Integral equation method for effecting Kinnersley-Chitre transformations

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The construction of spinning mass solutions of Einstein's vacuum field equations, which can be obtained by applying Kinnersley-Chitre transformations to known solutions, is facilitated by our discovery of a linear integral equation of the Cauchy type, the solution of which yields directly the generating function $F(t)$ of the Kinnersley-Chitre hierarchy of potentials associated with the transformed spacetime metric.

I. INTRODUCTION

After a decade of heroic but often frustrating attempts to obtain asymptotically flat stationary axially symmetric solutions of the vacuum Einstein field equations, corresponding to the exterior gravitational fields of rotating bodies, it appears that the *general* solution of this problem may well be at hand. Hoenselaers, Kinnersley, and Xanthopoulos[1] recently proposed a way to employ the Kinnersley-Chitre (KC) transformation theory[2-4] to construct asymptotically flat stationary axially symmetric vacuum metrics with arbitrary multipole moments. The direct manner of executing KC transformations which we shall describe in this paper should enable comprehension of these exciting but complicated developments by a wider audience.

Many are familiar with the idea[5] of associating with any given stationary vacuum spacetime metric a complex potential $\mathcal{E}_0$, and then subjecting that potential to an Ehlers transformation[6]

$$\mathcal{E} = (\mathcal{E}_0 + ia)/(1 + ia\mathcal{E}_0)$$

to get another complex potential $\mathcal{E}$ from which a new stationary vacuum metric can be constructed. The general character of the much more complicated KC transformations may be described in a similar way.

In Sec. II of Ref. 4 Kinnersley and Chitre demonstrated how from any given stationary axially symmetric vacuum spacetime ("seed metric") a certain complex 2×2 matrix potential $F_0(t)$ can be constructed.[7] This potential, which has been evaluated explicitly for an arbitrary static vaccuum metric, depends not only upon the nonignorable spacetime coordinates but also upon an additional variable t, which we permit to be complex. In the following we shall assume that $F_0(t)$ is analytic in an open neighborhood of $t=0$ and that $F_0(0)=i\epsilon$, where

$$\epsilon \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The main thesis of the present paper is that if $F_0(t)$ is the potential associated with a particular seed metric, then

$$F(t) \equiv [I + tf(t)]F_0(t) \tag{1.1}$$

is the potential associated with another stationary axially symmetric vacuum spacetime, providing that the 2×2 matrix function $f(t)$ is analytic in an open neighborhood of $t=0$ and satisfies the linear integral equation

$$f(t) + \frac{1}{2\pi i}\int_C ds\,\frac{[f(s)+s^{-1}I]K(s)}{s-t} = 0\,, \tag{1.2}$$

where the kernel $K(s)$ is given by

$$K(s) \equiv F_0(s)\tau(s)\epsilon[F_0(s)]^{-1}\,, \tag{1.3}$$

$$\tau(s)\epsilon \equiv \exp[\gamma(s)\epsilon] - I\,. \tag{1.4}$$

Here $\gamma(s)$ is an arbitrary spacetime-independent 2×2 Hermitian[8] matrix function of s, analytic in an annulus about $s=0$ inside the region of analyticity of $F_0(s)$ and $f(s)$. The contour C is any closed positively oriented contour surrounding $s=0$ and within this annulus. The point t in Eq. (1.2) lies in the interior of the region enclosed by C.

This integral-equation approach to KC transformations was developed only after we had mastered the original approach of Kinnersley and Chitre. It was indeed derived using their infinite hierarchy of potentials. Since, however, there is no reference to these potentials in the final calculational scheme and not even the KC function $G(s,t)$ appears, we are confident that a more direct derivation of our integral equation will eventually be developed, one which will avoid completely the infinite-hierarchy-of-potentials idea.

Before presenting our interim derivation we shall give several very simple examples of how the integral equation (1.2) can be solved, for many of our readers are likely to be more interested in how to apply our formalism to generate new solutions of the vacuum field equations than they are in studying our derivation.

II. A SUBGROUP WHICH LEAVES MINKOWSKI SPACE UNCHANGED

One approach which can be used in order to solve Eq. (1.2) is illustrated when one attempts to prove that the transformation induced by

$$\gamma(s)=\beta(s)\begin{pmatrix}1 & -is^{-1}\\ is^{-1} & s^{-2}\end{pmatrix} \tag{2.1}$$

maps Minkowski space into itself. In this expression $\beta(s)$ is a real[9] function of s, analytic everywhere (including infinity) except perhaps for an isolated singularity at $s=0$. In this case Eq. (1.4) gives

$$\tau(s)=\alpha(s)\begin{pmatrix}s & -i\\ i & s^{-1}\end{pmatrix}, \tag{2.2}$$

where

$$\alpha(s)\equiv(2i)^{-1}\{\exp[2i\beta(s)s^{-1}]-1\}. \tag{2.3}$$

The kernel

$$K(s)=[\det F_0(s)]^{-1}F_0(s)\tau(s)F_0(s)^T\epsilon \tag{2.4}$$

is easily evaluated for Minkowski space, where (cf. Ref. 4)

$$F_0(s)=F_0^{\mathrm{MS}}(s)$$
$$=\begin{pmatrix}-(\lambda-1+2sz)/2\lambda s & i(\lambda+1-2sz)/2\lambda\\ -i/\lambda & s/\lambda\end{pmatrix}. \tag{2.5}$$

Here and elsewhere in this paper

$$\lambda(s)\equiv[(1-2sz)^2+4s^2\rho^2]^{1/2}. \tag{2.6}$$

Substituting Eqs. (2.2) and (2.5) into Eq. (2.4), we obtain the kernel

$$K(s)=\alpha(s)\begin{pmatrix}2i & s^{-1}-2z\\ 0 & 0\end{pmatrix}, \tag{2.7}$$

where $\alpha(s)$ was defined in Eq. (2.3).

If the reader notices certain superficial differences between the expression $F_0^{\mathrm{MS}}(s)$ given in Eq. (2.5) and the corresponding expression to be found in Ref. 4, he should attribute these to the fact that we are staying with spacetime signature$(+++-)$, which we have always used, instead of switching to the signature $(---+)$ which was introduced by Kinnersley and Chitre. The sign differences in Eq. (2.5) are *not* due to our attributing mixed upper and lower indices to the elements of the matrix $F_0(s)$. We would write the elements of $F_0(s)$ in the form $F_0(s)_{AB}$ if we were ever to use indicial notation. Keeping this in mind, it will be seen that our stipulation concerning $F_0(0)$ is in exact agreement with that to be found in Eq. (2.24) of Ref. 2.

The form of Eq. (2.5) is also affected by the fact that we place the spacelike Killing vector before the timelike one, the opposite order to that used by Kinnersley and Chitre. This means that both the columns and the rows of 2×2 matrices such as $F_0(s)$ are interchanged. In particular, the KC matrix f_{AB} corresponds to our 2×2 matrix

$$h\equiv-\underline{K}\cdot\underline{K}^T,$$

where the minus sign arises from the use of the opposite signature, and where for us $\underline{K}$ is a 2×1 matrix whose upper and lower elements are, respectively, the spacelike and timelike Killing vectors.

Let us now return to the transformation of Minkowski space, noting that the kernel (2.7) vanishes at infinity and is analytic everywhere except for an essential singularity at $s=0$. The auxiliary kernel $K^-(s)$, constructed by reversing the sign of $\beta(s)$, shares these properties, and furthermore,

$$[I+K(s)][I+K^-(s)]=I. \tag{2.8}$$

In other words, the evaluation of the inverse of the matrix $I+K(s)$ is trivial.

Equation (1.2) can be expressed in the form

$$\frac{1}{2\pi i}\int_C ds\,\frac{f(s)[I+K(s)]}{s-t}=-\frac{1}{2\pi i}\int_C ds\,\frac{I+K(s)}{s(s-t)}. \tag{2.9}$$

However, if L is a positively oriented closed contour surrounding $t=0$ and $t=r$ but entirely within the contour C, then for s on C one has

$$\frac{1}{2\pi i}\int_L dt\,\frac{I+K^-(t)}{(s-t)(t-r)}=\frac{I+K^-(s)}{s-r}. \tag{2.10}$$

This can be seen by expanding the contour L out to infinity, where $K^-(t)$ vanishes. If one multiplies Eq. (2.9) by

$$\frac{1}{2\pi i}\,dt\,\frac{I+K^-(t)}{t-r}$$

and integrates over the contour L, Eqs. (2.8) and (2.10) permit one to infer that $f(r)=0$, thus establishing that Minkowski space is not altered by the transformation (2.1).

The subgroup of KC transformations which we have here identified is from our point of view somewhat simpler than the B group of Kinnersley and Chitre, to which it is closely related. Their B group corresponds to

$$\gamma(s)=\beta(s)\begin{pmatrix}1 & 0\\ 0 & s^{-2}\end{pmatrix}. \tag{2.11}$$

III. KERR-NUT (NEWMAN-UNTI-TAMBOURINO) METRIC FROM SCHWARZSCHILD METRIC

The same general approach is applicable when our version of the B group is applied to other seed metrics. If the seed metric is static, the potential $F_0(s)$ can be expressed in the form (cf. Ref. 1)

$$F_0(s)=\begin{pmatrix}e^{-\psi(0)} & 0\\ 0 & e^{\psi(0)}\end{pmatrix}F_0^{\mathrm{MS}}(s)\begin{pmatrix}e^{-\psi(s)} & 0\\ 0 & e^{\psi(s)}\end{pmatrix}, \tag{3.1}$$

where $\psi(s)$ is a generalization of Weyl's potential function ψ such that

$$d\psi(s)=[\lambda(s)]^{-1}[(1-2sz)-2s\rho^*]\,d\psi,\quad \psi(0)=\psi, \tag{3.2}$$

where the two-dimensional duality operator is denoted by $*$. In particular, for any Zipoy-Voorhees metric (in prolate spheroidal coordinates)

$$\psi(s)=\tfrac{1}{2}\delta\ln\left[\frac{x-2sy-\lambda(s)}{x-2sy+\lambda(s)}\right]. \tag{3.3}$$

We shall at this time consider the Schwarzschild metric ($\delta=1$).

We find that the kernel now has the form

$$K(s)=\frac{\alpha(s)}{s^2-\frac{1}{4}}B(s), \tag{3.4}$$

where

$$B(s)=\begin{pmatrix}-\dfrac{i}{2}\dfrac{(1+2sy)(x-2sy)}{x+1} & \dfrac{1+2sy}{4s}b(s)\\[2ex] \dfrac{s(x-2sy)}{(x+1)^2} & \dfrac{ib(s)}{2(x+1)}\end{pmatrix}$$

and

$$b(s)=-1+2sy(x-1)+4s^2(x+1-y^2).$$

Unlike the kernel (2.7) this kernel has simple poles at $s=\pm\frac{1}{2}$, points which lie outside the contour C and the contour L. Consequently, Eq. (2.10) must be replaced with

$$\frac{1}{2\pi i}\int_L dt\,\frac{I+K^-(t)}{(s-t)(t-r)}=\frac{I+K^-(s)}{s-r}+\frac{k_+}{(s-\frac{1}{2})(r-\frac{1}{2})}+\frac{k_-}{(s+\frac{1}{2})(r+\frac{1}{2})}, \tag{3.5}$$

where k_+ and k_- are independent of s and r.

Proceeding as in the Minkowski-space case, we obtain this time the solution

$$f(t)=\frac{f_+}{t-\frac{1}{2}}+\frac{f_-}{t+\frac{1}{2}}, \tag{3.6}$$

where f_+ and f_- are t-independent matrices, which may be identified by substituting Eq. (3.6) back into the integral equation. We find that

$$f_\pm=\begin{pmatrix}A_\pm & \pm i(x+1)(1\pm y)A_\pm\\ B_\pm & \pm i(x+1)(1\pm y)B_\pm\end{pmatrix}.$$

Here

$$A_\pm=i\alpha(\pm\tfrac{1}{2})(x+1)^{-1}(x\mp y)\times\{1\pm y\pm iC^{-1}[-\alpha(\tfrac{1}{2})(x-y)(1+y)+\alpha(-\tfrac{1}{2})(x+y)(1-y)]\},$$

$$B_\pm=i\alpha(\pm\tfrac{1}{2})(x+1)^{-2}(x\mp y)\times\{\pm i\pm C^{-1}[\alpha(\tfrac{1}{2})(x-y)+\alpha(-\tfrac{1}{2})(x+y)]\},$$

and

$$C=x+1+i\alpha(\tfrac{1}{2})(x-y)+i\alpha(-\tfrac{1}{2})(x+y).$$

From $f(t)$, using Eq. (1.1), one may easily evaluate $F(t)$ for the transformed spacetime. This should be very useful, for the transformed spacetime is Kerr-NUT (Newman-Unti-Tambourino) [simply Kerr if $\beta(s)$ is an even function of s and simply NUT if $\beta(s)$ is an odd function of s]. If one should desire, for example, to construct a Kerr-like solution with an arbitrary quadrupole moment by employing another KC transformation, it would be natural to begin with the Kerr metric as the seed metric. To our knowledge ours is the first determination of the potential $F(t)$ for the Kerr metric, and this is the basic ingredient which will allow one to perform subsequent KC transformations.

The construction of the metric from $f(t)$ is straightforward. One can choose the additive constants in the potential $F(t)$ so that

$$\frac{d}{dt}\left(\frac{F(t)+F(t)^\dagger}{2}\right)\Bigg|_{t=0}=h+iz\epsilon.$$

[Cf. Ref. 4 following Eq. (2.4).] Upon substituting into this equation the relation

$$F(t)=[I+tf(t)]F_0(t),$$

we obtain the following formula for the change in h:

$$\Delta h=\tfrac{1}{2}i[f(0)\epsilon+\epsilon f(0)^\dagger]. \tag{3.7}$$

However, $-h$ is one of the 2×2 blocks of the metric tensor. The other can be constructed by a variety of techniques, one of which was suggested in the work of Kinnersley.[10]

IV. KERNELS WITH NO ESSENTIAL SINGULARITY AT $s=0$

In the examples which we have considered thus far the kernel $K(s)$ had an essential singularity at $s=0$. A more direct method of solving the integral equation is available when the kernel has no essential singularity within C. Such is the case, for example, when

$$\tau(s)=\gamma(s)=\sum_{i=1}^{N}\frac{\alpha_i s}{s-u_i}\begin{pmatrix}0&0\\0&1\end{pmatrix}, \tag{4.1}$$

where α_i and u_i are real constants.

For any static seed metric $F_0(s)$ is given by Eq. (3.1), and the kernel $K(s)$ assumes the form

$$K(s)=\sum_{i=1}^{N}\frac{\alpha_i}{s-u_i}\,b(s)h[q(s)], \tag{4.2}$$

where

$$b(s)\equiv s^3[\lambda(s)]^{-1}e^{2\psi(s)},$$

$$q(s)\equiv\frac{\lambda(s)+1-2sz}{2s},$$

and

$$h[q]\equiv\begin{pmatrix} iq & q^2e^{-2\psi(0)}\\ e^{2\psi(0)} & -iq\end{pmatrix}.$$

The latter matrices have the useful property

$$h[q]h[q']+h[q']h[q]=(q-q')^2I. \tag{4.3}$$

In particular, $(h[q])^2=0$.

If one restricts attention to spacetime points sufficiently close to the origin $z=\rho=0$, then the branch points of the function $\lambda(s)$ lie outside the contour C, which we shall assume encloses only the poles at $s=u_i$ $(1\le i\le N)$ and at $s=t$. Thus the integral equation gives us

$$f(t)+[f(t)+t^{-1}I]K(t)+\sum_{i=1}^{N}\frac{\alpha_i b_i X_i}{u_i-t}=0, \tag{4.4}$$

where $b_i\equiv b(u_i)$ and $X_i\equiv X(u_i)$, with

$$X(s)\equiv[f(s)+s^{-1}I]h[q(s)].$$

Multiplying Eq. (4.4) by $[I-K(t)]$ and using the fact that $[K(t)]^2=0$, we obtain the solution

$$f(t)=-t^{-1}K(t)-\sum_{i=1}^{N}\frac{\alpha_i b_i X_i}{u_i-t}[I-K(t)] \tag{4.5}$$

of the integral equation. In particular, the change in the complex $\mathcal{E}$ potential is given by the lower right element of the matrix $f(0)i\epsilon$, that is, by

$$\Delta\mathcal{E}=-i\sum_{i=1}^{N}\frac{\alpha_i b_i}{u_i}[X_i]_{\text{lower left element}}. \tag{4.6}$$

The determination of the t-independent matrices X_i $(1\le i\le N)$ involves multiplying Eq. (4.4) by $h[q(t)]$ and then taking the limit $t\to u_j$ $(1\le j\le N)$. In this way we obtain a system of N linear equations

$$X_j(1+i\alpha_j a_j)+\sum_{i\ne j}\frac{\alpha_i b_i X_i}{u_i-u_j}h_j=u_j^{-1}h_j. \tag{4.7}$$

Here $h_i\equiv h[q(u_i)]$ and $a_i\equiv a(u_i)$, where

$$a(s)\equiv[s/\lambda(s)]^2q(s)e^{2\psi(s)}.$$

By taking advantage of the property (4.3) of the matrix $h[q]$, one can solve the Eqs. (4.7) without great difficulty. The solution can be expressed in terms of the auxiliary fields

$$Q_{ii}=\frac{1+i\alpha_i a_i}{-ib_i},$$

$$Q_{ij}=\frac{q_i-q_j}{u_i-u_j}\quad(i\ne j),$$

and

$$H_{ij}=\frac{h_i-h_j}{q_i-q_j}\quad(i\ne j).$$

We find that

$$X_1=(\det Q)^{-1}\begin{vmatrix} iu_1^{-1}I & -u_2^{-1}H_{12} & -u_3^{-1}H_{13} & \cdots\\ Q_{21} & Q_{22} & Q_{23} & \\ Q_{31} & Q_{32} & Q_{33} & \\ \vdots & & & \ddots \end{vmatrix}(b_1^{-1}h_1), \tag{4.8}$$

with analogous expressions for X_i $(i>1)$.

To evaluate the complex $\mathcal{E}$ potential using Eq. (4.6) we require the lower left component of the matrices X_i. One finds that

$$[X_1]_{\text{lower left element}} = i\, b_1^{-1} e^{2\psi(0)} (\det Q)^{-1} \begin{vmatrix} u_1^{-1} & u_2^{-1} & u_3^{-1} & \cdots \\ Q_{21} & Q_{22} & Q_{23} & \\ Q_{31} & Q_{32} & Q_{33} & \\ \vdots & & & \ddots \end{vmatrix}, \tag{4.9}$$

with analogous expressions for X_i $(i > 1)$.

In the simplest case, when $N = 1$, the complex potential is given by

$$\Delta \mathcal{E} = -i\, \frac{\alpha[u/\lambda(u)]^2 e^{2[\psi(u)+\psi(0)]}}{1 + i\alpha[u/\lambda(u)]q(u)e^{2\psi(u)}}, \tag{4.10}$$

in agreement with Eq. (4.3) of Ref. 1. In the same reference the $\mathcal{E}$ potential was worked out for $N = 2$ too, at least when the seed metric is Minkowski space. Aside from these special cases, we believe that our evaluation of $f(t)$ given in Eqs. (4.5) and (4.8) results in new solutions. The metrics corresponding to these solutions can be constructed in the manner suggested at the end of Sec. III or from the complex $\mathcal{E}$ potential, given in Eqs. (4.6) and (4.9). For the reasons discussed in Ref. 1 these solutions will be asymptotically flat (except for the possible inclusion of a NUT parameter).

Let us turn now to a description of the interim derivation of the integral equation (1.2). Readers who are familiar with Sec. III of Ref. 2 should have no trouble understanding the starting point of our derivation. Others may wish to consult the Appendix, where we outline how the infinitesimal KC transformation equations can be deduced.

V. EXPONENTIATING INFINITESIMAL KC TRANSFORMATIONS

Equations (3.1)–(3.3) of Ref. 2 can be expressed in the elegant matrix form

$$\delta T = L_{+-}(\gamma) + L_{++}(\gamma)T - TL_{--}(\gamma) - TL_{-+}(\gamma)T. \tag{5.1}$$

In the case of vacuum-vacuum transformations T is an $\infty \times \infty$ matrix consisting of 2×2 blocks $T^{(m,n)}$ $(-\infty < m < \infty,\ -\infty < n < \infty)$. In terms of the KC potentials, we have

$$\begin{aligned} T^{(0,n)} &= -iH^{(n)} \quad (n \geq 1), \\ T^{(m,n)} &= N^{(m,n)} \quad (m \geq 1,\ n \geq 1), \\ T^{(m,n)} &= 0 \quad \text{otherwise.} \end{aligned} \tag{5.2}$$

The $\infty \times \infty$ constant L matrices are in the vacuum-vacuum case given by

$$\begin{aligned} L_{++}(\gamma) &= \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} A^p P^\dagger \epsilon \gamma^{(q-p)} P A^{-q}, \\ L_{+-}(\gamma) &= -\sum_{p=0}^{\infty} \sum_{q=1}^{\infty} A^p P^\dagger \epsilon \gamma^{(-p-q)} \epsilon P A^{-q}, \\ L_{-+}(\gamma) &= -\sum_{p=1}^{\infty} \sum_{q=0}^{\infty} A^p P^\dagger \gamma^{(p+q)} P A^{-q}, \\ L_{--}(\gamma) &= \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} A^p P^\dagger \gamma^{(p-q)} \epsilon P A^{-q}, \end{aligned} \tag{5.3}$$

where P is a $2\times\infty$ matrix with 2×2 by blocks

$$P^{(n)} = \delta_{n,0} I,$$

$$A^{(m,n)} = \delta_{m,n+1} I,$$

and

$$[A^{-1}]^{(m,n)} = \delta_{m+1,n} I.$$

For each value of k $(-\infty < k < \infty)$, $\gamma^{(k)}$ is an arbitrary constant Hermitian[11] 2×2 matrix.

We associate with the infinite hierarchy of parameter matrices $\gamma^{(k)}$ the 2×2 Hermitian[8] matrix function of the complex variable t

$$\gamma(t) \equiv \sum_{k=-\infty}^{\infty} \gamma^{(k)} t^{-k}. \tag{5.4}$$

The L matrices may be regarded as functionals of $\gamma(t)$.

The thesis of this section of our paper is that if T_0 is the T matrix associated with a vacuum seed metric, then the T matrix of the solution which results from the finite KC transformation is given by

$$T[I + L_{-+}(\tau)T_0 + L_{--}(\tau)] = T_0 + L_{++}(\tau)T_0 + L_{+-}(\tau), \tag{5.5}$$

where

$$\tau(t)\epsilon \equiv \exp[\gamma(t)\epsilon] - I. \tag{5.6}$$

The derivation of Eq. (5.5) is facilitated by rewriting Eq. (5.1) in the even more concise form

$$\delta\sigma = -\sigma e L(\gamma)\sigma + L(\gamma)\sigma \ , \tag{5.7}$$

where

$$\sigma = \begin{pmatrix} T & 0 \\ I & 0 \end{pmatrix} ,$$

$$L = \begin{pmatrix} L_{++} & L_{+-} \\ L_{-+} & L_{--} \end{pmatrix} , \tag{5.8}$$

$$e = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} .$$

Equation (5.7) can be solved by expressing σ in the form

$$\sigma = N(\gamma)D(\gamma)^{-1}e, \quad D(0) = e \ . \tag{5.9}$$

Without loss of generality we may choose N and D so that

$$\delta N = LN, \quad \delta D = -LN \ . \tag{5.10}$$

Equations (5.10) are easily solved, and thus we discover that

$$\sigma(\gamma)\,[\,I - e(I - e^{L(\gamma)})\sigma(0)] = e^{L(\gamma)}\sigma(0) \ . \tag{5.11}$$

Furthermore, one can show that

$$e^{L(\gamma)} = I + L(\tau) \ , \tag{5.12}$$

where τ is defined in Eq. (5.6). Equation (5.5) results when one substitutes Eq. (5.12) into Eq. (5.11) and then reexpresses σ in terms of T using Eq. (5.8).

It should also be mentioned that if two KC transformations are performed in succession, then the σ which results from two applications of Eq. (5.11) can be obtained through a single application of Eq. (5.11) with $e^{L(\gamma)}$ replaced by the matrix product of the corresponding factors for the separate transformations. Thus, the matrices $e^{L(\gamma)}$ provide a representation of the KC group.

VI. DERIVATION OF THE INTEGRAL EQUATION

Our interim derivation of the integral equation (1.2) for the vacuum-vacuum case proceeds from a consideration of Eq. (5.5), which can be expressed in the form

$$T - T_0 = -T[\,L_{-+}(\tau)T_0 + L_{--}(\tau)] + L_{++}(\tau)T_0 + L_{+-}(\tau) \ . \tag{6.1}$$

In particular, for all $n \geq 1$, we conclude that

$$H^{(n)} - H_0^{(n)} = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} H^{(p)}\tau^{(p+q)}T_0^{(q,n)} - \sum_{p=0}^{\infty} H^{(p)}\tau^{(p-n)}\epsilon \ , \tag{6.2}$$

where it should be recalled that $H^{(0)} = i\epsilon$. In terms of the generating function

$$F(t) = \sum_{n=0}^{\infty} H^{(n)}t^n, \tag{6.3}$$

Eq. (6.1) implies that

$$F(t) - F_0(t) = \sum_{n=1}^{\infty} \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} H^{(p)}\tau^{(p+q)}T_0^{(q,n)}t^n - \sum_{n=1}^{\infty} \sum_{p=-\infty}^{\infty} H^{(p)}\tau^{(p-n)}\epsilon t^n ,$$

providing one sets $H^{(n)} = 0$ for all $n < 0$ and $T_0^{(m,n)} = 0$ for $m < 0$ or $n < 1$. Substituting

$$\tau^{(p)} = \frac{1}{2\pi i}\int_C ds\, s^{p-1}\tau(s) \ , \tag{6.4}$$

we obtain

$$F(t) - F_0(t) = \frac{1}{2\pi i}\int_C ds \left[F(s)\tau(s)s^{-1}T_0(s,t) - F(s)\tau(s)\epsilon\frac{t s^{-1}}{s-t}\right] \tag{6.5}$$

for t within the contour C. Here

$$T_0(s,t) \equiv \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} T_0^{(m,n)} s^m t^n \tag{6.6}$$

is the generating function for the $T_0^{(m,n)}$.

Our versions of Eqs. (2.21) and (2.22) of Ref. 4 permit us to conclude that

$$T_0(s,t) = \frac{t}{s-t}\{\epsilon - \epsilon[F_0(s)]^{-1}F_0(t)\} \tag{6.7}$$

and hence that

$$F(t) - F_0(t) = -\frac{1}{2\pi i}\int_C ds\, \frac{t s^{-1}}{s-t}F(s)\tau(s) \times \epsilon[F_0(s)]^{-1}F_0(t) \ .$$

This integral equation is completely equivalent to Eq. (1.2) for the function

$$f(t) = t^{-1}[F(t) - F_0(t)][F_0(t)]^{-1} \ .$$

VII. REMARKS

The linear integral equation (1.2) provides a new way to think about KC transformations. For the first time the full power of complex function theory can be brought to bear upon the stationary axially symmetric field problem. In addition, contact has been made with the well-developed field of linear integral equations, which have been studied widely in connection with many areas of physics, and concerning which there exists an extensive mathematical literature.

The transformations which were the subject of study in Ref. 1 correspond to especially simple analyticity properties for the kernel of the integral equation. Accordingly, for such transformations the solution of the integral equation presents no great difficulty. Since it appears from Ref. 1 that *all* asymptotically flat solutions can be constructed in this way, what remains to be done is to render the general solution in a particularly useful form. This will undoubtedly be the subject of intensive study.

Parallel to these efforts will be the actual construction, using KC transformations, of solutions of special physical significance. It should be remarked that in the case of static vacuum metrics, where Weyl provided the general solution long ago, little attention has yet been paid to the identification of solutions of special physical significance. Perhaps what is really needed is some progress on the interior problem. Now that there has been developed a constructive procedure for generating all asymptotically flat exterior fields, *the interior problem deserves more attention.*

In addition to trying to accomplish these practical objectives, we are attempting to develop a simpler, more direct derivation of our integral equation, and to generalize the equation to the electrovac-electrovac transformations.[12]

Finally, we should like to compliment Kinnersley and his co-workers on having discovered the key which unlocked the door to finding all spinning mass solutions of Einstein's vacuum field equations. We hope that our refinement of the KC transformation theory does indeed enable a wider audience of physicists to appreciate these exciting developments.

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APPENDIX

The basic equations (2.8)–(2.15) of Ref. 2 can be cast into a convenient matrix form

$$dE = -i\rho^{-1} h\epsilon * dE\ , \tag{A1}$$

$$dS = E^\dagger \epsilon\, dE, \tag{A2}$$

$$EA = EGS, \tag{A3}$$

where h is a 2×2 matrix which we introduced in Sec. II, A is an $\infty\times\infty$ matrix consisting of 3×3 blocks $A^{(m,n)} = \delta_{m,n+1} I\ (-\infty < m < \infty,\ -\infty < n < \infty)$, and E is a $2\times\infty$ matrix consisting of 2×3 blocks $E^{(n)}\ (-\infty < n < \infty)$ such that

$$\begin{aligned} &E^{(n)} = (H^{(n)}, \phi^{(n)}) \text{ for } n \geq 1, \\ &E^{(0)} = (i\epsilon, 0)\ , \\ &E^{(n)} = (0,0) \text{ for } n \leq -1\ . \end{aligned} \tag{A4}$$

The matrix S is an $\infty\times\infty$ matrix consisting of 3×3 blocks $S^{(m,n)}\ (-\infty < m < \infty,\ -\infty < n < \infty)$. It can be expressed as the sum of a constant part S_0, about which more will be said later, and a part T, which is defined as follows in terms of the KC hierarchy of potentials:

$$T^{(0,n)} = \begin{pmatrix} -iH^{(n)} & -i\phi^{(n)} \\ 0 & 0 \end{pmatrix} \quad (n \geq 1)\ ,$$

$$T^{(m,n)} = \begin{pmatrix} N^{(m,n)} & M^{(m,n)} \\ L^{(m,n)} & K^{(m,n)} \end{pmatrix} \ (m \geq 1,\ n \geq 1)\ , \tag{A5}$$

$$T^{(m,n)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ otherwise}\,.$$

The matrix G is an $\infty\times\infty$ matrix consisting of 3×3 blocks

$$G^{(m,n)} = \delta_{m1}\delta_{n1}\begin{pmatrix} 0 & 0 \\ 0 & 2i \end{pmatrix} + (\delta_{m0}\delta_{n1} + \delta_{m1}\delta_{n0})\begin{pmatrix} -\epsilon & 0 \\ 0 & 0 \end{pmatrix}\ . \tag{A6}$$

Subtracting from (A2) the adjoint of the same equation, we obtain the relation

$$S - S^\dagger - \mathfrak{E} = E^\dagger \epsilon E\ , \tag{A7}$$

where $\mathfrak{E}$ is a constant matrix of integration. This equation corresponds to Eqs. (2.16)–(2.18) of Ref. 2. While the constant matrix $\mathfrak{E}$ may be simplified by judicious choice of S_0, it cannot be eliminated altogether.

From Eq. (A3) we obtain the relation

$$A^{-1}E^\dagger = -S^\dagger G E^\dagger\ ,$$

so Eq. (A2) yields immediately the result

$$-A^{-1}dS = -A^{-1}E^\dagger\epsilon\, dE = S^\dagger G E^\dagger \epsilon\, dE = S^\dagger G\, dS\ .$$

Subtracting from this equation its adjoint, we obtain

$$S^\dagger A - A^{-1}S = S^\dagger G S - A^{-1}KA\ , \tag{A8}$$

where K is a constant matrix of integration, which like $\mathfrak{E}$ cannot be eliminated altogether, no matter how S_0 is chosen. Finally, using (A7), we obtain the relation

$$(S - \mathfrak{E})A - A^{-1}S = (S - \mathfrak{E})GS - A^{-1}KA, \tag{A9}$$

corresponding to Eqs. (2.19)–(2.22) of Ref. 2.

In Ref. 2 the choice which was actually made for S_0 was

$$S_0^{(m,n)} = \operatorname{sgn}(2n-1)\left[\delta_{m+n,1}\begin{pmatrix}0 & 0\\ 0 & i/2\end{pmatrix} + \delta_{m+n,0}\begin{pmatrix}-\epsilon & 0\\ 0 & 0\end{pmatrix}\right]. \tag{A10}$$

This corresponds to

$$\mathfrak{E}^{(m,n)} = \delta_{m0}\delta_{n0}\begin{pmatrix}\epsilon & 0\\ 0 & 0\end{pmatrix} \tag{A11}$$

and

$$K^{(m,n)} = \delta_{m1}\delta_{n1}\begin{pmatrix}0 & 0\\ 0 & -i/2\end{pmatrix} + (\delta_{m0}\delta_{n1} + \delta_{m1}\delta_{n0})\begin{pmatrix}\epsilon & 0\\ 0 & 0\end{pmatrix}. \tag{A12}$$

We found, however, that in later calculations it was convenient to use instead of (A10) the simpler form

$$S_0^{(m,n)} = (\delta_{m0}\delta_{n1} - \delta_{m1}\delta_{n0})\begin{pmatrix}0 & 0\\ 0 & i/2\end{pmatrix} + (\delta_{m,-1}\delta_{n1} - \delta_{m0}\delta_{n0} - \delta_{m1}\delta_{n,-1})\begin{pmatrix}-\epsilon & 0\\ 0 & 0\end{pmatrix}. \tag{A13}$$

This choice corresponds to $\mathfrak{E}$ given again by Eq. (A11), but now K is given by

$$K^{(m,n)} = (\delta_{m1}\delta_{n1} - \delta_{m0}\delta_{n2} - \delta_{m2}\delta_{n0})\begin{pmatrix}0 & 0\\ 0 & -i/2\end{pmatrix} + (\delta_{m0}\delta_{n1} + \delta_{m1}\delta_{n0} - \delta_{m,-1}\delta_{n2} - \delta_{m2}\delta_{n,-1})\begin{pmatrix}\epsilon & 0\\ 0 & 0\end{pmatrix}. \tag{A14}$$

Our derivation of Eq. (5.1) from Eqs. (A1), (A2), (A3), (A7), and (A9) entails first deducing a complete set of relations satisfied by the matrix S. These relations are the following:

$$2(\rho - z^*)dS + A^{-1*}dS = -(S - \mathfrak{E})G^*dS, \tag{A15}$$

$$(S - \mathfrak{E})A - A^{-1}S = (S - \mathfrak{E})GS - A^{-1}KA, \tag{A16}$$

$$dS = S^\dagger\, \mathfrak{E}\, dS\,, \tag{A17}$$

$$S - S^\dagger - \mathfrak{E} = S^\dagger\, \mathfrak{E} S\,. \tag{A18}$$

The first variation of each of these four relations is calculated, subject to the gauge condition $\delta S_0 = 0$, and the trial transformation

$$\delta S = C_{+-} + C_{++}S - SC_{--} - SC_{-+}S \tag{A19}$$

is substituted. Without making any further special assumptions concerning the form of S, we find that the constant matrices C_{+-}, C_{++}, C_{--}, and C_{-+} must satisfy certain algebraic relations. A lengthy analysis of these relations permits one to obtain their general solution. When these constants are substituted back into Eq. (A19), and the resulting equation is reexpressed in terms of the matrix T, we obtain Eq. (5.1), our version of Eqs. (3.1)–(3.3) of Ref. 2. In the present paper we presented the $\boldsymbol{L}$ matrices only for vacuum-vacuum transformations, saving for the future the consideration of electrovac-electrovac transformations.

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[7]Note that our signature is (+++−) and the lower right element of $F_0(t)$ is the time-time component.

[8]We mean that $\gamma(s)$ is Hermitian for real s.

[9]We mean that $\beta(s)$ is real for real s.

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[11]Our Hermitian $\gamma^{(k)}$ includes the Kinnersley-Chitre $\tau^{(k)}$ as well as their $\gamma^{(k)}$, which was real.

[12]*Added note*: We recently succeeded in obtaining an electrovac generalization of our integral equation. A manuscript is being prepared at this time. We are also well on the way to developing a more direct derivation of our integral equation, and to proving some of the conjectures which have been made by Kinnersley *et al.* and by ourselves.