Gravitational waves from rotating and precessing rigid bodies: Simple models and applications to pulsars

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An axially symmetric, torque-free rigid body, rotating and precessing, emits gravitational quadrupole radiation at two frequencies, ω and 2ω, corresponding to the $l = 2$, $m = 1,2$ spherical harmonics. We present explicitly the waveforms of the two polarizations at both frequencies. From observations of gravitational waves, one can derive information about the body's orientation and its precession amplitude. Electromagnetic radiation emitted by a spot fixed on the surface of the body arrives in pulses at a mean frequency Ω which is typically different from ω. If the body is not axially symmetric but the amplitude of the precession is small, the gravitational radiation at the lower frequency ω is split into two frequencies on either side of the electromagnetic pulse frequency. We present explicit waveforms for the two polarizations in this case also.

I. INTRODUCTION

Pulsars are widely interpreted as rotating, rather rigid neutron stars.[1] Some of the nearer, more rapidly spinning pulsars might be good sources of gravitational waves.[2-6] Experimental searches for these waves have already been made,[7,8] so far with negative results. In these experiments and in theoretical discussions of gravitational waves from pulsars, it has generally been assumed that the gravitational radiation is emitted at precisely twice the observed pulsar frequency. We point out here that this assumption is typically incorrect. The simplest pulsar model, an axially symmetric rigid body undergoing free precession, emits gravitational quadrupole radiation at two frequencies, ω and 2ω. The frequency ω and the radio pulsation frequency Ω differ by the precession frequency Ω_p; hence an attempt to resonate a high-Q gravitational-wave antenna with the pulsar's emissions, in order to build up a detectable signal, may fail if the radio pulses are used as a guide and if radio measurements have failed to determine the precession frequency. Also the gravitational radiation at frequency 2ω is usually much weaker than that at frequency ω.

In this paper we present explicit gravitational radiation waveforms for two of the simplest imaginable pulsar models: (1) a rigid, axisymmetric body undergoing free precession, and (2) a rigid asymmetric body, freely precessing with small wobble angle. Future papers will discuss more general models.

Section II of this paper outlines the assumptions and methods used here. Section III gives the results for the axisymmetric model and explains how a gravitational astronomer can deduce a pulsar's spin orientation, inclination, wobble angle, and ellipticity, from gravitational-wave observations. That section also explains the reasons for the difference between the fundamental gravitational-wave frequency and the electromagntic pulsar frequency. Section IV presents waveforms for the asymmetric model rotating with small mean wobble angle $\bar{\theta}$, and discusses how a gravitational astronomer can deduce information about a pulsar's orientation, oblateness, etc. in this case. Finally, Sec. V summarizes and reviews these results and other recent work on gravitational radiation from rigid bodies. That section also points out an error in Zimmermann's estimates[6] of gravitational luminosities for the Crab and Vela pulsars and gives corrected estimates.

II. METHOD

For the purposes of this paper, we model pulsars as torque-free, rigidly rotating bodies. Actually, radiation reaction, accretion, and other torques certainly exist, but simple estimates of their size suggest that their effects are likely to be small compared to the free precession.[9] Therefore we ignore them. Also, solid neutron-star matter is not perfectly rigid, so the precession rate calculated for a rigid body needs to be reduced somewhat, depending on the shear modulus and structure of the specific model being investigated. Fortunately the precessional equations of motion for a nonrigid body are isomorphic to the rigid-body equations, in the limit that the body's oblateness and wobble angle are small, and provided that the body acts as an elastic solid on precessional time scales.[9,10] The rigid-body gravitational radiation waveforms calculated below should therefore be correct for a nonrigid body, if the actual reduced precession rate is used in place of the theoretical rigid-body rate. Liquid-core neutron-star models typically precess slower than rigid bodies by factors ranging from 10^2 to 10^4; solid-core models of more massive neutron stars typically precess within a factor of 2 of the perfectly rigid precession rate.[9,11] Precession periods of ~20 hours for the Crab and of a few minutes for a solid-core Vela neutron star have been estimated.[11]

We take, as our theory of gravitation and mechanics, standard Newtonian theory (the weak-field, slow-motion, small-stress approximation to general relativity), augmented by the quadrupole-moment formalism for gravitational-wave generation.[12] (This formalism is discussed in most textbooks on general relativity; see, for example, Misner, Thorne, and Wheeler,[13] whose notation and conventions we use in this paper.) We are fairly sure, and shall attempt to prove in a subsequent paper, that the strong-field, slow-motion approximation[14] to general relativity (which is more nearly valid for neutron stars where $GM/Rc^2 \sim 0.2$) will give precisely the same waveform predictions as the weak-field formalism we use. The only difference to be expected is in the expressions for the body's moment of inertia and quadrupole-moment tensors as integrals over the body's mass and stress distributions.[14,15]

In our analysis the only relevant parameters from stellar structure are the three principal moments of inertia of the body and the wobble angle θ between the total angular momentum vector $\vec{J}$ and the body's third principal axis $\hat{x}_3$.

III. AXISYMMETRIC MODEL: WAVEFORMS AND ANALYSIS

We first consider a symmetric rigid body with moments of inertia $I_1 = I_2 \neq I_3$. The free precession of such an object in Newtonian theory is discussed in most classical mechanics texts.[16-18] It is straightforward to plug the resulting time-changing quadrupole-moment tensor into the gravitational radiation equations[13] and grind out the waves produced.

Suppose that the object's conserved angular momentum $\vec{J}$ has an "inclination angle" i relative to the plane of the observer's sky. (Inclination angle i is defined as astronomers do for binary star systems: $i = 0°$ means that $\vec{J}$ points toward the observer, $i = 90°$ means that $\vec{J}$ is perpendicular to the line of sight, $i = 180°$ means that $\vec{J}$ points away from the observer.) For an object at distance r, we find that the two polarizations[13] of gravitational waves have dimensionless amplitudes:

$$h_+ = \frac{2I_1\omega^2\epsilon\sin\theta}{r}[(1+\cos^2 i)\sin\theta\cos 2\omega t + \cos i \sin i \cos\theta \cos\omega t],$$

$$h_\times = \frac{2I_1\omega^2\epsilon\sin\theta}{r}(2\cos i \sin\theta \sin 2\omega t + \sin i \cos\theta \sin\omega t), \quad (1)$$

where the frequency is $\omega = J/I_1$, the ellipticity is $\epsilon = (I_3 - I_1)/I_1$, and we set $c = G = 1$.

A particular choice of coordinate axes and of the origin of time, $t=0$, was made by the observer to yield the above wave amplitudes: If $\hat{v}$ and $\hat{w}$ are orthogonal unit vectors chosen transverse to the direction of wave propagation, with $\hat{v} \times \hat{w} =$ (direction toward observer), then

$$h_+ = h^{TT}_{vv} = -h^{TT}_{ww} = (-1/r)(\ddot{I}_{vv} - \ddot{I}_{ww})$$

and

$$h_\times = h^{TT}_{vw} = (-2/r)\ddot{I}_{vw},$$

where TT refers to the "transverse-traceless" gauge, dots are time derivatives evaluated at the retarded time $t-r$, and the minus signs come from our use of

$$I_{ab} \equiv \int \rho(r^2\delta_{ab} - x_a x_b)\, d^3x$$

instead of the

$$\text{I\!\!\!-}_{ab} \equiv \int \rho(x_a x_b - \tfrac{1}{3}\delta_{ab} r^2)\, d^3x$$

of Ref. 13.

The observer can get into our "preferred" orientation by rotating his transverse axes $\hat{v}$ and $\hat{w}$ so as to maximize the observed ratio $|h_{+,2\omega}|/|h_{\times,2\omega}|$ (where $h_{+,2\omega}$ means the amplitude of h_+ at frequency 2ω, with its $\cos 2\omega t$ time dependence factored out, etc.). The same orientation of $\hat{v}$ and $\hat{w}$ must also maximize the independently observable ratio $|h_{\times,\omega}|/|h_{+,\omega}|$ if the waves come from a freely precessing, axially symmetric body. In this orientation, the projection of $\vec{J}$ into the plane of the sky lies along one of the directions $\hat{v}$, $\hat{w}$, $-\hat{v}$, or $-\hat{w}$. The quadrupole nature of the waves makes this 90° ambiguity unavoidable.

In Eqs. (1) the observer's origin of time $t=0$ is chosen so as to make the component of $h_\times$ at frequency ω proportional to $+\sin\omega t$, with a positive constant of proportionality. The same choice of $t=0$ must make the piece of $h_\times$ at 2ω proportional to $+\cos 2\omega t$ and

$$(h_+ \text{ at } \omega) \propto (\pm\cos\omega t),$$

$$(h_\times \text{ at } 2\omega) \propto (\pm\sin 2\omega t),$$

with the sign determined by the sign of $\cos i$. With this choice of time origin, it turns out that at (retarded time) $t=0$, the body's symmetry axis $\hat{x}_3$ lies in the plane defined by $\vec{J}$ and the direction to the observer. If the body's ellipticity ϵ is positive (oblate spheroid), $\hat{x}_3$ is at its farthest from the observer at $t=0$; if $\epsilon<0$ (prolate spheroid), $\hat{x}_3$ is at its nearest to the observer. (We use the convention that the constant of the motion $\hat{x}_3 \cdot \vec{J} = J\cos\theta$ is positive; in other words, θ lies between 0° and 90°. During free precession, $\hat{x}_3$ moves around $\vec{J}$ at angular rate ω.)

With his transverse axes aligned and his origin of time selected in the above manner, the observer can read off from his measured waveforms and Eqs. (1) the inclination and wobble angle of the gravitational wave source. The independent ratios $h_{\times,\omega}/h_{+,\omega}$ and $h_{\times,2\omega}/h_{+,2\omega}$ determine the inclination i in the range 0° to 180°; given i, the ratios $h_{+,2\omega}/h_{+,\omega}$ and $h_{\times,2\omega}/h_{\times,\omega}$ determine the wobble angle θ between 0° and 90°. Finally, the overall amplitude of the signals determines

$$|I_1\epsilon/r| = |(I_3-I_1)/r| .$$

If the distance r is known by other means, then a direct measure of the nonaxisymmetry $|I_3-I_1|$ follows. Note that gravitational observations alone cannot distinguish an oblate from a (perhaps improbable) prolate spheroid.

To compare the gravitational radiation waveforms with the electromagnetic pulsar signals, one needs a simple pulsar model. Suppose that something fixed on the surface of the neutron star (a magnetic pole, for example) at colatitude λ relative to the $\hat{x}_3$ body axis is associated with radio, optical, or other pulses observed once per turn of the star. The apparent rate of pulsation seen by a distant observer varies during the body's precession and depends on the precessional motion, on λ, and on the details of the pulsar radiation beam. Free precession would produce periodic peregrinations in the perceived pulse period, the mean pulse profile, and other pulsar parameters, such as pulse polarization.

Electromagnetic observations of pulsars have shown no evidence for precession.[19,20] In particular, any precession with a period between about 2 and 150 days must have an amplitude less than a few degrees[20] in the observed cases.

There are two scenarios which could explain the absence of observable precession. First, if the angle λ (between a pulsar's $\hat{x}_3$ body axis and the source of the radiation beam) were small compared to the wobble angle θ, then a pulse would be seen whenever the $\hat{x}_3$ pulsar axis passed sufficiently close to the observer's line of sight. The mean observed electromagnetic pulse frequency Ω would thus equal the gravitational-wave frequency $\omega = J/I_1$. But during the body's precession time

$$2\pi/\Omega_p = 2\pi I_3/\omega(I_1-I_3)\cos\theta ,$$

the observer would pass through the pulsar radiation beam from many different directions. For the precession to be invisible, the pulsar beam would have to be not only nearly axisymmetric, but also would have to be without observable linear polarization. Any net linear polarization would rotate through 360° during a precession time; this has not been observed.[1,21]

The second and much more plausible scenario to explain the lack of electromagnetic precession observations is that the pulsar's beam source is at an arbitrary angle λ, but that the wobble angle θ is small. In this case, the body-frame precessional angular velocity Ω_p adds to the inertial-space $\hat{x}_3$ angular velocity ω to give a mean electromagnetic pulse frequency $\Omega=\omega+\Omega_p$ different from the gravitational-wave frequency. (For an oblate body, $\Omega<\omega$.) The observer always passes through the pulsar beam from approximately the same direction, so no significant changes in pulse profile or polarization would be expected. A simple knife-beam model of the pulsar radiation pattern gives the result (for small θ) that during a precession time pulses arrive early and late by a phase of up to $\theta/\tan\lambda$, with sinusoidally varying phase shift.

Small (but nonzero) values of θ have been suggested in order to explain pulsar "glitches" (speed ups) and timing "noise" in terms of precession- and spin-down-induced starquakes.[11,22,23] Although the estimated fractional frequency difference between the electromagnetic pulses and the gravitational radiation is small, probably in the range 10^{-3} to 10^{-10} (Refs. 6, 11, 20, and 23), the fact that a difference may exist is critical for some gravitational-wave experiments. For instance, it has been suggested[6] that by controlling the frequency of a high-Q crystal to follow the radio pulsar emission, one might mechanically integrate up an observable gravitational-wave signal. Other proposals (Ref. 24 and references cited therein) involve heterodyne techniques to mechanically convert a monochromatic pulsar signal to zero frequency. These schemes clearly will fail for the simple freely precessing model described here, if the integration time needed to produce a measurable signal exceeds the reciprocal of the body-frame precession frequency. A more sophisticated broadband method of gravitational-wave detection is required. Any splitting between the gravitational and electromagnetic frequencies is a potential difficulty, but as compensation it provides another measure of the object's oblateness, including its sign (oblate vs prolate).

IV. TRIAXIAL MODEL WITH SMALL WOBBLE ANGLE: WAVEFORMS AND ANALYSIS

If the object lacks axial symmetry but its wobble angle is small enough, then its free precession and the resulting gravitational waves can still be expressed simply. Following a classical mechanics text,[16] let the body have principal moments of inertia $I_1 < I_2 < I_3$. Define two (not necessarily small) eccentricity parameters

$$e_1 = [2(I_3 - I_1)/I_1]^{1/2}$$

and

$$e_2 = [2(I_3 - I_2)/I_2]^{1/2}.$$

The mean ellipticity is $\epsilon = \frac{1}{2} e_1 e_2$. Let the precession amplitude be small, with $\vec{J}$ always near the $\hat{x}_3$ body axis and with mean wobble angle $\bar{\theta}$. To first order in $\bar{\theta}$, the mean electromagnetic pulsation frequency (from a spot fixed on the body, far from the $\hat{x}_3$ axis) is $\Omega = J/I_3$ and the precessional frequency is $\Omega_p \equiv \epsilon\Omega$. Define the small parameter $a \equiv 2\bar{\theta}/(e_1^2 + e_2^2)^{1/2}$. [Note that $\bar{\theta}$ must be much less than $\max(e_1, e_2)$ for a to be small and for this approximation to hold.] Then by plugging into the quadrupole radiation formulas,[13] we obtain

$$h_+ = \frac{2}{r}(1+\cos^2 i)(I_1 - I_2)\Omega^2 \cos 2\Omega t + \frac{\epsilon a \sin 2i}{2\sqrt{2}r}\left[(e_1 I_1 + e_2 I_2)\,\omega_+^2 \cos\omega_+ t + (e_1 I_1 - e_2 I_2)\omega_-^2 \cos\omega_- t\right],$$

$$h_\times = \frac{4}{r}\cos i\,(I_1 - I_2)\Omega^2 \sin 2\Omega t + \frac{\epsilon a \sin i}{\sqrt{2}\,r}\left[(e_1 I_1 + e_2 I_2)\omega_+^2 \sin\omega_+ t + (e_1 I_1 - e_2 I_2)\omega_-^2 \sin\omega_- t\right], \quad (2)$$

where $\omega_\pm \equiv (1 \pm \epsilon)\Omega$.

The above h_+ and $h_\times$ are defined using the same choice of transverse $\hat{v}$ and $\hat{w}$ vectors discussed following Eqs. (1). To get into the orientation and time origin of the waveforms (2), an observer can rotate his transverse axes to maximize $|h_{+,2\Omega}|/|h_{\times,2\Omega}|$. The same orientation will maximize $|h_{\times,\omega_+}|/|h_{+,\omega_+}|$ and $|h_{\times,\omega_-}|/|h_{+,\omega_-}|$, if the object producing the gravitational waves is indeed a precessing triaxial body with small wobble angle. An additional check is that the frequency of the 2Ω radiation must equal the sum of the frequencies of the other two components of the radiation (plus corrections of order $\bar{\theta}^2$). As before, one of the transverse axes $\hat{v}$, $\hat{w}$, $-\hat{v}$, or $-\hat{w}$ lies along the projection of $\vec{J}$ into the sky, but gravitational observations cannot resolve the 90° ambiguity. The choice of $t=0$ to make the time dependence of the measured h's agree with Eqs. (2) corresponds, as for the symmetric case earlier, to the body $\hat{x}_3$ axis lying in the plane defined by $\vec{J}$ and the direction toward the observer, with $\hat{x}_3$ at $t=0$ as far from the observer as it ever gets.

The object's inclination angle i is again defined unambiguously and redundantly by the ratios of the components of h_+ and $h_\times$ at the three frequencies. However, in making the small-$\bar{\theta}$ approximation, we have sacrificed the information $[O(\bar{\theta}^2)]$ necessary to derive the mean wobble angle $\bar{\theta}$ itself from the observations. The splitting between the various frequency components of the gravitational waves does enable one to measure the mean ellipticity ϵ. The relative amplitudes of the waves then give a variety of nonlinear combinations of the three moments of inertia and the wobble angle. If the distance to the object, r, is known by other means (so that the 2ω radiation gives a value for $I_1 - I_3$), then the observations are sufficient to determine all of the unknowns: I_1, I_2, I_3, and $\bar{\theta}$.

We view the results of the triaxial-rigid-body case [Eqs. (2) for small $\bar{\theta}$ and the waveforms for arbitrary $\bar{\theta}$ in a subsequent paper] not necessarily as predictions of actual gravitational waveforms to be expected, but as indications of the probable complexity and high information content of gravitational waves from astrophysical sources. Pulsars in nature are not perfectly rigid, and they are subject to significant electromagnetic radiation-reaction torques, accretion of matter, glitches, timing noise of uncertain origin, and other effects which we have omitted. Gravitational astronomy may be a powerful way to get a handle on the details of those effects.

V. CONCLUSIONS

Previous investigators have derived the correct energy and angular momentum loss equations for rigid rotating bodies in general relativity.[4,25-27] For the case $I_1 = I_2$, their result for the gravitational-wave luminosity is, in our notation

$$L_{GW} = \tfrac{2}{5}\epsilon^2 I_1^2 \omega^6 \sin^2\theta(16\sin^2\theta + \cos^2\theta),$$

where the $16\sin^2\theta$ term is from 2ω radiation and the $\cos^2\theta$ term is from ω radiation. To our knowledge, the fact that ω radiation exists and is significant has never been clearly pointed out. (Perhaps it has been overlooked because it vanishes when an object rotates about a principal axis.) For small wobble angles $\theta \ll 90°$, the radiation at frequency ω is in fact larger than the 2ω radiation for a sufficiently symmetric object. [The reason is simple: A body, such as an American football, wobbling by a small angle about its symmetry axis, has a large time-changing piece that "looks like itself" after a time $2\pi/\omega$, but only a small piece that "looks like itself" after time π/ω. In contrast, a football tumbling end-over-end ($\theta \sim 90°$) "repeats itself" every half revolution, and radiates gravitational waves most strongly at 2ω.] The possible difference between the frequency of the gravitational radiation (produced by the body's inertia tensor) and the mean electromagnetic pulsar frequency (produced by a spot fixed to the body's surface) is also significant.

A recent estimate by one of us (Zimmermann) of the actual astrophysical amplitude of the waves produced by pulsars, such as the Crab and Vela, found $h \sim 10^{-24}$ to 10^{-29} (Ref. 6) at frequency 2ω. Energy conservation, balancing spin-down and gravitational-wave luminosity, means that neither of these objects can have h at frequency ω much over 10^{-24}. But we must point out here that the formulas used for this estimate[6] are in error. For small θ, the gravitational luminosities ($\mathrm{erg\,sec^{-1}}$) calculated in Ref. 6 are too high by a factor of 16 and the bulk of the luminosity occurs at frequency ω, not 2ω. The actual mean wave amplitude h at frequency ω is a factor of 2 smaller than the values quoted; at frequency 2ω, the actual h is smaller by a factor of $\theta^{-1} \sim 100$, if θ is as small as estimated. (These errors are smaller than the astrophysically induced uncertainties in the estimates of Ref. 6.)

It is conceivable that experiments sensitive enough to detect sources with $h \sim 10^{-24}$ will be running within the next decade. Gravitational astronomers who do such experiments should be aware of the likelihood that the strongest radiation will be near but not at the radio pulsar frequency. Successful observations of these gravitational waves will yield new information about pulsar structure and spin alignment, information probably not obtainable by any other means.

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