Self-interaction of charged particles in the gravitational field

A. Vilenkin
Department of Physics, Case Western Reserve University, Cleveland, Ohio 44106
and Department of Physics, Tufts University, Medford, Massachusetts 02155
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The gravitationally induced self-interaction force is calculated at a large distance from a Schwarzschild black hole. If, instead of the electromagnetic field, the particle is coupled to a vector-meson field of vanishingly small, but nonzero mass, then it is shown that the self-force has the same magnitude but opposite direction. A sharp difference between massive and massless vector fields is a result of different boundary conditions at the horizon surface.

I. INTRODUCTION

The motion of a charged particle in the gravitational field is described by the equation[1,2]

$$m\frac{Du^\mu}{ds}=f_R^\mu+f_G^\mu\,, \tag{1}$$

where u^μ is the four-velocity of the particle,

$$f_R^\mu=\frac{2e^2}{3}\left(\frac{D^2u^\mu}{ds^2}-\frac{Du^\nu}{ds}\frac{Du_\nu}{ds}u^\mu\right) \tag{2}$$

is the radiation-reaction force, and f_G^μ is the gravitationally induced force of self-interaction. To explain the origin of the force f_G^μ, we note that Maxwell's equations in a curved space-time are analogous to the equations in a medium with dielectric and magnetic susceptibilities which are functions of coordinates and time (see, e.g., Ref. 3). The electromagnetic field of the particle polarizes the "medium," and this polarization results in an additional force on the particle, f_G^μ. It is clear that f_G^μ is a nonlocal function of the metric and of the previous trajectory of the particle.

DeWitt and DeWitt[4] calculated f_G^μ using the formalism developed by DeWitt and Brehme.[1] They considered a nonrelativistic charge in a weak static gravitational field. In harmonic coordinates,[5] the metric is given by

$$ds^2=(1+2U)dt^2-(1-2U)d\vec{x}^2\,, \tag{3}$$

where

$$U(\vec{x})=-\int\frac{\mu(\vec{x}\,')d\vec{x}\,'}{|\vec{x}-\vec{x}\,'|} \tag{4}$$

is the gravitational potential, $U(\vec{x})\ll 1$, and $\mu(\vec{x})$ is the mass density. In this case Eq. (1) takes the form

$$m(\dot{\vec{v}}+\vec{\nabla}U)=\vec{f}_R+\vec{f}_G\,, \tag{5}$$

where

$$\vec{f}_R=\tfrac{2}{3}e^2\ddot{\vec{v}} \tag{6}$$

is the nonrelativistic limit of Eq. (2) and[5,6]

$$\vec{f}_G=e^2\int\frac{(\vec{x}-\vec{x}\,')}{|\vec{x}-\vec{x}\,'|^4}\mu(\vec{x}\,')d\vec{x}\,'\,. \tag{7}$$

Note that $\vec{f}_G$ is not zero, even if the charge is at rest.

The purpose of this paper is to calculate $\vec{f}_G$ in the field of a Schwarzschild black hole at a large distance from the hole ($r\gg M$, where M is the mass of the hole). The result is

$$\vec{f}_G=e^2\frac{M\vec{x}}{r^4} \tag{8}$$

(the black hole is at the origin). It should be emphasized that Eq. (8) is not a trivial consequence of Eq. (7), since the latter equation has been obtained assuming that the gravitational field is weak everywhere, which is not so in the case of a black hole.

It will be shown also that $\vec{f}_G$ changes sign if the electric charge e is replaced by a source of a massive vector field having a vanishingly small but nonzero mass. A sharp difference between massive and massless vector fields is a result of different boundary conditions[7,8] at the horizon surface $r=M$.

II. WEAK STATIC GRAVITATIONAL FIELDS[9]

In this section we shall rederive the result of DeWitt and DeWitt [Eq. (7)] in a less rigorous but much simpler way. For simplicity, we shall consider a point charge at rest in a weak static gravitational field [(3) and (4)]. The electromagnetic field of the charge can be found from the Maxwell equations

$$(F^{\mu\nu}\sqrt{-g}\,)_{,\nu}=4\pi j^\mu\sqrt{-g}\,, \tag{9}$$

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu\,, \tag{10}$$

with

$$j^0\sqrt{-g}=e\delta(\vec{x}-\vec{x}_0),\quad j^k=0,\quad A^k=0.$$

Substituting Eqs. (3) and (4) in (9) and (10) and neglecting second and higher powers of U we get

$$(1-2U)\vec{\nabla}^2 A_0-2\vec{\nabla}U\cdot\vec{\nabla}A_0=-4\pi e\delta(\vec{x}-\vec{x}_0) \tag{11}$$

or, in the same approximation,

$$\vec{\nabla}^2 A_0-2\vec{\nabla}U\cdot\vec{\nabla}A_0=-4\pi e(1+2U_0)\delta(\vec{x}-\vec{x}_0). \tag{12}$$

Here

$$U\equiv U(\vec{x}),\quad U_0\equiv U(\vec{x}_0).$$

Now let us write A_0 in the form

$$A_0=(1+U_0+U)\frac{e}{\rho}+\psi(\vec{x}), \tag{13}$$

where $\rho=|\vec{x}-\vec{x}_0|$. Substituting this equation in Eq. (12) we get an equation for $\psi(\vec{x})$:

$$\vec{\nabla}^2\psi=-\frac{e}{\rho}\vec{\nabla}^2 U=-\frac{4\pi e}{\rho}\mu(\vec{x}), \tag{14}$$

where $\mu(\vec{x})$ is the mass density. ($\vec{\nabla}U\cdot\vec{\nabla}\psi$ is neglected because ψ is of order U, as will be clear *a posteriori*.) The solution of Eq. (14) is readily found in the form of the Poisson integral, and we obtain the final expression for A_0:

$$A_0=[1+U(\vec{x}_0)+U(\vec{x})]\frac{e}{|\vec{x}-\vec{x}_0|}+e\int\frac{\mu(\vec{x}')d\vec{x}'}{|\vec{x}-\vec{x}'|\,|\vec{x}'-\vec{x}_0|}. \tag{15}$$

The force of self-interaction can be defined as[10]

$$f_G^\mu=e\langle F^{\mu\nu}(x)\rangle u_\nu\big|_{x\to x_0}, \tag{16}$$

which in our case reduces to

$$\vec{f}_G=-e\langle\vec{\nabla}A_0(\vec{x})\rangle\big|_{\vec{x}\to\vec{x}_0}. \tag{17}$$

Here angular brackets mean averaging over all directions[11] in the three-plane orthogonal to the particle's world line. After a simple calculation we get

$$\vec{f}_G=-\frac{4}{3}\frac{e^2}{2a}\vec{\nabla}U-e\vec{\nabla}\psi, \tag{18}$$

where a is the cutoff radius. The first term on the right-hand side of this equation adds to the well-known kinetic term of Abraham[12] and Lorentz[13] to give

$$-\frac{4}{3}\frac{e^2}{2a}(\dot{\vec{v}}+\vec{\nabla}U). \tag{19}$$

Here $e^2/2a$ is the electromagnetic mass of the particle and $\frac{4}{3}$ is the well-known factor arising in all nonrelativistic calculations of the self-interaction.[12-14] The contribution (19) is absorbed in the mass renormalization in Eq. (5). The renormalized force is given by

$$\vec{f}_G=-e\vec{\nabla}\psi(\vec{x}_0)=e^2\int\frac{(\vec{x}_0-\vec{x}')}{|\vec{x}_0-\vec{x}'|^4}\mu(\vec{x}')d\vec{x}', \tag{20}$$

which coincides with the result of DeWitt and DeWitt [Eq. (7)].

III. BLACK HOLES

The electromagnetic field of a point charge at rest in the metric of a Schwarzschild black hole has been found by Cohen and Wald.[15] To allow a comparison with the previous section, we shall write this field in the harmonic coordinate system[4] (t, r, θ, ϕ), which is related to the Schwarzschild coordinates $(t,\ r_S,\ \theta,\ \phi)$ by $r_S=r+M$ (M is the mass of the black hole). The metric is then given by

$$ds^2=\left(\frac{r-M}{r+M}\right)dt^2-\left(\frac{r+M}{r-M}\right)dr^2-(r+M)^2d\Theta^2. \tag{21}$$

At large distances from the black hole ($r\gg M$)

$$ds^2=\left(1-\frac{2M}{r}\right)dt^2-\left(1+\frac{2M}{r}\right)(dr^2+r^2d\Theta^2), \tag{22}$$

which has the form of Eq. (3). The electrostatic potential is given by[15]

$$A_0=\begin{cases} e\sum_{l=0}^{\infty}G_l(r_0)F_l(r)P_l(\cos\theta), & r>r_0\\ e\sum_{l=0}^{\infty}F_l(r_0)G_l(r)P_l(\cos\theta), & r<r_0\end{cases} \tag{23}$$

where

$$G_l(r)=\begin{cases}1 & \text{for } l=0\\ \dfrac{2^l l!\,(l-1)!\,M^l}{(2l)!}(r-M)\dfrac{dP_l}{dr}\left(\dfrac{r}{M}\right) & \text{for } l\neq 0.\end{cases} \tag{24}$$

$$F_l(r)=-\frac{(2l+1)!}{2^l(l+1)!\,l!\,M^{l+1}}(r-M)\frac{dQ_l}{dr}\left(\frac{r}{M}\right). \tag{25}$$

P_l and Q_l are the two types of Legendre functions. Note that the functions G_l and F_l are different from g_l and f_l of Ref. 15, since we use another coordinate system. The charge is located at $(r,\ \theta)=(r_0,\ 0)$.

The asymptotics of G_l and F_l at large distances from the black hole ($r\gg M$) are given by

$$G_0(r)=1,$$
$$G_l(r)=r^l\left[1-\frac{M}{r}+O\left(\frac{M^2}{r^2}\right)\right],\quad l\geq 1, \tag{26}$$
$$F_l(r)=r^{-l-1}\left[1-\frac{M}{r}+O\left(\frac{M^2}{r^2}\right)\right].$$

Substituting Eqs. (26) in Eq. (23) and assuming that $r,\ r_0\gg M$, we find (for $r>r_0$)

$$A_0=\frac{e}{r}\left(1-\frac{M}{r}\right)+\frac{e}{r}\left(1-\frac{M}{r_0}-\frac{M}{r}\right)\sum_{l=1}^{\infty}\left(\frac{r_0}{r}\right)^l P_l(\cos\theta)$$
$$=\frac{e}{\rho}\left(1-\frac{M}{r_0}-\frac{M}{r}\right)+\frac{eM}{r_0 r}, \tag{27}$$

where ρ is the distance between the points (r, θ) and $(r_0, 0)$,

$$\rho=(r^2+r_0{}^2-2rr_0\cos\theta)^{1/2}.$$

Here we have used the well-known relation[16]

$$\sum_{l=0}^{\infty}\left(\frac{r_0}{r}\right)^l P_l(\cos\theta)=\frac{r}{\rho}. \tag{28}$$

To find A_0 at $M \ll r < r_0$, one has to interchange r_0 and r in the expression for A_0 at $r > r_0$ [this follows from Eq. (23)]. Equation (27) is invariant with respect to such interchange and therefore at $r<r_0$, A_0 is given by the same expression (27).

Equation (27) has the form of Eq. (15) with $U(\vec{x}) = -M/r$ and $\mu(\vec{x})=M\delta(\vec{x})$. The self-force is therefore given by Eq. (20) with $\mu(\vec{x})=M\delta(\vec{x})$:

$$\vec{f}_G=e^2M\vec{x}_0/r_0{}^4. \tag{29}$$

IV. MASSIVE VECTOR FIELDS AND BLACK HOLES

If the mass of the vector field A_ν is not exactly zero, then Eq. (9) has to be replaced by

$$(F^{\mu\nu}\sqrt{-g})_{,\nu}-\mu^2A^\mu\sqrt{-g}=-4\pi j^\mu\sqrt{-g}. \tag{30}$$

If the gravitational field is weak everywhere and the mass of the vector field is very small (i.e., μ^{-1} is much larger than the characteristic distance of the problem), then the massive term in Eq. (30) can be neglected and the self-force is given by Eq. (20). Thus there is a smooth transition from $\mu\neq 0$ to $\mu=0$. The situation is different in the case of a black hole. Let e be a charge of a massive vector field A_ν at a distance r_0 from the black hole and let $M\ll r_0 \ll \mu^{-1}$. Then it will be shown that the self-force is given by Eq. (29) but with an opposite sign. A sharp difference between massive and massless vector fields is a result of different boundary conditions at the horizon surface $r=M$.

If we require that the energy-momentum tensor of the field A_ν,

$$T_\sigma^\nu=-(4\pi)^{-1}[-\tfrac{1}{4}\delta_\sigma^\nu F^{\alpha\beta}F_{\alpha\beta}+F_{\sigma\alpha}F^{\nu\alpha}$$
$$+\mu^2(A_\sigma A^\nu-\tfrac{1}{2}\delta_\sigma^\nu A_\alpha A^\alpha)], \tag{31}$$

be nonsingular at the horizon, then the invariant $A_\nu A^\nu$ must be finite at $r=M$ and the potential A_0 must vanish at least like $(r-M)^{1/2}$ as $r\to M$. Thus all the physically meaningful solutions of the field equation (30) must satisfy the boundary condition[7,8]

$$A_0=0 \quad (r=M). \tag{32}$$

In the case of a massless field, the divergence of $A_\nu A^\nu$ at the horizon causes no difficulties (as long as the invariant $F_{\nu\sigma}F^{\nu\sigma}$ is finite). This is easily seen from Eq. (31) or from the gauge invariance of the massless vector field.

If $\mu^{-1}\gg M$, then the field in the region $r\ll\mu^{-1}$ can be approximated by a solution of the massless vector field equation (9) with the boundary condition (32). Such a solution is easily obtained from the solution of Cohen and Wald (23) which we shall denote A_{CW}. For $l\neq 0$, $G_l(r)\to 0$ as $r\to M$. Therefore

$$A_{CW}(r=M)=eF_0(r_0)=e(r_0+M)^{-1} \tag{33}$$

and

$$A_0=A_{CW}-\frac{2Me}{(r+M)(r_0+M)}. \tag{34}$$

The addition to A_{CW} in the last equation is just a spherical solution of the homogeneous Maxwell equations, namely, one proportional to $F_0(r)$. The second term in Eq. (34) indicates that the black hole acquires a charge $e'=-2Me(r_0+M)^{-1}$ in the presence of a point charge e at $r=r_0$. Note that $e+e'\to 0$ as $r_0\to M$, in agreement with the "no hair" conjecture.[17,7,8]

Returning now to the case $M\ll r_0\ll\mu^{-1}$ and using the asymptotic form (27) for A_{CW}, we get ($M\ll r \ll\mu^{-1}$)

$$A_0=\frac{e}{\rho}\left(1-\frac{M}{r_0}-\frac{M}{r}\right)-\frac{eM}{r_0 r}. \tag{35}$$

The self-force is given by

$$\vec{f}_G=-e^2M\vec{x}_0/r_0{}^4. \tag{36}$$

We see that $\vec{f}_G$ has the same magnitude as in the case of the electromagnetic field, but its direction is opposite. Electric charges are repelled from black holes,[18] while the charges of massive vector fields are attracted to them.

V. DISCUSSION

(1) It would be interesting to calculate the self-force at $r\sim M$, where the gravitational field is strong. However, an analytic calculation of f_G^μ in this case encounters formidable difficulties. The source of the difficulty lies in the fact that the Cohen-Wald solution has the form of an infinite series in which all terms are important for the calculation of f_G^μ. Fortunately at $r\gg M$ this series reduces to a form which can be summed explicitly [see Eq. (27)]. However, this does not happen at $r\sim M$.

(2) The self-interaction can be important in the physics of mini black holes and, in particular, in the Hawking radiation process.[19] Obviously, the

self-force cannot by itself give rise to particle production. However, it can result in a potential barrier suppressing the radiation, or it can reduce the existing potential barrier, which is due to gravitational and centrifugal forces and thus increase the emission rate. The potential energy corresponding to Eqs. (29) and (36) is given by

$$V(r) = \pm e^2 M/2r^2 , \tag{37}$$

where the upper and lower signs correspond to massless and massive vector fields, respectively. Taking a characteristic value $r \sim M$ for the distance r and extrapolating Eq. (37) to $r \sim M$, we find

$$V/T \sim \pm e^2 , \tag{38}$$

where $T = (8\pi M)^{-1}$ is the black-hole temperature. One expects that the effect of self-interaction is important if $V/T \gtrsim 1$, i.e., in the case of strong coupling ($e^2 \gtrsim 1$). The sign of $V(r)$ in Eq. (37) suggests that the particle emission rate is increased in the case of a massless vector field and is decreased for a massive vector field. It should be noted, however, that Eq. (37) has been derived for the case of a classical particle at rest at a large distance from the black hole and therefore it cannot serve as a basis for any reliable conclusion concerning the Hawking radiation. A reliable calculation of the particle emission rate with the self-interaction taken into account would require a full-fledged quantum theory of interacting fields in curved space times.

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[6]Equation (7) is rederived in Sec. II of this paper in a less rigorous but much simpler way than that of Ref. 4.

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