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Quantum numbers for Dirac spinor fields on a curved space-time

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The most general first-order differential operator that commutes with the Dirac operator and hence permits the construction of quantum numbers is given. Necessary and sufficient conditions for its existence are expressed in terms of the generalized Killing tensors of Yano. As a special case we obtain an extension to curved space-time of a covariant description of spin.

INTRODUCTION

The construction of first integrals of the evolution equations for the variables of a physical system plays a crucial role in the resolution of different problems in classical mechanics, classical field theory, and in quantum theory. When the equations of motion are derivable from a Lagrangian invariant under the action of a symmetry group, Noether's theorem allows one to associate a constant of the motion to each generator of the group. In general the symmetry group of a Lagrangian appears as the direct product of a subgroup of internal symmetries and a subgroup of geometrical symmetries. The geometrical symmetries most usually considered are the isometries. They are described locally by the so-called Killing vector fields that can be interpreted as representing the infinitesimal displacements of an isometry group.[1] The consequences of the existence of a Killing vector field are well known.[2] When the lines of the coordinate system being used are such that one of them coincides with the integral curves of a Killing vector field (a coordinate adapted to a symmetry) the corresponding coordinate does not appear in the Lagrangian. In this case the equations of motion admit a first separation of variables involving this particular coördinate defining a corresponding constant of motion. Moreover, and this is of considerable interest in quantum field theory, the derivative operator involving this privileged coordinate commutes with the field equation operator. In fact this derivative operator is the Lie derivative operator associated with the Killing vector field being considered. Its eigenvalues on the solution space of the field equations are the quantized values corresponding to the classical constant of motion. The average value of this quantity for a given state is obtained by calculating the flux of a conserved current across a spacelike hypersurface.

The constants of motion generated by Killing vector fields are not the only ones possible. Recently Carter[3] has developed a general formalism for this problem and has illustrated its use in the particular case of constants of motion for the equations of motion of a particle and of a scalar field in interaction with an electromagnetic field. This led him to distinguish the notions of Stäckel tensors and Killing tensors, second-order symmetric tensors by means of which are generalized the more familiar constants of motion which arise from Killing vectors. A particularly remarkable result due to the existence of these tensors is the separability of the Hamilton-Jacobi equation and the Klein-Gordon equation in the Kerr space-time which admits only *two* Killing vector fields. It was the discovery by Carter[4] of an unexpected fourth constant of motion in this space-time which was the starting point for the kind of study just described. More recently Chandrasekar[5] has devised a procedure (generalized by Page[6] and Güven[7]) of separation of variables for Dirac's equation on the Kerr space-time. An analysis of this method due to Carter and McLenaghan[8] shows that this separation procedure depends on the existence of a differential operator constructed from an antisymmetric second-rank tensor, the Penrose-Floyd[9] tensor, that commutes with the Dirac operator.

The aim of the present paper is to construct the most general first-order differential operator which commutes with the Dirac operator. We show that the tensors from which this operator is constructed must all satisfy the same kind of equation, namely the generalized Killing equations of Yano and Bochner.[10] Such tensors have also been studied in the context of general relativity by Collinson.[11] Each of these tensors is completely antisymmetric. In order to avoid confusion with the terminology introduced by Carter[12] we shall call them *Killing forms*. In particular, we show that in addition to the constants of motion obtained from Killing vectors and Penrose-Floyd tensors there exist first integrals depending on the existence of a completely antisymmetric third-rank tensor satisfying the equation of Yano and Bochner just mentioned. We shall call these tensors and their duals respectively tensors and vectors of Yano. Since it seems that the third-rank Yano tensors have been overlooked we present some of their properties. In particular, we show that they are intimately related to certain conformal-invariant properties of the space on which they are defined. We describe them briefly on Minkowski space, where their physical significance is the most transparent, and also in the de Sitter universe and on the Kerr-Newman space-time.

II. CONSTRUCTION OF THE CONSTANTS OF MOTION FOR THE SPINOR FIELD

The equation of the spinor field on a curved space-time in the presence of an electromagnetic field (with potential A_α) has the form

$$H\psi = m\psi\,, \tag{2.1}$$

where Dirac's operator H is given by

$$H = i\gamma^\alpha(\nabla_\alpha - ieA_\alpha)\,. \tag{2.2}$$

In this expression the Dirac matrices γ^α are defined in local coordinates up to a similarity transformation by the anticommutation relations

$$\{\gamma^\alpha, \gamma^\beta\} = 2g^{\alpha\beta}I\,, \tag{2.3}$$

while ∇_α denotes the canonical covariant derivative for spinors.[13] The essential properties of this derivative are summarized in the following formulas:

$$\nabla_\alpha \gamma^\beta = 0\,, \tag{2.4}$$

$$\nabla_{[\rho}\nabla_{\mu]} = \tfrac{1}{8}R_{\alpha\beta\rho\mu}\gamma^\alpha\gamma^\beta\,, \tag{2.5}$$

where $R_{\alpha\beta\rho\mu}$ denotes the components of the Riemannian-Christoffel curvature tensor.

The most general first-order differential operator

$$K = F^\alpha\nabla_\alpha + G \tag{2.6}$$

which commutes with Dirac's operator is obtained by setting separately equal to zero the symmetrized coefficients of the covariant derivatives of each order in the expansion of the commutator $[H,K]$.[14] This leads to a system of three necessary and sufficient conditions for K to be a constant of motion. The first of these conditions is

$$[\gamma^{(\alpha}, F^{\beta)}] = 0\,, \tag{2.7}$$

from which one finds on introducing an irreducible tensorial basis that F^α has the form

$$F^\alpha = B^\alpha I + C\gamma^\alpha + D^\alpha{}_\beta\gamma^5\gamma^\beta + E^\alpha{}_{\beta\gamma}\gamma^{[\beta}\gamma^{\gamma]}\,, \tag{2.8}$$

where

$$D_{\alpha\beta} = D_{[\alpha\beta]}\,, \quad E_{\alpha\beta\gamma} = E_{[\alpha\beta\gamma]} \tag{2.9}$$

and

$$\gamma^5 = \frac{1}{4!}\eta_{\alpha\beta\gamma\delta}\gamma^\alpha\gamma^\alpha\gamma^\gamma\gamma^\delta\,. \tag{2.10}$$

In the last equation $\eta_{\alpha\beta\gamma\delta}$ denotes the components of the volume element of the space-time. In natural coordinates it is given by

$$\eta_{\alpha\beta\gamma\delta} = 4!\sqrt{-g}\,\delta_{[\alpha}^0\delta_\beta^1\delta_\beta^2\delta_{\delta]}^3\,. \tag{2.11}$$

The second condition is

$$[G, \gamma^\alpha] - \gamma^\beta\nabla_\beta F^\alpha - ieA_\beta[F^\alpha, \gamma^\beta] = 0\,, \tag{2.12}$$

from which one deduces the conditions

$$C_{,\alpha} = 0\,, \tag{2.13}$$

$$B_{(\alpha;\beta)} = 0\,, \tag{2.14}$$

$$D_{\alpha(\beta;\gamma)} = 0\,, \tag{2.15}$$

$$E_{\alpha\beta(\gamma;\delta)} = 0\,. \tag{2.16}$$

These are just the equations satisfied by the Killing forms of Yano[15] defining respectively Killing vectors, Penrose-Floyd tensors, and Yano tensors.

The last two equations are equivalent to

$${}^*D_{\alpha\beta;\gamma} = \tfrac{2}{3}g_{\gamma[\beta}{}^*D_{\alpha]}{}^\delta{}_{;\delta}\,, \tag{2.17}$$

$$E_{\alpha;\beta} = \tfrac{1}{4}g_{\alpha\beta}E^\gamma{}_{;\gamma}\,, \tag{2.18}$$

where

$${}^*D_{\alpha\beta} = \tfrac{1}{2}\eta_{\alpha\beta\gamma\delta}D^{\gamma\delta}\,, \tag{2.19}$$

$$E_\alpha = \tfrac{1}{6}\eta_{\alpha\beta\gamma\delta}E^{\beta\gamma\delta} \tag{2.20}$$

denote the tensors dual to $D_{\alpha\beta}$ and $E_{\alpha\beta\gamma}$, respectively. We also may deduce from the condition (2.12) on taking account of the conditions (2.13) to (2.16) that only the trace of G remains undetermined. We obtain explicitly

$$G = \Phi I - \tfrac{3}{4}E^\alpha{}_{;\alpha}\gamma^5 + (\tfrac{1}{3}\,{}^*D^\alpha{}_{\beta;\alpha} - ieCA_\beta)\gamma^\beta + ieA_\alpha D_\beta{}^\alpha\gamma^5\gamma^\beta + (ieA_\alpha E_\beta{}^\alpha{}_\gamma - \tfrac{1}{4}B_{\beta;\gamma})\gamma^{[\beta}\gamma^{\gamma]}\,. \tag{2.21}$$

The third condition becomes on account of the Ricci identities (2.5) for the spinor covariant derivative

$$\tfrac{1}{8}\{F^{[\alpha}, \gamma^{\beta]}\}R_{\alpha\beta\gamma\delta}\gamma^\gamma\gamma^\delta + \gamma^\beta G_{;\beta} + ie(A_{\beta;\alpha}F^\alpha\gamma^\beta + A_\beta[G, \gamma^\beta]) = 0\,, \tag{2.22}$$

from which follow the additional conditions

$$\Phi_{,\alpha} + ie(B^\beta A_{\alpha;\beta} + A_\beta B^\beta{}_{;\alpha}) = 0\,, \tag{2.23}$$

$$F_{\alpha[\beta}D^\alpha{}_{\gamma]} = 0\,, \tag{2.24}$$

$$F_{\alpha[\beta}E^\alpha{}_{\gamma\delta]} = 0\,, \tag{2.25}$$

where

$$F_{\alpha\beta} = 2A_{[\beta;\alpha]} \tag{2.26}$$

is the electromagnetic field tensor.

The basic principles of quantum mechanics[16] require the operator K to be *formally self-adjoint* the same as Dirac's operator. Since any operator may be formally decomposed into self-adjoint and anti-self-adjoint parts this results in no loss of generality. Indeed the operator K is linear in the Killing forms. Thus the most general anti-self-adjoint operator may be deduced from the general self-adjoint operator by multiplication by i. For a spinor field the dual is obtained by the following operation (Dirac conjugation):

$$\psi \to \psi^\dagger \beta , \tag{2.27}$$

where β can be chosen to satisfy the relations

$$\nabla_\alpha \beta = 0, \quad \beta = \beta^\dagger . \tag{2.28}$$

Following Carter[17] an operator K will be formally self-adjoint if for arbitrary spinor fields ψ_1 and ψ_2. with compact support we have

$$\int \psi_1^\dagger \beta (F^\rho \nabla_\rho + G)\psi_2 dv = \int [(F^\rho\nabla_\rho + G)\psi_1]^\dagger \beta \psi_2 dv . \tag{2.29}$$

Since this equation must hold for arbitrary supports of the fields we may deduce the following two conditions. The first condition is

$$\beta F^\rho + F^{\rho\dagger}\beta = 0 , \tag{2.30}$$

from which one obtains the following conditions:

$$B^\alpha = -B^{\alpha *} , \tag{2.31}$$

$$C = -C^* , \tag{2.32}$$

$$D_{\alpha\beta} = D^*_{\alpha\beta} , \tag{2.33}$$

$$E_{\alpha\beta\gamma} = E^*_{\alpha\beta\gamma} . \tag{2.34}$$

The second condition is

$$\beta G - G^\dagger \beta = 0 , \tag{2.35}$$

which on account of the preceding conditions reduces to

$$\Phi = \Phi^* . \tag{2.36}$$

We now make the following substitutions in Eqs. (2.8) and (2.21):

$$\begin{aligned} &C \to ic, \quad B^\alpha \to ik^\alpha, \quad D_{\alpha\beta} \to f_{\alpha\beta}, \quad E_{\alpha\beta\gamma} \to y_{\alpha\beta\gamma} , \\ &E_\alpha \to y_\alpha , \quad \Phi \to \phi , \end{aligned} \tag{2.37}$$

where c is a *real* constant, k^α a *real* Killing vector, $f_{\alpha\beta}$ a *real* Penrose-Floyd tensor, $y_{\alpha\beta\gamma}$ a *real* Yano Killing form, and ϕ is a *real-valued* function.

We are now able to enunciate the following theorem: *The most general self-adjoint first-order differential operator which commutes with Dirac's operator is*

$$\begin{aligned} K = {} & [i(k^\alpha I + c\gamma^\alpha) + f^{\alpha\beta}\gamma^5\gamma_\beta + y^{\alpha\beta\gamma}\gamma_{[\beta}\gamma_{\gamma]}]\nabla_\alpha \\ & + \phi I - \tfrac{3}{4} y^\alpha{}_{;\alpha}\gamma^5 + (\tfrac{1}{3}\, {}^*f^{\alpha\beta}{}_{;\alpha} + ecA^\beta)\gamma_\beta \\ & + ieA_\alpha f^{\beta\alpha}\gamma^5\gamma_\beta + i(eA_\alpha y_\beta{}^\alpha{}_\gamma - \tfrac{1}{4}k_{\beta;\gamma})\gamma^{[\beta}\gamma^{\gamma]} , \end{aligned} \tag{2.38}$$

where k_α, $f_{\alpha\beta}$, and $y_{\alpha\beta\gamma}$ are antisymmetric tensors satisfying the Yano-Killing equations

$$c_{,\alpha} = k_{(\alpha;\beta)} = f_{\alpha(\beta;\gamma)} = y_{\alpha\beta(\gamma;\delta)} = 0 , \tag{2.39}$$

and the electromagnetic field tensor $F_{\alpha\beta}$ defined by Eq. (2.26) *satisfies*

$$eF_{\alpha[\beta}f^\alpha{}_{\gamma]} = 0 , \tag{2.40}$$

$$eF_{\alpha[\beta}y^\alpha{}_{\gamma\delta]} = 0 , \tag{2.41}$$

$$\phi_{,\alpha} - e(k^\beta A_{\alpha;\beta} + A_\beta k^\beta{}_{;\alpha}) = 0 . \tag{2.42}$$

III. DISCUSSION

First of all we note that Eq. (2.42) implies

$$e\underset{k}{\mathcal{L}} F_{\alpha\beta} = 0 , \tag{3.1}$$

where $\mathcal{L}_k$ represents the Lie derivative in the direction of the vector field k^α. This equation reflects the gauge invariance of the second kind of the Dirac equation. Clearly we can always choose ϕ constant by a gauge transformation. This remark elucidates the apparent lack of gauge invariance of the corresponding operator of Carter and McLenaghan[18] where the gauge is implicitly fixed by the assumption $\phi = 0$.

The constants of motion associated with Killing vectors and Penrose-Floyd tensors have been discussed by the previously mentioned authors. Thus we shall concentrate our attention on those associated with the tensors and dual vectors of Yano.

It follows from Eq. (2.18) by virtue of Poincaré's lemma that the Yano vector is derivable from a potential

$$y_\alpha = \partial_\alpha y . \tag{3.2}$$

In addition this vector is the generator of an infinitesimal conformal transformation since

$$y_{(\alpha;\beta)} = \tfrac{1}{4} g_{\alpha\beta} y^\gamma{}_{;\gamma} . \tag{3.3}$$

We propose to call such vectors the generators of *conformal translations*. Note that the orbits of these transformations are geodesic since

$$y_{\alpha;\beta}y^\beta = \tfrac{1}{4} y^\gamma{}_{;\gamma} y_\alpha . \tag{3.4}$$

If one differentiates Eq. (2.18) and applies the Ricci identities the following identities are obtained:

$$y_{\alpha;\beta\gamma} = -\tfrac{1}{3} g_{\alpha\beta} R_{\gamma\delta} y^{\delta}, \tag{3.5}$$

$$(R_{\alpha\beta\gamma\delta} - \tfrac{2}{3} g_{\gamma[\alpha} R_{\beta]\delta}) y^{\delta} = 0. \tag{3.6}$$

By contracting the latter equation with y^γ we deduce that *the Yano vector is an eigenvector of the Ricci tensor*:

$$R_{\alpha\beta}\, y^{\beta} = \lambda y_{\alpha}. \tag{3.7}$$

On an Einstein space Eq. (3.6) reduces to

$$C_{\alpha\beta\gamma\delta}\, y^{\delta} = 0, \tag{3.8}$$

where $C_{\alpha\beta\gamma\delta}$ denotes the Weyl tensor. It follows from this equation that *the existence of a Yano vector on an Einstein space implies that the space is conformally flat or of Petrov type N.* latter case the Yano vector is a characteristic null vector of the Weyl tensor. A further algebraic condition is obtained by writing Eq. (2.41) in the form

$$eF_{\alpha\beta}\, y^{\beta} = 0. \tag{3.9}$$

This condition implies that if the spinor field is effectively coupled to an electromagnetic field ($e \neq 0$) the latter field must be *singular* in order for a Yano vector to exist.

From the Yano tensor $y_{\alpha\beta\gamma}$ one may construct a Stäckel-Killing tensor by setting

$$a_{\mu\nu} = \tfrac{1}{2} y_{\mu\alpha\beta}\, y_{\nu}{}^{\alpha\beta} = y_{\mu} y_{\nu} - g_{\mu\nu} y_{\alpha} y^{\alpha}. \tag{3.10}$$

In fact as a consequence of Eqs. (2.18) and (3.9) this tensor satisfies the conditions[19]

$$a_{[\mu\nu]} = 0, \quad a_{(\mu\nu;\rho)} = 0, \quad \frac{e}{m} a_{\nu(\mu} F^{\nu}{}_{\rho)} = 0 \tag{3.11}$$

that are the necessary and sufficient conditions for the quadratic form

$$K = a_{\mu\nu} u^{\mu} u^{\nu} \tag{3.12}$$

to be a first integral of the classical equations of motion of a particle in interaction with an electromagnetic field:

$$u^{\alpha}{}_{;\beta} u^{\beta} = \frac{e}{m} F^{\alpha}{}_{\beta} u^{\beta}. \tag{3.13}$$

This is not the only link between the Yano-Killing forms and the constants of motion of these equations; Carter and McLenaghan[20] and Collinson[21] have exhibited an analogous property of the Penrose-Floyd tensor. In addition, Carter[22] has shown that when the source-free Einstein-Maxwell equations are satisfied on the space-time the Stäckel-Killing tensor defined by Eq. (3.11) is a Killing tensor. This means that the operator on scalar fields defined by

$$K = D_{\alpha} a^{\alpha\beta} D_{\beta}, \tag{3.14}$$

where

$$D_{\alpha} = \nabla_{\alpha} - ieA_{\alpha} \tag{3.15}$$

commutes with the Klein-Gordon operator

$$H = D_{\alpha} g^{\alpha\beta} D_{\beta}. \tag{3.16}$$

In this situation the integration of Eqs. (2.18) can be simplified by the following consideration. Indeed if

$$R^{\alpha}{}_{\beta} = x(F^{\alpha\rho}F_{\beta\rho} - \tfrac{1}{4}\delta^{\alpha}_{\beta} F^{\rho\sigma}F_{\rho\sigma}), \tag{3.17}$$

Eq. (2.18) implies

$$R^{\alpha}{}_{\beta;\gamma}\, y_{\alpha} = (R^{\alpha}{}_{\beta} y_{\alpha})_{;\gamma} - \tfrac{1}{4} R_{\gamma\beta}\, y^{\alpha}{}_{;\alpha}. \tag{3.18}$$

On the other hand, from Eq. (3.5) it follows that

$$R^{\gamma}{}_{[\alpha;\beta]}\, y_{\gamma} = 0. \tag{3.19}$$

Equations (3.17), (3.18), and (3.19) imply when $e \neq 0$

$$R^{\alpha}{}_{\beta;\gamma}\, y_{\alpha} = \tfrac{1}{4} x\, y_{[\gamma} \partial_{\beta]} (F_{\rho\sigma} F^{\rho\sigma}) = 0. \tag{3.20}$$

Thus in view of Eq. (3.2) we find that the potential of the Yano vector field is a function of the invariant $F_{\rho\sigma}F^{\rho\sigma}$ of the electromagnetic field. In particular, if the electromagnetic field in this model is radiative (a null field) there can exist no Yano vector.

To conclude, we note that the maximal number of independent solutions of Eq. (2.18) on an n-dimensional space is $n+1$. In particular, this number is attained on a space of constant curvature. In Minkowski space the general form of the Yano vector in rectilinear coordinates is

$$y^{\alpha} = \tfrac{1}{2}(a^{\alpha} + b x^{\alpha}), \tag{3.21}$$

where a^α and b are five arbitrary constants. The corresponding operator given by Eq. (2.38) gives a covariant description of the operators of helicity and spin[23] when $b = 0$ and is diagonalizable simultaneously with the energy and momentum operators. This is no longer true when b is not equal to zero. In this case the interpretation of the operator is less clear, although its structure suggests that it could be related to the operator of spin-orbit coupling.

Likewise in the de Sitter universe, which in Nachtmann[24] coordinates has the metric

$$ds^2 = R^2 \lambda^{-2}\left(d\lambda^2 - \sum_{i=1}^{3} (dx^i)^2\right), \tag{3.22}$$

the general form of the Yano vector is given by

$$y^{i} = \tfrac{1}{2} R^{-1} \lambda (a^{i} + b x^{i}), \tag{3.23a}$$

$$y^{0} = \tfrac{1}{2} R^{-1} \sum_{i=1}^{3} [a^{i} x^{i} + \tfrac{1}{2} b (x^{i})^2] + \tfrac{1}{4} R^{-1} b (\lambda^2 - R^2) + \tfrac{1}{2} a^{0}. \tag{3.23b}$$

It also depends on five arbitrary constants which have been chosen in such a manner so as to reduce to the Minkowskian expression (3.18) in the limit when the curvature of the de Sitter universe tends to zero ($R \to \infty$). Finally it can be shown that the Kerr-Newman solution does *not* admit a nonzero Yano vector.

CONCLUSION

The novelty of the method developed here stems from the fact that first integrals of Dirac's equation have been obtained without any explicit reference to the existence of a particular isometry group. This technique is important for the study of wave equations on a curved space-time. Indeed if in flat space-time constants of motion of a system can be interpreted as proper values of operators defined from the generators of the Poincaré group, new constants of motion independent of the isometry group can appear in curved space-time. The equations that we have written here generalize in this sense Killing's equations and, it seems, should play a crucial role in the field theory whether in the frame of general relativity or in a situation describing an interaction with an external field in flat space-time. In particular, we have been able to give a seemingly new geometrical interpretation of the concept of spin for the Dirac field; our approach leads directly to a covariant formulation which would be interesting to compare with others.[25]

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