Star-product representation of path integrals

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It is shown that Feynman's path integral can be written as a Fourier transform of a function defined on the classical phase space with the help of the star product recently introduced by Bayen, Flato, Fronsdal, Lichnerowicz, and Sternheimer. This gives a well-defined meaning to path integrals and may prove to be a useful tool in their actual evaluation.

I. INTRODUCTION

It has recently been shown by Bayen *et al.*[1] that the transition from classical to quantum mechanics, in other words, the process of "quantization," corresponds to replacing the algebra of observables (i.e., functions over classical phase space) with the usual product by the algebra defined by an associative product, called the star product, and defined as

$$(f*g)(q,p)=f\exp\left[\frac{i\hbar}{2}\left(\frac{\overleftarrow{\partial}}{\partial q}\frac{\overrightarrow{\partial}}{\partial p}-\frac{\overleftarrow{\partial}}{\partial p}\frac{\overrightarrow{\partial}}{\partial q}\right)\right]g$$
$$=\sum_{r,s=0}^{\infty}\frac{1}{r!s!}\left(\frac{i\hbar}{2}\right)^{r+s}(-1)^s \times\frac{\partial^{r+s}f}{\partial q^r\partial p^s}\frac{\partial^{r+s}g}{\partial p^r\partial q^s}, \qquad (1.1)$$

where $f(q,p)$ and $g(q,p)$ are the observables. It is checked in a straightforward manner that the star product is associative, and that the quantity $(f*g-g*f)/i\hbar$ is just the Moyal bracket introduced long ago.[2] This means that, among other things, one can do quantum-mechanical calculations within the classical framework without using any Hilbert-space formalism. However, the technical advantage of using the star product is yet to be explored fully. In this paper we want to point out the relation of the star product with the Feynman path integral; and show that the path integral is just the Fourier transform over the momentum variable of the well-defined function $\exp^*(-itH/\hbar)(q,p)$ where

$$\exp^*(f)(q,p)=1+f(q,p)+\frac{1}{2!}(f*f)(q,p)+\cdots . \qquad (1.2)$$

To be precise,

$$\int\prod_{\tau}\frac{dq(\tau)dp(\tau)}{2\pi\hbar}\exp\left[\frac{i}{\hbar}\int_0^t(p\dot{q}-H)d\tau\right]$$
$$=\int\frac{dp}{2\pi\hbar}\exp\left[\frac{i}{\hbar}p(q''-q')\right]\times\exp^*\left(-\frac{i}{\hbar}Ht\right)\left(\frac{q''+q'}{2},p\right), \qquad (1.3)$$

where the path integral on the left-hand side is over all paths $q(\tau)$ with $q(0)=q'$, $q(t)=q''$, and all $p(\tau)$.

In Sec. II we prove a lemma for a type of function occurring typically in path integrals, namely, those with a semigroup property. It is shown how the semigroup property is related to the star product. In Sec. III this relation is used to derive the representation (1.3) for the path integral as a special case. In Sec. IV we discuss an alternative, though less general, method for arriving at the same result using Wigner's distribution function. In Sec. V we apply (1.3) to get the path integral for the standard case of harmonic oscillator as an illustration, and conclude with a few remarks.

II. SEMIGROUP PROPERTY

Lemma. Let $F(\xi,\eta)$ and $G(\xi,\eta)$ be two functions which can be Taylor-expanded in the variable ξ anywhere on the real line and which go to zero, along with all the partial derivatives with respect to η as $\eta\to\pm\infty$. Define the following functions:

$$\mathfrak{F}_{21}(q_2,q_1)=\int_{-\infty}^{\infty}\frac{dp}{2\pi\hbar}\exp\left[\frac{i}{\hbar}p(q_2-q_1)\right]\times F\left(\frac{q_1+q_2}{2},p\right),$$
$$\mathfrak{F}_{32}(q_3,q_2)=\int_{-\infty}^{\infty}\frac{dp'}{2\pi\hbar}\exp\left[\frac{i}{\hbar}p'(q_3-q_2)\right]\times G\left(\frac{q_3+q_2}{2},p'\right). \qquad (2.1)$$

We shall call F and G the kernels of $\mathfrak{F}_{21}$ and $\mathfrak{F}_{32}$, respectively. Let S be the class of all functions of the form (2.1) whose kernels satisfy the conditions mentioned above. Then the function

$$\mathfrak{F}_{31}(q_3,q_1)\equiv\int dq_2\mathfrak{F}_{32}(q_3,q_2)\mathfrak{F}_{21}(q_2,q_1)$$

also belongs to S, and is given by

$$\mathfrak{F}_{31}(q_3,q_1)=\int\frac{dp}{2\pi\hbar}\exp\left[\frac{i}{\hbar}p(q_3-q_1)\right]\times(G*F)\left(\frac{q_3+q_1}{2},p\right). \tag{2.2}$$

In other words, functions belonging to S have the semigroup property (2.2) with their kernels $F((q_1+q_2)/2,p)$ obeying the semigroup law with respect to the star product.

To see how (2.2) is obtained, we substitute the expressions (2.1) for $\mathfrak{F}_{32}$ and $\mathfrak{F}_{21}$ on the left-hand side of (2.2), and Taylor-expand both G and F about $(q_3+q_1)/2$:

$$G\left(\frac{q_3+q_2}{2},p\right)=G\left(\frac{q_3+q_1}{2}+\frac{q_2-q_1}{2},p\right)=\sum_r\frac{1}{r!}\frac{\partial^r}{\partial\xi^r}G(\xi,p)\bigg|_{\xi=(q_3+q_1)/2}\left(\frac{q_2-q_1}{2}\right)^r,$$

$$F\left(\frac{q_2+q_1}{2},p\right)=\sum_s\frac{1}{s!}\frac{\partial^s}{\partial\xi^s}F(\xi,p)\bigg|_{\xi=(q_3+q_1)/2}\left(\frac{q_2-q_3}{2}\right)^s. \tag{2.3}$$

We get

$$\int dq_2\mathfrak{F}_{32}(q_3,q_2)\mathfrak{F}_{21}(q_2,q_1)=\int dq_2\int\frac{dp}{2\pi\hbar}\sum_{r,s}\frac{1}{r!s!}\frac{(-1)^s}{2^{r+s}}(q_3-q_2)^s\exp\left[\frac{i}{\hbar}p(q_3-q_2)\right]\frac{\partial^r}{\partial\xi^r}G(\xi,p)\bigg|_{\xi=(q_3+q_1)/2}$$
$$\times\int\frac{dp'}{2\pi\hbar}(q_2-q_1)^r\exp\left[\frac{i}{\hbar}p'(q_2-q_1)\right]\frac{\partial^s}{\partial\xi^s}F(\xi,p')\bigg|_{\xi=(q_3+q_1)/2}. \tag{2.4}$$

Expressing

$$(q_3-q_2)^s\exp\left[\frac{i}{\hbar}p(q_3-q_2)\right]$$

as

$$(-i\hbar)^s\frac{\partial^s}{\partial p^s}\exp\left[\frac{i}{\hbar}p(q_3-q_2)\right],$$

and doing an integration by parts, we transfer the derivatives acting on the exponential to G recalling that G and its derivatives with respect to η vanish at $\pm\infty$. We do the same thing to F, and obtain

$$\int dq_2\mathfrak{F}_{32}(q_3,q_2)\mathfrak{F}_{21}(q_2,q_1)=\int dq_2\int\frac{dp}{2\pi\hbar}\sum_{r,s}\frac{(-1)^s}{r!s!}\left(\frac{i\hbar}{2}\right)^{r+s}\frac{\partial^r}{\partial\xi^r}\frac{\partial^s}{\partial\eta^s}G(\xi,\eta)\bigg|_{\substack{\xi=(q_3+q_1)/2\\ \eta=p}}$$
$$\times\exp\left[\frac{i}{\hbar}p(q_3-q_2)\right]\int\frac{dp'}{2\pi\hbar}\frac{\partial^s}{\partial\xi^s}\frac{\partial^r}{\partial\eta^r}F(\xi,\eta)\bigg|_{\substack{\xi=(q_3+q_1)/2\\ \eta=p'}}\exp\left[\frac{i}{\hbar}p'(q_2-q_1)\right]. \tag{2.5}$$

Taking the q_2 integration inside and integrating we get $\delta(p-p')$, and performing the p' integration we obtain (2.2), with the definition of the star product given by (1.1). Obviously $(G*F)(\xi,\eta)$ satisfies the conditions of differentiability with respect to ξ, and behavior as $\eta\to\pm\infty$ if G and F separately do. The statement (2.2) is thus proved.

III. THE FEYNMAN PATH INTEGRAL

As an application of (2.2), consider

$$F_{i+1,i}(q,p)=\exp\left[-\frac{i}{\hbar}(t_{i+1}-t_i)H(q,p)\right], \tag{3.1}$$

where $(t_0=0,\ t_1,\ldots,t_N=t)$ is a partitioning of the interval $(0,t)$ and $H(q,p)$ is the Hamiltonian. Let

$$\mathfrak{F}_{i+1,i}(q_{i+1},q_i)=\int\frac{dp_{i+1}}{2\pi\hbar}\exp\left\{\frac{i}{\hbar}\left[p_{i+1}(q_{i+1}-q_i)-(t_{i+1}-t_i)H\left(\frac{q_{i+1}+q_i}{2},p_{i+1}\right)\right]\right\}, \tag{3.2}$$

where we have called the integration variable p_{i+1}. Then Eq. (2.2) reads explicitly in this case

$$\int dq_i \mathfrak{F}_{i+1,i}(q_{i+1},q_i)\mathfrak{F}_{i,i-1}(q_i,q_{i-1}) = \int \frac{dp_{i+1}dq_i dp_i}{(2\pi\hbar)^2}\exp\left\{\frac{i}{\hbar}\left[p_{i+1}(q_{i+1}-q_i)+p_i(q_i-q_{i-1}) - (t_{i+1}-t_i)H\left(\frac{q_{i+1}+q_i}{2},p_{i+1}\right)-(t_i-t_{i-1})H\left(\frac{q_i+q_{i-1}}{2},p_i\right)\right]\right\}$$

$$= \int \frac{dp}{2\pi\hbar}\exp\left[\frac{i}{\hbar}p(q_{i+1}-q_{i-1})\right](F_{i+1,i}*F_{i,i-1})\left(\frac{q_{i+1}+q_{i-1}}{2},p\right). \quad (3.3)$$

Repeating this process over and over again, and using the associativity of the star product, we get

$$\int \frac{dp_N dq_{N-1}\cdots dq_1 dp_1}{(2\pi\hbar)^N}\exp\left\{\frac{i}{\hbar}\left[p_N(q''-q_{N-1})+\cdots+p_1(q_1-q')-(t_N-t_{N-1})H\left(\frac{q''+q_{N-1}}{2},p_N\right)-\cdots-(t_1-t_0)H\left(\frac{q_1+q'}{2},p_1\right)\right]\right\}$$

$$= \int \frac{dp}{2\pi\hbar}\exp\left[\frac{i}{\hbar}p(q''-q')\right](F_{N,N-1}*\cdots*F_{2,1}*F_{1,0})\left(\frac{q''+q'}{2},p\right). \quad (3.4)$$

Now as the partition $\{t_i\}$ is made finer and finer, the left-hand side of the above equation tends to the Feynman path integral over all paths $p(\tau)$ and $q(\tau)$ with $q(0)=q'$ and $q(t)=q''$.

On the right-hand side we get repeated star products of factors such as

$$\exp\left[-\frac{i}{\hbar}(t_{i+1}-t_i)H(q,p)\right].$$

For infinitesimally small $t_{i+1}-t_i$, the difference between the star exponential

$$\exp^*\left[-\frac{i}{\hbar}(t_{i+1}-t_i)H\right](q,p)$$

and the ordinary exponential is of order $(t_{i+1}-t_i)^2$, and therefore

$$\lim_{N\to\infty}(F_{N,N-1}*\cdots*F_{1,0})\left(\frac{q''+q'}{2},p\right) = \exp^*\left(-\frac{i}{\hbar}tH\right)\left(\frac{q''+q'}{2},p\right), \quad (3.5)$$

again, by invoking the associativity of the star product. Thus the representation (1.3) is established.

Generalization of the formula to many degrees of freedom and the field theory is straightforward. See remark (3) in Sec. V.

IV. CONNECTION WITH WIGNER DISTRIBUTION FUNCTION

The connection (2.2) between functions $\mathfrak{F}\in S$ with the semigroup property, and kernels obeying the semigroup law with respect to star product, is quite general. In fact, one can consider arbitrary functions $F_{i+1,i}(\xi,\eta)$ satisfying the conditions of the lemma (2.2) and construct a functional integral by using (2.2) repeatedly and taking the limit. Thus, a functional integral gets related to a continuous star product quite generally. For simple enough cases such as (3.1), the continuous star product reduces neatly to a single factor (3.5).

In this section we consider an alternative derivation of (1.3) which is extremely simple, though somewhat less general in the following sense: One has to assume the existence of the quantized theory first, and then relate it to functions on the phase space through Wigner's distribution function,[3] whereas the whole idea behind theories with a star product or path integrals is to use these entities in the classical phase space only, thereby defining a process of quantization.

Recall that if $\hat{A}$ is an observable in the Hilbert space, then there corresponds to it a function $A(q,p)$ on the phase space defined by

$$A(q,p) = \int \langle q''|\hat{A}|q'\rangle \exp\left[\frac{i}{\hbar}(q'-q'')p\right] \times \delta\left(q-\frac{q''+q'}{2}\right)dq'dq''. \quad (4.1)$$

In particular, $\hat{A}$ could be the projection operator $\hat{\rho}_\psi = |\psi\rangle\langle\psi|$ on the state $|\psi\rangle$, and the corresponding function $\rho_\psi(q,p)$ is the Wigner distribution function for $|\psi\rangle$. It follows easily that

$$\langle\psi|\hat{A}|\psi\rangle = \mathrm{Tr}(\hat{\rho}_\psi\hat{A}) = \int \frac{dp\,dq}{2\pi\hbar}\rho_\psi(q,p)A(q,p), \quad (4.2)$$

and that for an arbitrary matrix element,

$$\langle\phi|A|\psi\rangle = \tfrac{1}{2}\mathrm{Tr}[\{\hat{\rho}_{\phi+\psi}-i\hat{\rho}_{\phi+i\psi}+(i-1)(\hat{\rho}_\phi+\hat{\rho}_\psi)\}\hat{A}] \equiv \mathrm{Tr}[\hat{\rho}_{\phi,\psi}\hat{A}] = \int\frac{dp\,dq}{2\pi\hbar}\rho_{\phi,\psi}(q,p)A(q,p). \quad (4.3)$$

The star product naturally comes into the picture when we try to define the phase-space function corresponding to $\hat{A}\hat{B}$. We shall omit the straightforward and simple calculation and just note that if $A(q,p)$ and $B(q,p)$ are the phase-space functions defined by (4.1), then

$$(A*B)(q,p)=\int\langle q''|\hat{A}\hat{B}|q'\rangle\exp\left[\frac{i}{\hbar}(q'-q'')p\right]\times\delta\left(q-\frac{q'+q''}{2}\right)dq'dq''. \qquad (4.4)$$

Therefore, the phase-space function corresponding to

$$\exp\left(-\frac{i}{\hbar}t\hat{H}\right)=1-\frac{it}{\hbar}\hat{H}+\left(-\frac{it}{\hbar}\right)^2\frac{1}{2!}\hat{H}\hat{H}+\cdots$$

is the star exponential

$$\exp^*\left(-\frac{i}{\hbar}H\right)(q,p)=1-\frac{it}{\hbar}H+\left(-\frac{it}{\hbar}\right)^2\frac{1}{2!}(H*H)(q,p)+\cdots. \qquad (4.5)$$

Now the Feynman path integral in quantum mechanics is just the matrix element $\langle q''(t)|q'(0)\rangle$, where $|q'(\tau)\rangle$ is the "moving representation" in the Heisenberg picture,[4] and

$$\langle q''(t)|q'(0)\rangle=\langle q''(0)|\,e^{-(it/\hbar)\hat{H}}|q'(0)\rangle. \qquad (4.6)$$

Our formula (1.3) follows immediately from (4.3) and the statement (4.5) when we realize that $\rho_{\phi,\psi}(q,p)$ for $|\phi\rangle=|q''(0)\rangle$ and $|\psi\rangle=|q'(0)\rangle$ is just

$$\exp\left[\frac{i}{\hbar}p(q''-q)\right]\delta\left(q-\frac{q'+q''}{2}\right).$$

V. ILLUSTRATION AND REMARKS

For a harmonic oscillator with $H=\frac{1}{2}(p^2+q^2)$, $\exp^*[-(it/\hbar)H](q,p)$ turns out to be[1]

$$\exp^*\left(-\frac{it}{\hbar}H\right)(q,p)=(\cos\tfrac{1}{2}t)^{-1}\exp\left[\frac{p^2+q^2}{i\hbar}\tan(\tfrac{1}{2}t)\right]. \qquad (5.1)$$

Substituting in (1.3) and carrying out the momentum integration, we immediately recover the standard expression for the path integral for the harmonic oscillator,[5] with m and ω equal to 1:

$$\left(\frac{1}{2\pi i\sin t}\right)^{1/2}\exp\left\{\frac{i}{2\hbar\sin t}[(q'^2+q''^2)\cos t-2q'q'']\right\}.$$

We conclude this paper with a few remarks:

(1) The analysis of Sec. II shows that there is an intimate connection between path integrals and star products. Thus, given any path integral $\mathfrak{F}(q'',t'';q',t')$ over all paths $q(\tau)$ with initial value q' and final value q'', we must have the semigroup property

$$\mathfrak{F}(q'',t'';q',t')=\int dq\,\mathfrak{F}(q'',t'';q,t)\mathfrak{F}(q,t;q',t'), \qquad (5.2)$$

where $t'<t<t''$. This is the least we can expect from any reasonable definition of the path integral, because the set of all paths can be equivalently considered to be the set of all paths restricted to value q at t; this restricted value is then varied over all the domain. If we are able to calculate the given path integral $\mathfrak{F}(q,t,q',t')$ for the *infinitesimal* interval $(t-t')$, then by inverting the Fourier transform (2.1) we get the corresponding kernel

$$F_{t,t'}\left(\frac{q+q'}{2},p\right)=\int d(q-q')\mathfrak{F}\left(\frac{q+q'}{2}+\frac{q-q'}{2},t;\frac{q+q'}{2}-\frac{q-q'}{2},t\right)\exp\left[-\frac{i}{\hbar}p(q-q')\right]. \qquad (5.3)$$

By (2.2), the path integral for the finite interval is therefore a continuous star product of infinitesimal kernels (5.3). In those cases where the star products over infinitesimal kernels dovetail to give a single factor, as they do for the ordinary quantum-mechanical case (3.1), the problem of path integration is reduced to calculating a well-defined expression in terms of the star product.

(2) It is well known that path integrals require the so-called "$i\epsilon$ prescription" for their convergence. In the above formalism it occurs as the condition on $F(\xi,\eta)$ as $\eta\to\pm\infty$, stated in the beginning of Sec. II. In order that the quantum-mechanical kernel (3.1) satisfy it, the coefficient of p^2 in $H(q,p)$ should have the correct $i\epsilon$.

(3) The generalization to field theory should be interesting, because through the star product defined by the generalization of the Poisson bracket operator

$$\left(\frac{\overleftarrow{\partial}}{\partial q}\frac{\overrightarrow{\partial}}{\partial p}-\frac{\overleftarrow{\partial}}{\partial p}\frac{\overrightarrow{\partial}}{\partial q}\right)$$

to the expression

$$\int d^3x\left(\frac{\overleftarrow{\delta}}{\delta\phi(x)}\frac{\overrightarrow{\delta}}{\delta\pi(x)}-\frac{\overleftarrow{\delta}}{\delta\pi(x)}\frac{\overrightarrow{\delta}}{\delta\phi(x)}\right),$$

we can reduce the path integral over all fields $\phi(\vec{x},t)$, $\pi(\vec{x},t)$ to a single path integral over $\pi(\vec{x},t=0)$. This corresponds to the single integral $\int dp$ of (1.3). The interesting question is: Can this process of reducing the path integral to the star product be repeated once (or three times) more? That is, is it possible to reduce the four-dimensional path integral entirely in terms of star products and ordinary integration?

(4) Any transition from classical to quantum mechanics faces the problem of ordering, i.e., although the usual product on phase space is commutative, the star product is not. The connection of the star-product path integral with the Wigner distribution function given in Sec. IV shows that the ordering involved in the Weyl correspondence[6] [the inverse to Wigner correspondence (4.1)], and is given by

$$f(q,p)\to\hat{f}=\int\tilde{f}(u,v)\exp\left(\frac{u\hat{p}+v\hat{q}}{i\hbar}\right)du\,dv\,,$$

where $\tilde{f}(u,v)$ is the Fourier transform of $f(q,p)$ and $\hat{q}$, $\hat{p}$, and $\hat{f}$ are the quantum-mechanical operators corresponding to classical q, p, and $f(q,p)$. However, as remarked in Sec. II, part II of Ref. 1, other orderings can also be equivalently discussed in star-product formalism.

(5) Finally, we would like to emphasize that the main advantage of the star-product formalism is that it gives a definite meaning to the rather notoriously ill-defined path-space measure. The expression on the right-hand side of (1.3) is a well-defined expression in terms of $H(q,p)$ and its derivatives. At present we do not have enough algorithms to calculate and approximate the star exponential $\exp^*[-(it/\hbar)H(q,p)]$. But when this is done, we would have an alternative and unambiguous way to calculate the path integral.

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