Perturbation technique for quantum fields in curved space

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We derive a concrete expression for the vacuum expectation value of the stress tensor for a massless, nonconformal scalar field propagating in a background spatially flat Robertson-Walker spacetime, up to second order in perturbation theory in the conformal-breaking parameter. The result, which is valid for an arbitrary Robertson-Walker scale factor (subject only to vanishing scalar curvature at some moment) is manifestly nonlocal, yet can still be written as an integral expression in closed form. The method should be extendible to the massive field and anisotropic spacetime cases.

I. INTRODUCTION

The further development of quantum field theory in curved spacetime is hampered by the paucity of models that are exactly soluble in terms of known functions. Most of the tractable cases have been investigated by now, both for computing particle production rates and the expectation values of stress tensors. The soluble models by their very simplicity hide the full nonlocal structure of the theory, which can permit particles produced in one region of spacetime to propagate to another region.

The situation resembles that of atomic physics in the 1920's, and as with that subject the time comes when efficient perturbation and approximation techniques are essential. This is particularly true if quantum stress tensors are to be used in Einstein's equations for back-reaction calculations on the dynamics of the universe, or for evaporating black holes. Stress-tensor expressions for restricted classes of spacetimes are of no help if the induced dynamics of the spacetime causes it to evolve out of that class. This is true of the nonconformal massless scalar field in a Robertson-Walker spacetime. Exact solutions were found in closed form by Bunch and Davies[1] for the case of power-law expansion, but the general evolution of such a model universe would not always remain of this form; this is clearly true if singularity avoidance takes place.

In this paper we present a "once-and-for-all" calculation of the quantum stress tensor for a nonconformal massless scalar field in a spatially flat Robertson-Walker background spacetime that is completely general except for the assumption that the scalar curvature vanishes at some moment in the past. The technique should be extendible to include mass and anisotropy. We use a perturbation technique where the conformal coupling parameter $(\xi-\frac{1}{6})$ is treated as small.[2,3] Our results are developed to second order in this parameter, but higher-order terms should prove easy to calculate if required. The chief difficulty in our calculation is the need to covariantly regularize the stress tensor $\langle T_{\mu\nu}\rangle$ to this order. Higher-order terms are all finite.

We find

$$\begin{aligned}\langle T_{\mu\nu}\rangle = (2880\pi^2)^{-1}\Big\{&-\tfrac{1}{6}\,{}^{(1)}H_{\mu\nu}+{}^{(3)}H_{\mu\nu}+10\Lambda\,{}^{(1)}H_{\mu\nu}+180\Big[\Lambda^2\,{}^{(1)}H_{\mu\nu}(1+\ln\mu C^{1/2})+\Lambda^2 C_{\mu\nu}\\&+\Lambda\mathcal{H}_{\mu\nu}\Big(C^{-1}\int_{-\infty}^{\eta} g'(\eta_1)\ln|\eta-\eta_1|\,d\eta_1\Big)\\&+\tfrac{1}{2}\epsilon_{\mu\nu}C^{-1}g(\eta)\int_{-\infty}^{\eta} g'(\eta_1)\ln|\eta-\eta_1|\,d\eta_1\\&-\tfrac{1}{2}\epsilon_{\mu\nu}C^{-1}\int_{-\infty}^{\eta}d\eta_1\int_{-\infty}^{\eta}d\eta_2\,g'(\eta_1)g'(\eta_2)\ln|\eta_1-\eta_2|\Big]\Big\},\end{aligned}\tag{1.1}$$

where $\Lambda\equiv\xi-\frac{1}{6}$, the $H_{\mu\nu}$ tensors are in standard notation [see Eqs. (4.15) and (4.16)], $g=\Lambda RC$, C is the conformal scale factor, and R is the scalar curvature. The traceless tensor $\epsilon_{\mu\nu}$ has components $e_{00}=1$, $e_{ii}=\frac{1}{3}$. The factor μ is an arbitrary scale factor. The tensor operator $\mathcal{H}_{\mu\nu}$ is a generalization of ${}^{(1)}H_{\mu\nu}$, defined and discussed in Sec. V. The local but nongeometrical tensor $C_{\mu\nu}$ is defined by Eqs. (4.17) and (4.18). The primes denote derivatives with respect to the arguments, and all quantities without explicit arguments are understood to be evaluated at conformal time η. (The result is homogeneous in space.)

A direct calculation shows that (1.1) is covariantly conserved, and has the following trace:

$$(2880\pi^2)^{-1}\Big\{(1-60\Lambda-1080\Lambda^2-1080\Lambda^2\ln\mu C^{1/2})\Box R-(R_{\alpha\beta}R^{\alpha\beta}-\tfrac{1}{3}R^2)+1080\Lambda^2 C_\alpha{}^\alpha - 1080\Lambda^2\Box\left[C^{-1}\frac{\partial}{\partial\eta}\int_{-\infty}^{\eta} g(\eta_1)\ln|\eta-\eta_1|\,d\eta_1\right]\Big\}. \tag{1.2}$$

The lowest order (zeroth order in Λ) is just the famous conformal trace anomaly, the first-order term is purely geometrical, but the second-order term contains a nonlocal contribution, having the form of the integral of a geometrical quantity with a logarithmic kernel over all the past history of the universe back to the "in" region where the vacuum state is defined. The appearance of the nonlocal terms in (1.1) and (1.2) is the reward for all the work invested. That they can be written in such a compact and general form is very gratifying. It is these terms which probe the full nonconformal structure of the theory, and hence take account of particle production as well as vacuum polarization throughout the history of the cosmological model.

The particle creation has been computed in this approximation scheme by Zeldovich and Starobinski,[2] who find for the local rate of particle production per unit volume

$$\frac{\Lambda^2R^2}{288\pi}, \tag{1.3}$$

which is a purely local, geometrical quantity.

The plan of this paper is as follows. In Sec. II we outline the basic model, and in Sec. III we compute the two-point (Green's) function which is the basic quantum object of the theory. Passing from these to the stress tensor is mathematically extremely arduous, but by now a routine procedure in these sorts of calculations. We sketch the details in Sec. IV. Section V is devoted to computing the vital nonlocal terms in the stress tensor.

II. DETAILS OF THE MODEL

We treat a massless scalar field ϕ which satisfies the equation

$$(\Box+\xi R)\phi=0, \tag{2.1}$$

where ξ is an arbitrary parameter. The case $\xi=\frac{1}{6}$ corresponds to conformally invariant coupling, so we shall be interested in the case where $\Lambda\equiv(\xi-\frac{1}{6})\ll 1$.

The background spacetime is chosen to be spatially flat Robertson-Walker, with the metric

$$ds^2=C(\eta)(d\eta^2-dx_1{}^2-dx_2{}^2-dx_3{}^2). \tag{2.2}$$

In this spacetime, Eq. (2.1) possesses normal-mode solutions of the form

$$\phi_k\propto C^{-1/2}e^{i\vec{k}\cdot\vec{x}}\psi_k(\eta), \tag{2.3}$$

where ψ_k satisfies the equation

$$\frac{\partial^2\psi_k}{\partial\eta^2}+(k^2+\Lambda RC)\psi_k=0. \tag{2.4}$$

Henceforth we shall write ΛRC as $g(\eta)$ and assume

$$g(\eta),\ g'(\eta),\ g''(\eta)\to 0 \tag{2.5}$$

at some moment in the past. For example, we could have past asymptotic flatness

$$C(\eta)\to 1 \quad \text{as } \eta\to-\infty \tag{2.6a}$$

or choose the radiation-dominated, big-bang Friedmann universe

$$C(\eta)\propto\eta^2 \quad \text{as } \eta\to 0. \tag{2.6b}$$

In that region of spacetime for which (2.5) holds, Eq. (2.4) possesses standard exponential solutions

$$\psi_k(\eta)\propto e^{-ik\eta}. \tag{2.7}$$

We construct a quantum vacuum state based on these modes, in the usual way. For case (2.6a) this will coincide with the conventional Minkowski-space vacuum of ordinary quantum field theory, and for case (2.6b) we will obtain the usual conformal vacuum.[4] In what follows we shall write $-\infty$ as the lower bound on η, corresponding to (2.6a). The treatment may be used for the other case by simply replacing $-\infty$ by 0.

The essential feature of our treatment is to assume that $g(\eta)$ is small (i.e., $\Lambda\ll 1$), specifically

$$\int_{-\infty}^{\eta} g(\eta_1)(\eta-\eta_1)d\eta_1\ll 1, \tag{2.8}$$

and to seek approximate solutions to (2.4) that reduce to (2.7) in the remote past. This can be achieved by converting (2.4) to the integral equation

$$\psi_k(\eta) = e^{-ik\eta} - k^{-1}\int_{-\infty}^{\eta} g(\eta_1)\psi_k(\eta_1)\sin k(\eta-\eta_1)d\eta \tag{2.9}$$

and solving for ψ_k in a perturbation series in powers of $g(\eta)$. By iteration, we obtain for the first two orders in g

$$\psi_k(\eta) \simeq e^{-ik\eta} - k^{-1}\int_{-\infty}^{\eta} g(\eta_1)e^{-ik\eta_1}\sin k(\eta-\eta_1)d\eta_1 + k^{-2}\int_{-\infty}^{\eta} d\eta_1\int_{-\infty}^{\eta_1} d\eta_2\, g(\eta_1)g(\eta_2)e^{-ik\eta_2} \times \sin k(\eta_1-\eta_2)\sin k(\eta-\eta_1)\,. \tag{2.10}$$

The crucial property of the perturbation series which makes this approach so powerful is that successive terms have greater inverse powers of k. This means that when computing formally divergent field expectation values, such as $\langle T_{\mu\nu}\rangle$, the higher-order perturbation corrections are all finite. Indeed, only quadratic and logarithmic divergences occur in the first two orders. Thus, the orders that we treat are the only ones that require regularization. The finite remainder can be routinely computed with ease to any required order of the perturbation series.

III. CALCULATING THE TWO-POINT FUNCTION

We first calculate the object

$$G(x'', x') \equiv \tfrac{1}{2}\langle \phi(x'')\phi(x') + \phi(x')\phi(x'')\rangle\,, \tag{3.1}$$

where $\langle\ \rangle$ denotes the expectation value in the conventional vacuum state in the "in" region and x denotes the spacetime point (η, x_1, x_2, x_3). Because of the isotropy of the spacetime (2.2) we may choose, without loss of generality, the points x'', x' to lie in the η-x_1 plane. Inserting the appropriate normalization constant in (2.3) and substituting a complete set of mode solutions ϕ_k into (3.1) yields

$$G(x'', x') = \frac{C^{-1/2}(\eta'')C^{-1/2}(\eta')}{8\pi^2|\Delta x_1|} \times \int_0^{\infty} [\psi_k(\eta'')\psi_k^*(\eta') + \psi_k^*(\eta'')\psi_k(\eta')] \times \sin k|\Delta x_1| dk\,, \tag{3.2}$$

where $\Delta x_1 = x_1'' - x_1'$.

From the solution (2.10) we obtain

$$\begin{aligned}\psi_k(\eta'')\psi_k^*(\eta') + \psi_k^*(\eta'')\psi_k(\eta') = {} & \cos k\Delta\eta - \frac{2}{k}\int_{-\infty}^{\eta''} g(\eta_1)\sin k(\eta''-\eta_1)\cos k(\eta'-\eta_1)d\eta_1 \\ & + \frac{2}{k^2}\int_{-\infty}^{\eta''} d\eta_1\int_{-\infty}^{\eta_1} d\eta_2\, g(\eta_1)g(\eta_2)\cos k(\eta_2-\eta')\sin k(\eta_1-\eta_2)\sin k(\eta''-\eta_1) \\ & + \frac{1}{k^2}\int_{-\infty}^{\eta''} d\eta_1\int_{-\infty}^{\eta'} d\eta_2\, g(\eta_1)g(\eta_2)\cos k(\eta_2-\eta_1)\sin k(\eta'-\eta_2)\sin k(\eta''-\eta_1) \\ & + (\eta'' \leftrightarrow \eta')\,,\end{aligned} \tag{3.3}$$

where $\Delta\eta = \eta'' - \eta'$.

Equation (3.3) reveals the second attractive feature of the perturbation technique. All the k integrals in (3.2) consist of products of sin and cos divided by powers of k. All can be reduced to the standard forms

$$\int_0^{\infty} \frac{\sin ax \sin bx}{x}\, dx = \tfrac{1}{2}\ln\left|\frac{a+b}{a-b}\right| \tag{3.4}$$

and

$$\int_0^{\infty} \frac{\sin ax \sin bx \sin cx}{x^2}\, dx = \tfrac{1}{4}(a+b)\ln\left|\frac{a+b+c}{a+b-c}\right| + \tfrac{1}{4}c\ln|(a+b)^2-c^2| - (b \to -b)\,, \tag{3.5}$$

where in (3.5) $a \geq b > 0$, $c > 0$. Thus, the major obstacle to computing concrete expressions for $\langle T_{\mu\nu}\rangle$, namely, evaluating the mode integrals in terms of known functions, is surmounted. It is this fact which enables us to calculate $\langle T_{\mu\nu}\rangle$ for a *general* Robertson-Walker spacetime, (i.e., a general function C).

The first term on the right of (3.3) is the zeroth-order, conformally trivial term, treated in detail by Davies *et al.*[4] We shall not repeat that work in what follows, but shall use some of their results. The second and third terms of (3.3) may be rearranged as follows:

$$-\frac{2}{k}\int_{-\infty}^{\bar\eta} g(\eta_1)\sin 2k(\bar\eta-\eta_1)d\eta_1+\frac{4}{k^2}\int_{-\infty}^{\bar\eta}d\eta_1\int_{-\infty}^{\eta_1}d\eta_2\, g(\eta_1)g(\eta_2)\sin k(\bar\eta+\eta_1-2\eta_2)\sin k(\bar\eta-\eta_1)$$
$$+\frac{4}{k^2}\int_{-\infty}^{\bar\eta}d\eta_1\int_{-\infty}^{\eta_1}d\eta_2\, g(\eta_1)g(\eta_2)\cos 2k(\eta_1-\eta_2)\sin^2\left(\frac{k\Delta\eta}{2}\right)$$
$$+\text{infinitesimal integrals which vanish as } \Delta\eta\to 0\,, \tag{3.6}$$

where $\bar\eta=\frac{1}{2}(\eta''+\eta')$, and we defer discussion of a number of integrals of the form $\int_{\eta}^{\eta+\Delta\eta/2}$ until later. Henceforth, we assume that the separation between the points x'' and x' is small, so that $\Delta\eta$ and Δx_1 are infinitesimal quantities. Note that the first two terms in (3.6) are the only terms to survive in the limit of spacelike point separation, $\Delta\eta\to 0$. To arrive at (3.6) we have used the property

$$\int_{-\infty}^{\bar\eta}\int_{-\infty}^{\eta_1} f(\eta_1)f(\eta_2)d\eta_1 d\eta_2=\frac{1}{2}\int_{-\infty}^{\bar\eta}\int_{-\infty}^{\bar\eta} f(\eta_1)f(\eta_2)d\eta_1 d\eta_2 \tag{3.7}$$

to collapse together the two sets of double integrals.

Expression (3.6) is now substituted into (3.2), the order of integration interchanged, and the k integrals evaluated. The first term yields

$$-\frac{C^{-1/2}(\eta'')C^{-1/2}(\eta')}{8\pi^2|\Delta x_1|}\int_{-\infty}^{\bar\eta} g(\eta_1)\ln\left|\frac{\bar\eta-\eta_1+|\Delta x_1|/2}{\bar\eta-\eta_1-|\Delta x_1|/2}\right|d\eta_1. \tag{3.8}$$

The second-order terms are more complicated and require some thoughtful manipulation to avoid infrared divergences. We omit the details here.

As $|\Delta x_1|$ is small, expression (3.8) may be expanded in a power series. To avoid divergent η_1 integrals, three successive integrations by parts must first be performed. In the end we arrive at

$$\frac{C^{-1/2}(\eta'')C^{-1/2}(\eta')}{16\pi^2}\left[2g(\bar\eta)\ln\left|\frac{\Delta x_1}{2}\right|-2g(\bar\eta)-2\int_{-\infty}^{\bar\eta} g'(\eta_1)\ln|\eta-\eta_1|\,d\eta_1+\frac{(\Delta x_1)^2}{12}g''(\bar\eta)\left(\ln\left|\frac{\Delta x_1}{2}\right|-\frac{11}{6}\right)\right.$$
$$\left.-\frac{(\Delta x_1)^2}{12}\int_{-\infty}^{\bar\eta} g'''(\eta_1)\ln|\bar\eta-\eta_1|\,d\eta_1\right]+O((\Delta x_1)^4). \tag{3.9}$$

This is the first-order correction to the conformal two-point function when the point separation is restricted to be spacelike. The equivalent second-order expression is similar, but contains double integral terms as well.

However, we are not finished yet. The full two-point function G requires the additional timelike pieces. These arise from the third term in (3.6) and from the infinitesimal integrals omitted from that expression, which are

$$-\frac{2}{k}\int_{\bar\eta}^{\bar\eta+\Delta\eta/2} g(\eta_1)\sin k\left(\bar\eta-\eta_1+\frac{\Delta\eta}{2}\right)\cos k\left(\bar\eta-\eta_1-\frac{\Delta\eta}{2}\right)d\eta_1$$
$$+\frac{1}{k^2}\int_{\bar\eta}^{\bar\eta+\Delta\eta/2}d\eta_1\int_{\bar\eta}^{\bar\eta-\Delta\eta/2}d\eta_2\, g(\eta_1)g(\eta_2)\cos k(\eta_2-\eta_1)\sin k\left(\bar\eta-\eta_2-\frac{\Delta\eta}{2}\right)\sin k\left(\bar\eta-\eta_1+\frac{\Delta\eta}{2}\right)$$
$$+\frac{2}{k^2}\int_{\bar\eta}^{\bar\eta+\Delta\eta/2}d\eta_1\int_{-\infty}^{\eta_1}d\eta_2\, g(\eta_1)g(\eta_2)\cos k\left(\eta_2-\bar\eta+\frac{\Delta\eta}{2}\right)\sin k\left(\bar\eta-\eta_1+\frac{\Delta\eta}{2}\right)\sin k(\eta_1-\eta_2)+(\Delta\eta\to-\Delta\eta)\,. \tag{3.10}$$

Once again, the expression is substituted into (3.2) and the k integration is performed first. As a result, a variety of logarithmic functions replace the trigonometric factors in (3.10). These, and the g factors, are then laboriously expanded in Taylor series about $\bar\eta$ up to order $(\Delta\eta)^2$, with integrations by parts over η_1 and η_2 performed if necessary. Eventually, the η integrations can be carried out explicitly. The final term of (3.10), which involves an infinite integral also, must be handled carefully. The integral range is first broken into $(-\infty,\bar\eta)$ and $(\bar\eta,\eta_1)$, enabling the order of integration of the former piece to be reversed and yielding a single integral term in the final answer. The other piece is a double infinitesimal piece, similar to the second term of (3.9). These long and tedious manipulations will not be reproduced here. Their effect is to replace $(\Delta x_1)^2$ in the logarithms by $|\Delta\eta^2-\Delta x_1{}^2|$ and to contribute some additional terms proportional to $\Delta\eta^2$.

The final expression for the two-point function up to order Λ^2, $\Delta\eta^2$, and $\Delta\vec{x}^2$ turns out to be

$$G(x'',x')=\frac{C^{-1/2}(\eta'')C^{-1/2}(\eta')}{16\pi^2}\Bigg\{-\frac{4}{\Delta\eta^2-\Delta\vec{x}^2}+[g+\tfrac{1}{24}g''\Delta\vec{x}^2-\tfrac{1}{8}g^2(\Delta\eta^2-\Delta\vec{x}^2)]\ln\tfrac{1}{4}|\Delta\eta^2-\Delta\vec{x}^2|$$
$$-2g+\Delta\eta^2(\tfrac{1}{24}g''+\tfrac{3}{8}g^2)-\tfrac{11}{24}\Delta\vec{x}^2(\tfrac{1}{3}g''+g^2)$$
$$+\int_{-\infty}^{\bar\eta}[-2g'(\eta_1)+\tfrac{1}{2}\Delta\eta^2 g(\bar\eta)g'(\eta_1)-\tfrac{1}{2}\Delta\vec{x}^2 g(\eta_1)g'(\eta_1)-\tfrac{1}{12}\Delta\vec{x}^2 g'''(\eta_1)]\ln|\bar\eta-\eta_1|\,d\eta_1$$
$$+\int_{-\infty}^{\bar\eta}d\eta_1\int_{-\infty}^{\eta_1}d\eta_2[4g(\eta_1)g(\eta_2)+\tfrac{1}{6}\Delta\vec{x}^2 g(\eta_1)g''(\eta_2)]\ln\left|\frac{\eta-\eta_2}{\eta_1-\eta_2}\right|$$
$$-\tfrac{1}{2}\Delta\eta^2\int_{-\infty}^{\bar\eta}d\eta_1\int_{-\infty}^{\eta_1}d\eta_2 g(\eta_1)g''(\eta_2)\ln|\eta_1-\eta_2|\Bigg\}, \tag{3.11}$$

where all quantities without explicit arguments are understood to be evaluated at $\bar\eta$.

The first term in the braces of Eq. (3.11) is the familiar expression for the zeroth-order (conformal) two-point function (see for example Davies *et al.*[4]). It is quadratically divergent as $\Delta\eta$ and $\Delta\vec{x}\to 0$, leading to a quartic divergence in the stress tensor. The second term contains logarithmic divergences leading to quadratic and logarithmically divergent terms in the stress tensor. These terms are all, as expected, local objects. The remaining terms are finite as $\Delta\eta$ and $\Delta\vec{x}\to 0$. It is necessary to compute G to order $\Delta\eta^2$ and $\Delta\vec{x}^2$ to obtain the finite terms of $\langle T_{\mu\nu}\rangle$ by differentiation.

A useful check on the algebra leading to (3.11) can be made by substituting G into Eq. (2.1) and ensuring that it is satisfied order by order in Λ.

IV. CONSTRUCTING THE STRESS TENSOR

The most convenient method of obtaining $\langle T_{\mu\nu}\rangle$ from (3.11) is the method of geodesic point separation.[4-6] We assume the points x'', x' lie a proper distance 2ϵ apart along a geodesic in the background spacetime, whose unit tangent vector at the midpoint is t^μ. This midpoint, which we shall label $(\eta,\vec{x})$, is the spacetime point of interest where $\langle T_{\mu\nu}(x)\rangle$ is to be evaluated. It is *not* the same as the point $(\bar\eta,\vec{x})$, but differs from it by a factor of order ϵ^2.

To effect the point-separation procedure, it is first necessary to both transform (3.11) from $(\bar\eta,\vec{x})$ to (ϵ,t^μ) coordinates and to evaluate all functions at η rather than $\bar\eta$. This involves a double expansion, for which we need the following series:

$$\bar\eta=\eta-\tfrac{1}{4}\epsilon^2 D[(t^0)^2+(t^1)^2]\ , \tag{4.1}$$

$$C^{-1/2}(\eta'')C^{-1/2}(\eta')=C^{-1}(\eta)\{1+\epsilon^2[(-\tfrac{1}{2}\dot D+\tfrac{1}{4}D^2)(t^0)^2+\tfrac{1}{4}D^2(t^1)^2]\}, \tag{4.2}$$

$$\Delta\eta^2=4\epsilon^2(t^0)^2[1+\tfrac{1}{3}\epsilon^2(t^0)^2(-\tfrac{1}{2}\dot D+\tfrac{1}{2}D^2)+\tfrac{1}{3}\epsilon^2(t^1)^2(-\tfrac{1}{2}\dot D+\tfrac{3}{2}D^2)]\,, \tag{4.3}$$

$$\Delta x_1^{\,2}=4\epsilon^2(t^1)^2[1+\tfrac{1}{3}\epsilon^2(t^0)^2(-\dot D+\tfrac{3}{2}D^2)+\tfrac{1}{6}(t^1)^2\epsilon^2 D^2]\,, \tag{4.4}$$

where $D\equiv D(\eta)\equiv\dot C C^{-1}\equiv C^{-1}\partial C/\partial\eta$, and the t^μ vector is restricted to the $(0,1)\equiv(\eta,x_1)$ plane, as explained.

Substituting (4.1)–(4.4) into (3.11) one obtains, after more tedious algebra,

$$[16\pi^2C(\eta)]^{-1}\Bigg\{\text{conformal term}+g\ln(\epsilon^2C^{-1})-2g+\epsilon^2(t^0)^2[-(\tfrac{1}{2}\dot D-\tfrac{1}{4}D^2)g-\tfrac{1}{4}Dg'-\tfrac{1}{2}g^2]\ln\epsilon^2C^{-1}$$
$$+\epsilon^2(t^1)^2(\tfrac{1}{4}D^2g-\tfrac{1}{4}Dg'+\tfrac{1}{6}g''+\tfrac{1}{2}g^2)\ln\epsilon^2C^{-1}+\epsilon^2(t^0)^2[(\tfrac{5}{6}\dot D-\tfrac{1}{3}D^2)g+\tfrac{1}{2}Dg'+\tfrac{1}{6}g''+\tfrac{3}{2}g^2]$$
$$+\epsilon^2(t^1)^2(-\tfrac{1}{3}D^2g+\tfrac{1}{2}Dg'-\tfrac{11}{18}g''-\tfrac{11}{6}g^2)$$
$$-2\int_{-\infty}^{\eta}d\eta_1\ln|\eta-\eta_1|\left[1+\epsilon^2(t^0)^2\left(-\tfrac{1}{2}\dot D+\tfrac{1}{4}D^2-\tfrac{1}{4}D\frac{\partial}{\partial\eta_1}-g\right)+\epsilon^2(t^1)^2\left(\tfrac{1}{4}D^2-\tfrac{1}{4}D\frac{\partial}{\partial\eta_1}+\frac{1}{6}\frac{\partial^2}{\partial\eta_1^{\,2}}+g(\eta_1)\right)\right]g'(\eta_1)$$
$$+\epsilon^2D[(t^0)^2+(t^1)^2]\int_{-\infty}^{\eta}g^2(\eta_1)\ln|\eta-\eta_1|\,d\eta_1$$
$$+4\int_{-\infty}^{\eta}d\eta_1\int_{-\infty}^{\eta_1}d\eta_2\left[1+\epsilon^2(t^0)^2\left(-\tfrac{1}{2}\dot D+\tfrac{1}{4}D^2-\tfrac{1}{4}D\frac{\partial}{\partial\eta_2}\right)+\epsilon^2(t^1)^2\left(\tfrac{1}{4}D^2-\tfrac{1}{4}D\frac{\partial}{\partial\eta_2}+\frac{1}{6}\frac{\partial^2}{\partial\eta_2^{\,2}}\right)\right]\ln|\eta-\eta_2|g(\eta_1)g(\eta_2)$$
$$-4\int_{-\infty}^{\eta}d\eta_1\int_{-\infty}^{\eta_1}d\eta_2\left[1+\epsilon^2(t^0)^2\left(-\tfrac{1}{2}\dot D+\tfrac{1}{4}D^2+\frac{1}{2}\frac{\partial^2}{\partial\eta_2^{\,2}}\right)+\epsilon^2(t^1)^2\left(\tfrac{1}{4}D^2+\frac{1}{6}\frac{\partial^2}{\partial\eta_2^{\,2}}\right)\right]\ln|\eta_1-\eta_2|\,d\eta_1\,d\eta_2\Bigg\}. \tag{4.5}$$

We do not bother to write out the conformal term, as it has been treated elsewhere. All terms whose arguments are not explicit are understood to be evaluated at η.

The calculation has now reached a nadir of complexity. It is not manifestly covariant, and considerable simplification occurs if we make use of the explicit definition of g as ΛRC, written in terms of D, $\dot{D}$, etc., and group the various combinations of D's into known geometrical objects. These will be $R_{;\mu\nu}$, $\Box R g_{\mu\nu}$, $R^{\alpha\beta}R_{\alpha\beta}g_{\mu\nu}$, $R^2 g_{\mu\nu}$, and $R_{\alpha\mu}R^{\alpha}{}_{\nu}$. Expressions for these geometrical objects in terms of D and its derivatives are given in the appendices of Refs. 4 and 7, and it is a straightforward matter to fit the coefficients in expression (4.5). As a final check on the algebra, we applied the wave equation (2.1) again, but in the ϵ, t^{μ} coordinates. This is easily accomplished by taking the trace of what is called $\langle T_{\mu\nu}^{(3)}\rangle$ in Refs. 4 and 7, computed using (4.5), and adding to it the ξR term. The relevant formula for $\langle T_{\mu\nu}^{(3)}\rangle$ is given in Ref. 7.

To discuss the result, we shall first consider the local pieces of (4.5). These can be written

$$(16\pi)^{-1}\{\text{conformal term}+\Lambda[R\ln(\epsilon^2C^{-1})-2R+\tfrac{1}{6}(-2RR_{\alpha\beta}t^\alpha t^\beta+R_{;\alpha\beta}t^\alpha t^\beta-\Box R-3R^2\Lambda)\epsilon^2\ln(\epsilon^2C^{-1}) +\epsilon^2A_{\alpha\beta}t^\alpha t^\beta+\epsilon^2\Lambda B_{\alpha\beta}t^\alpha t^\beta]\}\,, \quad (4.6)$$

where the local, but nongeometrical, tensors $A_{\mu\nu}$ and $B_{\mu\nu}$ have the following components in the (η, x_1) coordinates:

$$A_{00}=C^{-1}(\tfrac{1}{2}D+2\ddot{D}D+3\dot{D}^2+\tfrac{7}{4}\dot{D}D^2-\tfrac{1}{2}D^4)\,, \quad (4.7)$$
$$A_{11}=C^{-1}(-\tfrac{11}{6}D-\tfrac{1}{3}\ddot{D}D-\tfrac{11}{6}\dot{D}^2+\tfrac{1}{2}\dot{D}D^2-\tfrac{1}{2}D^4)\,,$$

$$B_{00}=\tfrac{27}{2}C^{-1}(\dot{D}^2+\dot{D}D^2+\tfrac{1}{4}D^4)\,,$$
$$B_{11}=-\tfrac{33}{2}C^{-1}(\dot{D}^2+\dot{D}D^2+\tfrac{1}{4}D^4)\,. \quad (4.8)$$

We are now ready to compute the geometrical piece of $\langle T_{\mu\nu}(x)\rangle$. To do this involves differentiating (4.6) and letting $\epsilon\to 0$. This is by far the hardest part of the calculation, as the process is not just conventional straightforward differentiation. Fortunately, however, the procedure has been done once and for all in earlier work[7] for a completely general $G(x'', x')$, and so we merely have to plug (4.6) into the general formula and read off $\langle T_{\mu\nu}\rangle$. Because of the $\ln\epsilon^2$ terms in (4.6), the result is both quadratically and logarithmically divergent. This was expected, and a useful check on this part of our calculation is to ensure that the divergent terms are identical to those calculated by Christensen[5] for an arbitrary spacetime. They are.

To obtain a finite (regularized) $\langle T_{\mu\nu}\rangle$, it is necessary to subtract Christensen's two-point function, which is purely local and geometrical, from our G (see, for example, Refs. 7 and 8). The result is

$$(16\pi^2)^{-1}[c+\epsilon^2(e_{\alpha\beta}t^\alpha t^\beta+f)]\,, \quad (4.9)$$

where

$$c=-\tfrac{1}{18}R-\Box R(2+\ln\mu^2C)\,, \quad (4.10)$$

$$e_{\alpha\beta}=\tfrac{1}{180}(-2R_\alpha{}^\lambda R_{\lambda\beta}+\tfrac{16}{3}RR_{\alpha\beta}-R_{;\alpha\beta}) +\Lambda[A_{\alpha\beta}+\Lambda B_{\alpha\beta}+\tfrac{1}{6}\ln\mu^2C(2RR_{\alpha\beta}+R_{;\alpha\beta})]\,, \quad (4.11)$$

$$f=\tfrac{1}{180}(-\tfrac{2}{3}R^2+\Box R+R^{\alpha\beta}R_{\alpha\beta}) +\Lambda[(\tfrac{1}{6}\Box R+\tfrac{1}{2}\Lambda R^2)\ln\mu^2C-\tfrac{1}{6}\Box R-\tfrac{1}{2}\Lambda R^2]\,. \quad (4.12)$$

In arriving at expressions (4.10)–(4.12) we have inserted the results of Davies *et al.*[4] for the conformal term. The factor μ^2 is an arbitrary scale factor which always arises in the regularization of a nonconformal massless field stress tensor.

The final step is to plug (4.9)–(4.12) into the formula for the stress tensor, as given by Bunch.[7] This is

$$\langle T_{\mu\nu}\rangle=(\tfrac{1}{12}-\Lambda)c_{;\mu\nu}+(\Lambda+\tfrac{1}{24})g_{\mu\nu}\Box c-(\Lambda+\tfrac{1}{6})cG_{\mu\nu} -\tfrac{1}{2}e_{\mu\nu}+\tfrac{1}{4}e_\alpha{}^\alpha g_{\mu\nu}+\tfrac{1}{2}fg_{\mu\nu}\,. \quad (4.13)$$

After some algebra, we find, for the local piece of $\langle T_{\mu\nu}\rangle$ up to order Λ^2,

$$(2880\pi^2)^{-1}\{-\tfrac{1}{6}\,{}^{(1)}H_{\mu\nu}+{}^{(3)}H_{\mu\nu}+10\Lambda\,{}^{(1)}H_{\mu\nu} +180\Lambda^2[\tfrac{1}{2}\,{}^{(1)}H_{\mu\nu}\ln(\mu^2C)+{}^{(1)}H_{\mu\nu}+C_{\mu\nu}]\}\,, \quad (4.14)$$

where

$${}^{(1)}H_{\mu\nu}=2R_{;\mu\nu}-2(\Box R)g_{\mu\nu}+2(RR_{\mu\nu}-\tfrac{1}{4}R^2g_{\mu\nu}) \quad (4.15)$$

and

$${}^{(3)}H_{\mu\nu}=-R^{\alpha\beta}R_{\alpha\mu\beta\nu}+\tfrac{1}{12}R^2g_{\mu\nu} \quad (4.16)$$

are known to the only *geometrical* conserved tensors of this adiabatic order in the Robertson-

Walker spacetime.[4] The tensor $C_{\mu\nu}$ is local, but nongeometrical. Its appearance in the second-order term arises because of the presence of the $\ln C$ term, which gives a local but nongeometrical expression when covariantly differentiated. Thus, without $C_{\mu\nu}$ the local Λ^2 term would not be conserved. The components of $C_{\mu\nu}$ in these coordinates are

$$C_{00}=C^{-1}(-\tfrac{9}{2}\dot{D}D^2-\tfrac{9}{4}D^4)\,, \tag{4.17}$$

$$C_{11}=C^{-1}(6\ddot{D}D+\tfrac{9}{2}\dot{D}^2+\tfrac{9}{2}\dot{D}D^2-\tfrac{15}{8}D^4)\,. \tag{4.18}$$

The conformal term in (4.14) is the by now familiar expression.[4] The first-order term, curiously, does not contain ${}^{(3)}H_{\mu\nu}$ or logarithmic terms. Also, by fortune, the nongeometrical pieces which come from differentiating $\ln C$ in (4.13) combine with $A_{\mu\nu}$ to give a purely geometrical result. This does not happen in the Λ^2 term.

In arriving at (4.14) we have also had to take account of the celebrated conformal anomaly.[9] Although the theory is not conformally invariant above zeroth order, there will still be "anomalous" terms to order Λ and Λ^2. These are given from Christensen's two-point function[5] as

$$-(2880\pi^2)^{-1}(15\Lambda\,\Box R+45\Lambda^2R^2)g_{\mu\nu}\,, \tag{4.19}$$

and have been included in (4.14) along with the other terms from (4.10)–(4.12) and, of course, the terms arising from the familiar zeroth-order anomaly.

V. THE NONLOCAL PIECES

We now return to (4.5) and deal with the more interesting nonlocal pieces which must be separately conserved.

Consider the first two nonlocal terms in (4.5). They may be written

$$-\frac{\Lambda}{8\pi^2C}\int_{-\infty}^{\eta}d\eta_1\ln|\eta-\eta_1|\,\frac{\partial}{\partial\eta_1}\Big[1+\epsilon^2(t^0)^2\Big(-\tfrac{1}{2}\dot{D}+\tfrac{1}{4}D^2-\tfrac{1}{4}D\frac{\partial}{\partial\eta_1}-g(\eta)\Big)$$
$$+\epsilon^2(t^1)^2\Big(\tfrac{1}{4}D^2-\tfrac{1}{4}D\frac{\partial}{\partial\eta_1}+\frac{1}{6}\frac{\partial^2}{\partial\eta_1{}^2}+g(\eta_1)\Big)\Big]R(\eta_1)C(\eta_1)\,. \tag{5.1}$$

First we note that

$$\frac{\partial}{\partial\eta}\int_{-\infty}^{\eta}f(\eta_1)\ln|\eta-\eta_1|\,d\eta_1=\int_{-\infty}^{\eta}f'(\eta_1)\ln|\eta-\eta_1|\,d\eta_1\,, \tag{5.2}$$

which may be proved by first integrating the logarithm by parts. Moreover, as (5.2) is a function of η only, any spacelike derivatives may also be taken through, formally, onto $f(\eta_1)$, along with any Christoffel symbols that arise from covariant differentiation, as they are all evaluated at η. These quantities also commute with the $\partial/\partial\eta_1$ derivative in (5.1). Hence we may apply formula (4.13) directly to the expression in square brackets in (5.1), inside the integral.

When the Λ-independent part of (4.13) is applied to the Λ-independent terms inside the square brackets in (5.1) we obtain a surprising result—zero. In fact, this can be anticipated by noting that the $\ln\mu^2$ factors in the geometrical piece can be absorbed into the $\ln|\eta-\eta_1|$ factor of (5.1): The coefficients are identical to that of the local logarithmic pieces. Therefore, inspecting (5.1) we see that with μ replacing $|\eta-\eta_1|$ we may integrate trivially and obtain a coefficient of $\ln\mu^2$ identical to that of the $\ln C$ local term which, as mentioned in the previous section, vanishes in $\langle T_{\mu\nu}\rangle$ to order Λ.

However, there will also be a contribution from (5.1) to order Λ^2. The Λ terms in (4.13) are

$$-\Lambda(\nabla_\mu\nabla_\nu-g_{\mu\nu}\Box+R_{\mu\nu}-\tfrac{1}{2}Rg_{\mu\nu})c\,, \tag{5.3}$$

so, if $c=\Lambda R$, then comparison with (4.15) shows that (5.3) is

$$-\tfrac{1}{2}\Lambda^2\,{}^{(1)}H_{\mu\nu}+\tfrac{1}{4}\Lambda^2R^2g_{\mu\nu}\,. \tag{5.4}$$

If, on the other hand, $c=\Lambda R(\eta_1)C(\eta_1)C^{-1}(\eta)$, as is the case in (5.1), then we must replace ${}^{(1)}H_{\mu\nu}$ by the object

$$2(\nabla_\mu\nabla_\nu-g_{\mu\nu}\Box+R_{\mu\nu}-\tfrac{1}{4}Rg_{\mu\nu})\times\Big(C^{-1}\int_{-\infty}^{\eta}g'(\eta_1)\ln|\eta-\eta_1|\,d\eta_1\Big)\,. \tag{5.5}$$

It is convenient to abbreviate the operator in small parentheses as $\mathcal{H}_{\mu\nu}$ and to regard (5.5) as a sort of nonlocal generalization of ${}^{(1)}H_{\mu\nu}$. Note that

$$\mathcal{H}_{\mu\nu}[R(\eta)]={}^{(1)}H_{\mu\nu}(\eta)\,. \tag{5.6}$$

Next consider the first double integral term in

(4.5). We note the following property:

$$\frac{\partial}{\partial\eta}\int_{-\infty}^{\eta} d\eta_1 \int_{-\infty}^{\eta_1} d\eta_2 f(\eta_1) f(\eta_2) \ln|\eta-\eta_2|$$
$$= \int_{-\infty}^{\eta} d\eta_1 \int_{-\infty}^{\eta_1} d\eta_2 f(\eta_1) f'(\eta_2) \ln|\eta-\eta_2| + f(\eta)\int_{-\infty}^{\eta} f(\eta_1)\ln|\eta-\eta_1|\, d\eta_1 - \int_{-\infty}^{\eta} f^2(\eta_1)\ln|\eta-\eta_1|\, d\eta_1 . \quad (5.7)$$

Thus, as far as the double integral part of the above result is concerned, the effect of $\partial/\partial\eta$ is to operate on $f(\eta_2)$ as $\partial/\partial\eta_2$ in the integrand. This is the exact analog of (5.2), and a comparison of the first double integral with (5.1) shows that the operators inside the square brackets are identical to first order (i.e., ignoring the g terms in the latter). Hence we may apply the result found in the single integral case, namely, that the contribution of this integral to the above double integral part of $\langle T_{\mu\nu}\rangle$ is zero. There will, of course, be some single integral pieces that originate from the final two terms in (5.7).

Most of the final term in (4.5) also gives zero when (4.13) is applied. The only surviving double integral piece to order Λ^2 in $\langle T_{\mu\nu}\rangle$ is, in $(\eta, \vec{x})$ coordinates,

$$-\frac{1}{32\pi^2 C(\eta)}\begin{bmatrix} 1 & & & \\ & \frac{1}{3} & & \\ & & \frac{1}{3} & \\ & & & \frac{1}{3}\end{bmatrix}\int_{-\infty}^{\eta}\int_{-\infty}^{\eta} g'(\eta_1) g'(\eta_2) \times \ln|\eta_1-\eta_2|\, d\eta_1\, d\eta_2 , \quad (5.8)$$

which is actually traceless. In arriving at (5.8) we have performed an integration by parts, and used the property (3.7) to symmetrize the double integral.

The only remaining contributions to $\langle T_{\mu\nu}\rangle$ are single integral pieces. These come from the final single integral in (4.5), from single integrals of the type shown in (5.7), from the extra term corresponding to the second term of (5.4), from the $g(\eta)$ and $g(\eta_1)$ terms in (5.1), and from the partial integration leading to (5.8). Combining all these pieces together results in much cancellation, leaving only

$$\frac{1}{32\pi^2 C(\eta)}\begin{bmatrix} 1 & & & \\ & \frac{1}{3} & & \\ & & \frac{1}{3} & \\ & & & \frac{1}{3}\end{bmatrix} g(\eta)\int_{-\infty}^{\eta} d\eta_1 g'(\eta_1)\ln|\eta-\eta_1| . \quad (5.9)$$

The sum of (4.14), (5.5), (5.8), and (5.9) yields the full stress tensor up to order Λ^2 which we gave in Sec. I.

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[1]T. S. Bunch and P. C. W. Davies, J. Phys. A 11, 1315 (1978).

[2]Ya. B. Zeldovich and A. A. Starobinsky, Zh. Eksp. Teor. Fiz. Pis'ma Red. (to be published).

[3]M. V. Fischetti, J. B. Hartle, and B. L. Hu, Phys. Rev. D (to be published); J. B. Hartle and B. L. Hu, *ibid.* (to be published).

[4]P. C. W. Davies, S. A. Fulling, S. M. Christensen, and T. S. Bunch, Ann. Phys. (N.Y.) 109, 108 (1977).

[5]S. M. Christensen, Phys. Rev. D 14, 2490 (1976).

[6]S. L. Adler, J. Lieberman, and Y. J. Ng, Ann. Phys. (N.Y.) 106, 279 (1977).

[7]T. S. Bunch, Ph.D. thesis, King's College, London, 1977 (unpublished).

[8]T. S. Bunch and P. C. W. Davies, Proc. R. Soc. London A354, 59 (1978).

[9]D. M. Capper and M. J. Duff, Nuovo Cimento A23, 173 (1974); S. Deser, M. J. Duff, and C. J. Isham, Nucl. Phys. B111, 45 (1976).