1}{2}\Omega''(0)\Delta^2 D''(\bar{q}) + O(\Delta^3) , \tag{3.11a}$$

$$a^{\Omega}(q',q) = A^{\Omega}(\bar{q}) + i\Omega'(0)\Delta A^{\Omega\prime}(\bar{q}) + O(\Delta^2) , \tag{3.11b}$$

$$v^{\Omega}(q',q) = V^{\Omega}(\bar{q}) + O(\Delta) . \tag{3.11c}$$

Note that we have written d^{Ω} up to $O(\Delta^3)$, a^{Ω} up to $O(\Delta^2)$, and v^{Ω} up to $O(\Delta)$; it will be clear from what follows that this is all we need.

B. Correspondence rules and discretizations

We can now compute $\langle Q | U(t,t_0) | Q_0 \rangle$ as in Sec. II but starting from $\hat{H}$ written as in (3.1). Owing to (3.5) one has to replace in (2.12) the function h^{α} by h^{Ω} given by (3.10) and one has ($q_0 = Q_0$, $q_{n+1} = Q$)

$$\langle Q | U(t,t_0) | Q_0 \rangle = \int \prod_{i=1}^{n} dq_i \prod_{j=1}^{n+1} dp_j \exp\left\{ i\epsilon \sum_{j=1}^{n+1} \left[p_j \frac{\Delta_j}{\epsilon} - h^{\Omega}(p, q_j, q_{j-1}) \right] \right\} \tag{3.12}$$

which we write formally in the limit $n \to \infty$ ($\epsilon \to 0$) as

$$\langle Q | U(t,t_0) | Q_0 \rangle = \int_{\gamma(\Omega)} Dq\,Dp \exp\left\{ i \int_{t_0}^{t} d\tau [p\dot{q} - H^{\Omega}(p(\tau), q(\tau))] \right\} \delta(q(t_0) - Q_0)\,\delta(q(t) - Q) , \tag{3.13}$$

since $h^{\Omega}(p,q',q) \to H^{\Omega}(p,q)$, $q' \to q$. The subscript $\gamma(\Omega)$ now stands for the discretization $\gamma(\Omega)$ defined by (3.12), i.e., by the function h^{Ω}. In order to see when one can consider two different discretizations equivalent, we perform in (3.12) the Gaussian integration over dp_j. One obtains

$$\langle Q | U(t,t_0) | Q_0 \rangle = \int \prod_{i=1}^{n} dq_i \prod_{j=1}^{n+1} \left[\frac{1}{[2\pi\epsilon d^{\Omega}(q_j, q_{j-1})]^{1/2}} \exp\left(-\frac{\Delta_j^2}{2\epsilon d^{\Omega}(q_j,q_{j-1})} - \Delta_j \frac{a^{\Omega}(q_j,q_{j-1})}{d^{\Omega}(q_j,q_{j-1})} - \epsilon \frac{a^{\Omega}(q_j,q_{j-1})^2}{2d^{\Omega}(q_j,q_{j-1})} - \epsilon v^{\Omega}(q_j,q_{j-1}) \right) \right] . \tag{3.14}$$

One should remember now that owing to the dominant term $(-\Delta_j^2/2\epsilon d^{\Omega})$ in the exponential, Δ_j is $O(\sqrt{\epsilon})$, and also that only terms up to $O(\epsilon)$ are needed. Equation (3.14) then tells us that $d^{\Omega}(q_j,q_{j-1})$ is only needed up to terms $O(\Delta_j^3)$, a^{Ω} up to terms $O(\Delta_j^2)$, and v^{Ω} up to terms $O(\Delta_j)$, i.e., just what we have anticipated in writing (3.11a)–(3.11c). Moreover, this indicates that we can replace the discretization $\gamma(\Omega)$ by an equivalent one $\tilde{\gamma}(\Omega)$ [equivalent in the sense that the value of the n-dimensional integral in (3.13) is unchanged when $n \to \infty$] by changing h^{Ω} to $\tilde{h}^{\Omega}$, replacing d^{Ω} by $\tilde{d}^{\Omega}$, a^{Ω} by $\tilde{a}^{\Omega}$, v^{Ω} by $\tilde{v}^{\Omega}$, such that

$$\tilde{d}^{\Omega}(q',q) - d^{\Omega}(q',q) = O(\Delta^3) , \tag{3.15a}$$

$$\tilde{a}^{\Omega}(q',q) - a^{\Omega}(q',q) = O(\Delta^2) , \tag{3.15b}$$

$$\tilde{v}^{\Omega}(q',q) - v^{\Omega}(q',q) = O(\Delta) . \tag{3.15c}$$

In view of our purpose which is to specify the cancellation mechanism between the tadpoles and the Ω-dependent vertices in H^Ω, we take the liberty of using an $\tilde{h}^\Omega$ with $\tilde{d}^\Omega$, $\tilde{a}^\Omega$, and $\tilde{v}^\Omega$ given by

$$\tilde{d}^\Omega(q',q)=(1-c_1)D(\bar{q}+c_2\Delta)+c_1D(\bar{q}+c_3\Delta)\,, \tag{3.16a}$$

$$\tilde{a}^\Omega(q',q)=A^\Omega(\bar{q}+c_4\Delta)\,, \tag{3.16b}$$

$$\tilde{v}^\Omega(q',q)=V^\Omega(\bar{q})\,. \tag{3.16c}$$

One can easily check that (3.15) is satisfied if

$$(1-c_1)c_2+c_1c_3=i\Omega'(0)\,, \tag{3.17a}$$

$$(1-c_1)c_2{}^2+c_1c_3{}^2=-\Omega''(0)\,, \tag{3.17b}$$

and

$$c_4=i\Omega'(0)\,. \tag{3.17c}$$

From the first two equations c_2 and c_3 can be solved in terms of c_1, $\Omega'(0)$, and $\Omega''(0)$. When $\Omega''(0)\leq\Omega'(0)^2$ (which is the case for all common correspondence rules) there is a solution for any c_1 strictly between 0 and 1. Then $\tilde{h}^\Omega=-\frac{1}{2}ip^2\tilde{d}^\Omega-p\tilde{a}^\Omega-i\tilde{v}^\Omega$ defines an equivalent discretization $\tilde{\gamma}(\Omega)$.

We now proceed now as before writing $H^\Omega=H_0^\Omega+H_1^\Omega$, with

$$H_0^\Omega(p,q)=-\tfrac{1}{2}icp^2-\mu pq-i\tfrac{1}{2}\omega^2q^2+\bar{\Omega}'(0)\mu\,, \tag{3.18a}$$

$$H_1^\Omega(p,q)=-\tfrac{1}{2}ip^2D_1(q)-pA_1^\Omega(q)-iV_1^\Omega(q)\,, \tag{3.18b}$$

and

$$A_1^\Omega=A_1-i\bar{\Omega}'(0)D_1',\ V_1^\Omega=V_1+i\bar{\Omega}'(0)A_1'+\tfrac{1}{2}\Omega''(0)D_1''\,.$$

Instead of (2.17) we have now

$$\langle Q\,|\,U(t,t_0)\,|\,Q_0\rangle=\exp\left[-i\int_{t_0}^t d\tau H_1^\Omega\left(\frac{1}{i}\frac{\delta}{\delta j^*(\tau)},\frac{1}{i}\frac{\delta}{\delta j(\tau)}\right)\right]Z_0[j,j^*]\Big|_{j=j^*=0} \tag{3.19}$$

with

$$Z_0^\Omega[j,j^*]=\int_{\gamma(\Omega)}DqDp\exp\left\{i\int_{t_0}^t d\tau[p\dot{q}+i\tfrac{1}{2}cp^2+\mu pq+\tfrac{1}{2}i\omega^2q^2-\bar{\Omega}'(0)\mu+jq+j^*p]\right\}\delta(q(t_0)-Q_0)\delta(q(t)-Q)\,. \tag{3.20}$$

But by construction $Z_0^\Omega[j,j^*]=Z_0[j,j^*]$ and we see now that we have in the expansion of (3.19) the same problem as with the expansion of (2.17). The way out is again to notice that the discretization $\tilde{\gamma}(\Omega)$ tells us that the right-hand side of (3.19) should be interpreted as

$$\begin{aligned}\exp\Bigg(\int_{t_0}^t d\tau\{&-\tfrac{1}{2}p(\tau)^2[(1-c_1)D_1((\tfrac{1}{2}-c_2)q(\tau-\epsilon)+(\tfrac{1}{2}+c_2)q(\tau+\epsilon))+c_1D_1((\tfrac{1}{2}-c_3)q(\tau-\epsilon)+(\tfrac{1}{2}+c_3)q(\tau+\epsilon))]\\&+ip(\tau)A_1^\Omega((\tfrac{1}{2}-c_4)q(\tau-\epsilon)+(\tfrac{1}{2}+c_4)q(\tau+\epsilon))-V_1^\Omega(q(\tau))\}\Bigg)\Bigg|_{\substack{p=\delta/i\delta j^*\\q=\delta/i\delta j}}Z_0[j,j^*]\Bigg|_{\substack{j=j^*=0\\\epsilon\to0^+}}\,.\end{aligned} \tag{3.21}$$

This formula defines again unambiguously the expansion of (3.19). The prescription for the tadpoles that one obtains from (3.21) and using (3.17a) and (3.17b) is now as follows: (a) The double tadpole coming from $p(\tau)^2D_1(q(\tau))$ takes the value

$$(1-c_1)[B(\tau)+(\tfrac{1}{2}+c_2)i]^2+c_1[B(\tau)+(\tfrac{1}{2}+c_3)i]^2=[B(\tau)+\tfrac{1}{2}i-\Omega'(0)]^2+\Omega''(0)-\Omega'(0)^2\,.$$

(b) A single tadpole coming from $p(\tau)^2D_1(q(\tau))$ takes the value

$$(1-c_1)[B(\tau)+(\tfrac{1}{2}+c_2)i]+c_1[B(\tau)+(\tfrac{1}{2}+c_3)i]\,,$$

one coming from $p(\tau)A_1^\Omega(q(\tau))$, the value $B(\tau)+(\frac{1}{2}+c_4)i$. In both cases this is equal to $B(\tau)+(\frac{1}{2}i-\Omega'(0))$.

C. Examples

The results of Sec. II correspond to the choice

$$\Omega(u,v)=(1-\alpha)\exp(\tfrac{1}{2}iuv)+\alpha\exp(-\tfrac{1}{2}iuv)\,.$$

Indeed, H^Ω coincides with H^α. Moreover, since $\Omega'(0)=(\frac{1}{2}-\alpha)i$ and $\Omega''(0)=-\frac{1}{4}$, the condition $\Omega''(0)\leq\Omega'(0)^2$ is satisfied and the prescription for the double tadpole reduces to $(1-\alpha)B(\tau)^2+\alpha[B(\tau)+i]^2$ while a single tadpole has the value $B(\tau)+\alpha i$, which are the results we have obtained before.

The choice

$$\Omega(u,v)=\exp[i(\tfrac{1}{2}-\alpha)uv]\,,$$

which for $\alpha=\frac{1}{2}$ reduces to Weyl correspondence, gives the discretization used in Ref. 5. In this case all the tadpoles take the value $B(\tau)+\alpha i$, which is a result obtained in Ref. 11 using a generalized Wick theorem. One has to take in Ref. 11 the vector $\vec{\lambda}$ as $\vec{\lambda}=(0,0,1-2\alpha)$. The first two components must be zero since the Ω functions we use

TABLE I. Values of $\Omega(u,v)$ for some common rules of correspondence together with the associated rules for the tadpoles.

Rule of correspondence	$\Omega(u,v)$	Double tadpole	Single tadpole
Weyl	1	$[B(\tau)+\frac{1}{2}i]^2$	$B(\tau)+\frac{1}{2}i$
Standard	$\exp(\frac{1}{2}iuv)$	$[B(\tau)]^2$	$B(\tau)$
Anti-standard	$\exp(-\frac{1}{2}iuv)$	$[B(\tau)+i]^2$	$B(\tau)+i$
Symmetric	$\cos(\frac{1}{2}uv)$	$[B(\tau)+i]^2-\frac{1}{4}$	$B(\tau)+\frac{1}{2}i$
Born-Jordan	$\frac{\sin(\frac{1}{2}uv)}{\frac{1}{2}uv}$	$[B(\tau)+\frac{1}{2}i]^2-\frac{1}{12}$	$B(\tau)+\frac{1}{2}i$

satisfy $\Omega(0,v)=\Omega(u,0)=1$, which is necessary for the analysis of discretization in (p,q) space. The case when the two first components of $\vec{\lambda}$ are different from zero is related to discretizations in $(a,a^\dagger)$ space with $a=(q+ip)/\sqrt{2}$, $a^\dagger=(q-ip)/\sqrt{2}$, a problem that we shall consider in a forthcoming work. For completeness we list in Table I the values of $\Omega(u,v)$ for some common rules of correspondence together with the associated rules for the tadpoles.

IV. THE INVERSE PROBLEM

A. Variation of the value of a functional integral with the discretization

The problem we treat in this section is in a certain sense the inverse of the one treated in the previous sections. We consider a functional integral

$$J(Q,t;Q_0,t_0)=\int_\gamma DqDp\exp\left\{i\int_{t_0}^t d\tau[p\dot{q}-H(p,q)]\right\}\times\delta(q(t_0)-Q_0)\delta(q(t)-Q) \quad (4.1)$$

and we want to compute its value in the form

$$\langle Q|\exp[-i(t-t_0)\hat{H}^\gamma(\hat{p},\hat{q})]|Q_0\rangle,$$

that is, we want to calculate the operator $\hat{H}^\gamma(\hat{p},\hat{q})$ given the function $H(p,q)$ and the discretization γ.

The definition of the functional integral in (4.1) is

$$J=\lim_{n\to\infty}\int\left(\prod_{i=1}^n dq_i\prod_{j=1}^{n+1}\frac{dp_j}{2\pi}\right)\times\exp\left\{i\epsilon\sum_{j=1}^{n+1}\left[p_j\frac{\Delta_j}{\epsilon}-h^\gamma(p_j,q_j,q_{j-1})\right]\right\}, \quad (4.2)$$

where the function $h^\gamma(p,q',q)\to H(p,q)$, $q'\to q$, is known since the discretization γ is given. We should remark that as we have explained before, the value of J can be computed in the form of a formal series, and this without ambiguities, because the discretization γ [i.e., the function $h^\gamma(p,q',q)$] fixes the prescription for the tadpoles. We consider the case in which $H(p,q)$ is quadratic in p, i.e., it is of the form

$$H(p,q)=-\tfrac{1}{2}ip^2D(q)-pA(q)-iV(q). \quad (4.3)$$

Consequently $h^\gamma(p,q',q)$ is of the form

$$h^\gamma(q,q',q)=-\tfrac{1}{2}ip^2d(q',q)-pa(q',q)-iv(q',q), \quad (4.4)$$

with $d(q',q)\to D(q)$, $a(q',q)\to A(q)$, $v(q',q)\to V(q)$, $q'\to q$.

If we now do the Gaussian integrations over dp_j in (2), we shall obtain formula (3.14) with $d^\Omega(q',q)$ replaced by $d(q',q)$, $a^\Omega(q',q)$ by $a(q',q)$, and $v^\Omega(q',q)$ by $v(q',q)$. We now develop these functions as

$$d_i(q)=\left.\frac{\partial^i d(q',q)}{\partial q'^i}\right|_{q'=q}, \quad i=1,2$$

$$a_1(q)=\left.\frac{\partial a(q',q)}{\partial q'}\right|_{q'=q},$$

$$d(q',q)=D(q)+\Delta d_1(q)+\tfrac{1}{2}\Delta^2 d_2(q)+O(\Delta^3), \quad (4.5a)$$

$$a(q',q)=A(q)+\Delta a_1(q)+O(\Delta^2), \quad (4.5b)$$

$$v(q',q)=V(q)+O(\Delta), \quad (4.5c)$$

where $\Delta\equiv q'-q$ and replace in the n-dimensional functional integral over dq_i, keeping only terms up to $O(\epsilon)$. [We recall that $\Delta_j^2=O(\epsilon)$.] One obtains (the limit $n\to\infty$ is of course to be understood)

$$J=\int\prod_{i=1}^{n}dq_i\prod_{j=1}^{n+1}\frac{1}{\left[2\pi\epsilon D(q_{j-1})\left(1+\Delta_j\frac{d_1(q_{j-1})}{D(q_{j-1})}+\frac{\Delta_j^2}{2}\frac{d_2(q_{j-1})}{D(q_{j-1})}\right)\right]^{1/2}}$$

$$\times\exp\left[-\frac{\Delta_j^2}{2\epsilon D(q_{j-1})\left(1+\Delta_j\frac{d_1(q_{j-1})}{D(q_{j-1})}+\frac{\Delta_j^2}{2}\frac{d_2(q_{j-1})}{D(q_{j-1})}\right)}-\Delta_j\frac{A(q_{j-1})+\Delta_j a_1(q_{j-1})}{D(q_{j-1})\left(1+\Delta_j\frac{d_1(q_{j-1})}{D(q_{j-1})}\right)}-\epsilon\frac{A(q_{j-1})^2}{2D(q_{j-1})}-\epsilon V(q_{j-1})\right]. \quad (4.6)$$

In fact what we have done is just to change γ to an equivalent discretization developing everything around q_{j-1}. Keeping terms up to $O(\epsilon)$, formula (6) can be written as

$$J=\int\prod_{i=1}^{n}dq_i\prod_{j=1}^{n+1}\frac{1}{[2\pi\epsilon D(q_{j-1})]^{1/2}}\exp\left(-\frac{\Delta_j^2}{2\epsilon D(q_{j-1})}\right)\left[1-\Delta_j\left(-\frac{d_1}{2D}-\frac{A}{D}\right)+\epsilon\left(-\frac{1}{2}\frac{A^2}{D}-V\right)\right.$$
$$+\Delta_j^2\left(-\frac{d_2}{4D}+\frac{3d_1^2}{8D^2}+\frac{A^2}{2D^2}-\frac{a_1}{D}+\frac{3Ad_1}{2D^2}\right)$$
$$\left.+\frac{\Delta_j^3}{\epsilon}\left(\frac{d_1}{2D^2}\right)+\frac{\Delta_j^4}{\epsilon}\left(\frac{d_2}{4D^2}-\frac{3d_1^2}{4D^3}-\frac{Ad_1}{2D^3}\right)+\frac{\Delta_j^6}{\epsilon^2}\left(\frac{d_1^2}{8D^4}\right)\right]. \quad (4.7)$$

One can now use the following replacements [valid under the n-dimensional functional integral in the sense that its value will be unchanged when $n\to\infty$ (Ref. 2)]

$$\Delta_j^2\doteq\epsilon D(q_{j-1}),\quad \frac{\Delta_j^3}{\epsilon}\doteq 3\Delta_j D(q_{j-1}),\quad \frac{\Delta_j^4}{\epsilon}\doteq 3\epsilon D(q_{j-1})^2,\quad \frac{\Delta_j^6}{\epsilon}\doteq 15\epsilon D(q_{j-1})^3\,, \quad (4.8)$$

where the symbol $\doteq$ was introduced by DeWitt[13] to denote the stochastic equivalence under the multidimensional integral. After using (4.8) one obtains for (4.7) [putting everything back in the exponential and keeping only terms up to $O(\epsilon)$]

$$J=\int\prod_{i=1}^{n}dq_i\prod_{j=1}^{n+1}\frac{1}{[2\pi\epsilon D(q_{j-1})]^{1/2}}\exp\left[-\frac{\Delta_j^2}{2\epsilon D(q_{j-1})}-\Delta_j\frac{\bar{A}(q_{j-1})}{D(q_{j-1})}-\epsilon\frac{\bar{A}(q_{j-1})^2}{2D(q_{j-1})}-\epsilon\bar{V}(q_{j-1})\right], \quad (4.9)$$

with

$$\bar{A}(q)=A(q)-d_1(q)\,, \quad (4.10a)$$

$$\bar{V}(q)=V(q)+a_1(q)-\tfrac{1}{2}d_2(q)\,. \quad (4.10b)$$

We can now reintroduce the integral over the variables p_j and write (4.9) in the form

$$J=\int\left(\prod dq_i\prod\frac{dp_i}{2\pi\epsilon}\right)\exp\left\{i\epsilon\sum_{j=1}^{n+1}\left[p_j\frac{\Delta_j}{\epsilon}+\tfrac{1}{2}ip_j^2D(q_{j-1})+p_j\bar{A}(q_{j-1})+i\bar{V}(q_{j-1})\right]\right\}. \quad (4.11)$$

But recalling the definition of the discretization $\gamma(0)$ (Sec. II) we see that (4.11) implies

$$J=\int_{\gamma_1(0)}DqDp\exp\left\{i\int_{t_0}^{t}d\tau[p\dot{q}-\bar{H}^{\gamma}(p(\tau),q(\tau))]\right\}\delta(q(t_0)-Q_0)\delta(q(t)-Q) \quad (4.12)$$

with $\bar{H}^{\gamma}$ given by

$$\bar{H}^{\gamma}(p,q)=-\tfrac{1}{2}ip^2D(q)-p\bar{A}(q)-i\bar{V}(q)\,, \quad (4.13)$$

and consequently the operator $\hat{H}^{\gamma}(\hat{p},\hat{q})$ we are looking for is

$$\hat{H}^{\gamma}(\hat{p},\hat{q})=-\tfrac{1}{2}i\hat{p}^2D(\hat{q})-\hat{p}\bar{A}(\hat{q})-i\bar{V}(\hat{q})\,, \quad (4.14)$$

and (4.1) is equal to $\langle Q|\exp[-i(t-t_0)\hat{H}^{\gamma}]|Q_0\rangle$.

The calculation we have presented shows clearly that $\hat{H}^{\gamma}(\hat{p},\hat{q})$ is a function of the equivalence classes of discretizations γ as defined in Sec. III, i.e., the value of (4.1) changes when γ is changed to an inequivalent discretization $\tilde{\gamma}$.

B. Comment

As an illustration of the techniques we have exposed and of the use of the notion of discretization we want to comment here on a calculation done by Mizrahi.[9]

One starts with a given phase-space function $H(p,q)$; then one associates it with an operator $\hat{H}^{\Omega}(\hat{p},\hat{q}) = \Omega H(p,q)$ (see Appendix B) for each possible function $\Omega(u,v)$ by the formula

$$\hat{H}^{\Omega}(\hat{p},\hat{q}) = \int dp\,dq\,H(p,q)\Delta^{(\Omega)}(q-\hat{q},p-\hat{p})\,. \quad (4.15)$$

One uses now the function

$$\Omega_\alpha(u,v) = (1-\alpha)\exp(\tfrac{1}{2}iuv) + \alpha\exp(-\tfrac{1}{2}iuv)$$

to associate with each operator $\hat{H}^{\Omega}(\hat{p},\hat{q})$ a new phase-space function $H_\alpha^{\Omega}(p,q)$ by [compare (3.3)]

$$H_\alpha^{\Omega}(p,q) = \overline{\Omega}_\alpha\left(i\frac{\partial}{\partial q}, i\frac{\partial}{\partial p}\right)H_{AS}^{\Omega}(p,q)\,, \quad (4.16)$$

where

$$\langle p|\hat{H}^{\Omega}(\hat{p},\hat{q})|q\rangle = H_{AS}^{\Omega}(p,q)\langle p|q\rangle\,.$$

From the results of Secs. II and III it follows immediately that one can write for the propagator associated with the operator $\hat{H}^{\Omega}(\hat{p},\hat{q})$, i.e., for $\langle Q|U^{\Omega}(t,t_0)|Q_0\rangle$ with

$$i\frac{\partial U^{\Omega}(t',t)}{\partial t'} = \hat{H}^{\Omega}U^{\Omega}(t',t)\,,\;\; U^{\Omega}(t,t)=1\,,$$

the functional integral representations [remember $\gamma(\alpha)$ is the discretization associated with the use of $\Omega_\alpha(u,v)$]

$$\langle Q|U^{\Omega}(t,t_0)|Q_0\rangle = \int_{\gamma(\alpha)} Dq\,Dp\,\exp\left\{i\int_{t_0}^{t} d\tau[p\dot{q} - H_\alpha^{\Omega}(p,q)]\right\} \times\delta(q(t_0)-Q_0)\delta(q(t)-Q)\,. \quad (4.17)$$

In the case $\alpha=1$ (which corresponds to standard ordering, i.e., the operators $\hat{p}$ to the right of the operators $\hat{q}$) this calculation is the one of Ref. 9 [where the function $H_1^{\Omega}(p,q)$ is called $H_{n0}(p,q)$].

The prescription for the tadpoles in the formal series expansion of (4.17) is fixed by the knowledge of $\gamma(\alpha)$ and given in Sec. II. For $\alpha=1$ it reduces simply to say that all tadpoles take the value $B(\tau)+i$. Remembering that $\Delta^{12}(\tau,\tau+\epsilon) = B(\tau)+i$, $\epsilon\to 0^+$, we see that in each term of the formal series expansion, the function $\Delta^{12}(\tau,\tau)$ is to be defined as the limit $\Delta^{12}(\tau,\tau')$, $(\tau'-\tau)\to 0^+$, which is the prescription given in Ref. 9. The calculation we have just presented consists then in writing one of the many possible functional integral representations [namely, the one corresponding to the discretization $\gamma(1)$ for the propagator $\langle Q|U^{\Omega}(t,t_0)|Q_0\rangle$, where the operator $\hat{H}^{\Omega}(\hat{p},\hat{q})$ determining $U^{\Omega}(t,t_0)$ is one of the possible operators one can associate with a given phase-space function $H(p,q)$ by the transformation $\hat{H}^{\Omega}(\hat{p},\hat{q}) = \Omega H(p,q)$].

V. CONCLUSIONS

We have shown that the notion of discretization γ, or more precisely of equivalence classes of discretizations γ with respect to the equivalence relation of Sec. III, removes all ambiguities in the definition of functional integrals in phase space as limits of discretized expressions. We have also seen that the definition called without limiting procedure in terms of a formal series expansion has the same ambiguities as the definition through discretization; in fact we have proved that the ambiguities of the two methods are in one-to-one correspondence, since knowledge of the discretization γ fixes the prescription one needs in order that all terms in the formal series expansion are well defined. As it is clear from the text one can trace back the need to introduce the concept of discretization to the stochastic property $\Delta_j^{\,2} = O(\epsilon)$ of the paths. The relation of all this to the ordering problems of noncommuting operators has also been carefully considered.

We have studied elsewhere[2] the notion of discretization in q space, as well as its relation to the Feynman definition[14] and to the problem of the most probable path.[10] The covariance problems related to a general change of variables in our starting equation (2.1) and of its consequences for the functional integral representations have also been discussed.[2]

APPENDIX A: CALCULATION OF $Z_0[j,j^*]$

We have

$$Z_0[j,j^*] = \int_{\gamma(\alpha)} Dq\,Dp\,\exp\left[i\int_{t_0}^{T} d\tau(p\dot{q} + \tfrac{1}{2}icp^2 + \mu pq + \tfrac{1}{2}i\omega^2q^2 - i\alpha\mu + J^*p + Jq)\right]\delta(q(t_0)-Q_0)\delta(q(T)-Q)\,. \quad (\text{A1})$$

This generating functional can easily be calculated using the differential equations for $\delta Z_0/\delta j(t)$, $\delta Z_0/\delta j^*(t)$ with boundary conditions

$$\left.\frac{\delta Z_0}{\delta j(t)}\right|_{t=T} = iQZ_0\,,\;\; \left.\frac{\delta Z_0}{\delta j(t)}\right|_{t=t_0} = iQ_0Z_0$$

(see e.g., a similar calculation in Ref. 15). Instead of doing this we present here a different calculation to illustrate how one can integrate p's and q's at the same time, and where one has to be careful if one uses continuous instead of lattice expressions. Introducing the vector $x(t)=(p(t),q(t))$, one can write the exponential in the form

$$-\tfrac{1}{2}\iint (x(t),A(t,t')x(t'))dt\,dt'+\int (b(t),x(t))dt$$

which would give after integration an expression of the form

$$(\det A)^{-1/2}\exp\left[\tfrac{1}{2}\iint (b(t),A^{-1}(t,t')b(t'))dt\,dt'\right].$$

To see to what extent this formal procedure is correct, one goes back to the discrete version of (A1), e.g., in $\gamma(\frac{1}{2})(q_0=Q_0,\, q_{n+1}=Q)$

$$\int\sum_{i=1}^{n}dq_i\prod_{j=1}^{n+1}\frac{dp_j}{2\pi}\exp\left[-\tfrac{1}{2}\sum_{j,k=1}^{n+1}\epsilon^2\left(p_jp_kc\frac{\delta_{jk}}{\epsilon}+q_jq_k\omega^2\frac{\delta_{jk}}{\epsilon}\right)-\tfrac{1}{2}\sum_{j=1}^{n+1}\epsilon(-2ip_j)\left(\frac{q_j-q_{j-1}}{\epsilon}+\mu\frac{q_j+q_{j-1}}{2}\right)+iJ_j^*p_j+iJ_jq_j+\tfrac{1}{2}\epsilon\mu\right]. \tag{A2}$$

Rewriting the term in $p\dot{q}$ we obtain the exponential in the form $-\frac{1}{2}(x,Ax)+(b,x)$, where

$$\begin{aligned}
&A=\begin{pmatrix}A^{11} & A^{12}\\ A^{21} & A^{22}\end{pmatrix},\\
&A_{ij}^{11}=c\frac{\delta_{ij}}{\epsilon},\quad i,j=1,\ldots,n+1\\
&A_{ij}^{22}=\omega^2\frac{\delta_{ij}}{\epsilon},\quad i,j,\ldots,n\\
&A_{ij}^{12}=-i\left(\frac{1}{\epsilon^2}+\frac{\mu}{2\epsilon}\right)\delta_{ij}+\left(-\frac{1}{\epsilon^2}+\frac{\mu}{2\epsilon}\right)\delta_{i,j+1},\quad i=1,\ldots,n+1,\; j=1,\ldots,n\\
&A_{ij}^{21}=-i\left(\frac{1}{\epsilon^2}+\frac{\mu}{2\epsilon}\right)\delta_{ij}+\left(-\frac{1}{\epsilon^2}+\frac{\mu}{2\epsilon}\right)\delta_{i,j-1},\quad i=1,\ldots,n,\; j=1,\ldots,n+1\\
&b=\left(iJ_j^*-iQ_0\left(1-\frac{\epsilon\mu}{2}\right)\frac{\delta_{1j}}{\epsilon}+iQ\left(1-\frac{\epsilon\mu}{2}\right)\frac{\delta_{n+1,j}}{\epsilon},\,iJ_k\right),\quad j=1,\ldots,n+1,\quad k=1,\ldots,n.
\end{aligned} \tag{A3}$$

The matrix A formally tends to

$$A(t,t')=\begin{bmatrix}c\delta(t-t') & -i\left(-\dfrac{\partial}{\partial t'}+\mu\right)\delta(t-t')\\ i\left(-\dfrac{\partial}{\partial t'}-\mu\right)\delta(t-t') & \omega^2\delta(t-t')\end{bmatrix} \tag{A4}$$

and can also be obtained directly from (A1). Note, however, that

$$2\int p(\dot{q}+\mu)=\int p(\dot{q}+\mu)-\int q(\dot{p}-\mu)+pq\Big|_{t_0}^{T}$$

only gives half of the correct boundary terms. The finite matrix A has a unique inverse G while boundary conditions are required to compute the inverse of $A(t,t')$. We will see that the justification of (A4) results from the specification of the correct boundary condition. Indeed one of the four systems of equations for the inverse Δ of A is, from (A3),

$$\begin{aligned}
&c\Delta_{1k}^{11}-i\epsilon\left(\frac{1}{\epsilon^2}+\frac{\mu}{2\epsilon}\right)\Delta_{1k}^{21}=\delta_{1k},\\
&c\Delta_{ik}^{11}-i\epsilon\left[\left(\frac{1}{\epsilon^2}+\frac{\mu}{2\epsilon}\right)\Delta_{ik}^{21}+\left(\frac{-1}{\epsilon^2}+\frac{\mu}{2\epsilon}\right)\Delta_{i-1,k}^{21}\right]=\frac{\delta_{ik}}{\epsilon},\\
&\qquad i=2,\ldots,n,\\
&c\Delta_{n+1,k}^{11}-i\epsilon\left(\frac{1}{\epsilon^2}+\frac{\mu}{2\epsilon}\right)\Delta_{nk}^{21}=\frac{\delta_{n+1,k}}{\epsilon},\quad k=1,\ldots,n+1.
\end{aligned} \tag{A5}$$

If the second equation were valid for $i=1,\ldots,n+1$, we could replace (A5) by the differential equation obtained from (A3) in the limit. Adding the missing terms proportional to Δ_{0k}^{21} and $\Delta_{n+1,k}^{21}$ respectively in the first and the last equation, one remarks that the resulting equations containing these new variables, which do not occur in the finite matrices, are consistent with (A5) provided $\Delta_{0k}^{21}=\Delta_{n+1,k}^{21}=0$. In this way we obtain $\Delta^{21}(0,t')=\Delta^{21}(T,t')=0$ as boundary conditions for (A4). Inspection of the other equations yields $\Delta^{22}(0,t')=\Delta^{22}(T,t')=0$, giving us all the boundary conditions required to solve the system of coupled differential equations. It turns out that Δ^{22} and Δ^{11} are proportional to Green's functions solutions of $(\partial^2/\partial t^2-\rho^2)$, where $\rho^2=\mu^2+c\omega^2$, and are symmetric in (t',t).

$$\Delta^{12}(t,t')=\Delta^{21}(t',t)=\frac{i}{c}\left(\frac{\partial}{\partial t}+\mu\right)\Delta^{22}(t,t')$$

has a jump at the diagonal. More specifically

$$\Delta^{22}(t,t')=\frac{c}{\rho\sinh\rho(T-t_0)}[\theta(t-t')\sinh\rho(t-t_0)\sinh\rho(T-t')+\theta(t'-t)\sinh\rho(t'-t_0)\sinh\rho(T-t)]\,,$$

$$\Delta^{11}(t,t')=\frac{\omega^2}{\rho(\rho^2-\mu^2)\sinh\rho(T-t_0)}[\theta(t-t')(\rho\cosh\rho(T-t')-\mu\sinh\rho(T-t'))(\rho\cosh\rho(t-t_0)+\mu\sinh\rho(t-t_0))$$
$$+\theta(t'-t)(\rho\cosh\rho(t'-t_0)+\mu\sinh\rho(t'-t_0))(\rho\cosh\rho(T-t)-\mu\sinh\rho(T-t))]\,,$$

$$\Delta^{21}(t,t')=\frac{i}{\sinh\rho(T-t_0)}[\theta(t-t')\sinh\rho(T-t')(\rho\cosh\rho(t-t_0)+\mu\sinh\rho(t-t_0))$$
$$-\theta(t'-t)\sinh\rho(t'-t_0)(\rho\cosh\rho(T-t)-\mu\sinh\rho(T-t))]\,. \qquad \text{(A6)}$$

Using these solutions one immediately obtains for $Z_0[J,J^*]$

$$Z_0[J,J^*]=K\exp\left\{-\tfrac{1}{2}\left[\int_{t_0}^{T}\int_{t_0}^{T}(J^*-Q_0\delta(t-t_0)+Q\delta(t-T))\Delta^{11}(t,t')(J^*-Q_0\delta(t'-t_0)+Q\delta(t'-T))\right.\right.$$
$$\left.\left.+2\int_{t_0}^{T}\int_{t_0}^{T}(J^*(t)-Q_0\delta(t-t_0)+Q\delta(t-T))\Delta^{12}(t,t')J(t')+\int_{t_0}^{T}\int_{t_0}^{T}J(t)\Delta^{22}J(t')\right]\right\}. \qquad \text{(A7)}$$

The normalization factor is $Z_0[J,J^*]|_{J=J^*=Q=Q_0=0}$ and is given by

$$K=\left[\frac{\rho}{2c\pi\sinh\rho(T-t_0)}\right]^{1/2}\exp[\tfrac{1}{2}\mu(T-t_0)]\,. \qquad \text{(A8)}$$

Note that the calculation in $\gamma(\alpha)$ is exactly the same up to (A7), the α dependence being canceled in the calculation of the determinant (see Ref. 1).

APPENDIX B: CORRESPONDENCE BETWEEN OPERATORS AND PHASE-SPACE FUNCTIONS

The problem of expressing an operator $\hat{B}(\hat{p},\hat{q})$, function of two noncommuting operators in some ordered form is equivalent to the problem of mapping it onto a c-number function $B(p,q)$. Agarwal and Wolf[8] associated with each such mapping a mapping operator Ω and its inverse Θ such that

$$B(p,q)=\Theta\hat{B}(\hat{p},\hat{q})\,, \qquad \text{(B1)}$$

$$\hat{B}(\hat{p},\hat{q})=\Omega B(p,q)\,, \qquad \text{(B2)}$$

and characterized these mappings by a function $\Omega(u,v)$ satisfying certain conditions. For the convenience of the reader who is more acquainted with the work of Cohen[16] we remark that the function $\Omega(u,v)$ of Agarwal and Wolf and the function $F(\xi,\eta)$ of Cohen are related by $\Omega(u,v)=F(v,u)$.

Basic definitions of Ω and θ are

$$B^{\Omega}(p,q)=2\pi\,\mathrm{Tr}[\hat{B}(\hat{p},\hat{q})\Delta^{(\tilde{\Omega})}(q-\hat{q},p-\hat{p})]\,, \qquad \text{(B3)}$$

$$\hat{B}(\hat{p},\hat{q})=\int dp\,dq\,B^{\Omega}(p,q)\Delta^{(\Omega)}(q-\hat{q},p-\hat{p})\,, \qquad \text{(B4)}$$

in which

$$\Delta^{(\Omega)}(q-\hat{q},p-\hat{p})=(2\pi)^{-2}\int\Omega(u,v)\exp(\tfrac{1}{2}iuv)\times\exp[-iu(q-\hat{q})]\exp[-iv(p-\hat{p})]du\,dv \qquad \text{(B5)}$$

and

$$\tilde{\Omega}(u,v)=[\Omega(-u,-v)]^{-1}\,. \qquad \text{(B6)}$$

The way to show

$$H^{\Omega}(p,q)=\bar{\Omega}\left(i\frac{\partial}{\partial q},i\frac{\partial}{\partial p}\right)H_{AS}(p,q)$$

[formula (3.3)] is by writing (B3) as

$$H^{\Omega}(p,q)=2\pi\int dp'dq'\langle p'|\hat{H}(\hat{p},\hat{q})|\hat{q}'\rangle\frac{1}{(2\pi)^2}\int du\,dv\times\langle q'|e^{iu\hat{q}}e^{iv\hat{p}}|p'\rangle\,\tilde{\Omega}(u,v)e^{(i/2)uv}e^{-iuq-ivp}\,, \qquad \text{(B7)}$$

and by then introducing

$$\langle p|\hat{H}(\hat{p},\hat{q})|q\rangle = H_{AS}(p,q)\langle p|q\rangle \tag{B8}$$

and

$$\bar{\Omega}(u,v)=\tilde{\Omega}(u,v)\exp(\tfrac{1}{2}iuv)\,. \tag{B9}$$

The proof of

$$h^{\Omega}(p,q',q)=\Omega\left(-i\frac{\partial}{\partial\bar{q}},-\Delta\right)H^{\Omega}(p,\bar{q})$$

[formula (3.6)], $\bar{q}=\frac{1}{2}(q'+q)$, $\Delta=(q'-q)$ goes as follows: By definition one has

$$\langle q'|\hat{H}(\hat{p},\hat{q})|q\rangle=\int\frac{dp}{2\pi}e^{ip(q'-q)}h^{\Omega}(p,q',q)\,, \tag{B10}$$

but after an easy calculation also

$$\langle q'|\hat{H}(\hat{p},\hat{q})|q\rangle = \int\frac{dp}{2\pi}e^{ip(q'-q)}\frac{1}{2\pi}\int du\,dv\;\Omega(u,-\Delta)e^{iuv}H^{\Omega}(p,\bar{q}-v)\,. \tag{B11}$$

Then, defining $w=\bar{q}-v$,

$$h^{\Omega}(p,q',q)=\frac{1}{2\pi}\int du\,dw\;\Omega\left(-i\frac{\partial}{\partial\bar{q}},-\Delta\right)\times\exp[iu(\bar{q}-w)H^{\Omega}(p,w)] \tag{B12}$$

which reduces to (3.6).

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