Comments on "Anharmonic oscillator"

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It is pointed out that the quantization conditions derived by Lu, Wald, and Young in a paper on an anharmonic oscillator are the same as the well-known JWKB quantization condition, particularized to JWKB approximations of the third and fifth orders, which can be derived in a much simpler and more general way. A number of corrections to the above-mentioned paper by Lu, Wald, and Young are also pointed out.

The Miller and Good[1] method, as extended by Lu[2] to include higher-order terms, was used in a paper by Lu, Wald, and Young[3] to calculate the energy levels of an anharmonic oscillator. In the present note we point out that, as particular cases of the well-known JWKB-quantization condition, one immediately obtains the quantization conditions derived in a more complicated way by Lu, Wald, and Young.[3] We also point out a number of corrections to their paper.

Since Lu, Wald, and Young[3] use the harmonic oscillator, corresponding to $p_2{}^2=2m[(n+\frac{1}{2})\hbar(k/m)^{1/2}-\frac{1}{2}kx^2]$, as the solved problem, the right-hand member of their Eq. (6) as well as that of their Eq. (16) is easily seen to be equal to $2\hbar(n+\frac{1}{2})\pi$. One therefore realizes that their quantization conditions (6) and (16) to the orders $\hbar^2$ and $\hbar^4$, respectively, are the same as the usual JWKB-quantization condition of the third order and of the fifth order, respectively. Simple, transparent ways of deriving this quantization condition in arbitrary order have recently been reviewed by Fröman and Fröman.[4]

We also point out that the following corrections should be introduced in the paper by Lu, Wald, and Young.[3] In Eq. (4) the term $-(1-\alpha)S$ in the square brackets should be replaced by $-(1-\alpha)S^2$. On the right-hand side of Eq. (5b), the numerator $(a^2\hbar^2/m)^{1/2}$ should be replaced by $(a^2\hbar^2/m)^{1/3}$. On the right-hand side of Eq. (19) inside the square brackets, the last term in the parentheses multiplying $F(\frac{1}{2}\pi,K)$ should not be $-3a^4b^2$ but $-3a^2b^4$. On the right-hand side of Eq. (21), inside the square brackets, the next-to-last term in the parentheses multiplying $F(\frac{1}{2}\pi,K)$ should not be $-345a^2b^2$ but $-345a^2b^6$. On the right-hand side of Eq. (22) the factor $224(1-\alpha^2)$ in front of the first integral should be replaced by $224(1-\alpha)^2$, in the integrand of the fifth integral $(a^2+S^2)^{7/2}$ should be replaced by $(a^2+S^2)^{5/2}$, and in the parentheses in front of the sixth (i.e., the last) integral the first term should not be $-224(1-\alpha^2)/\alpha^{21/4}$ but $-224(1-\alpha)^2/\alpha^{21/4}$.

From what was said in the beginning of this note it is clear that by calculating the energy levels by means of the JWKB-quantization condition in the third and in the fifth orders, we should reproduce the figures in columns (b) and (c), respectively, of Table I in the paper by Lu, Wald, and Young.[3] Such a check shows that the values in

TABLE I. Corrected values in columns (c) of Table I in the paper by Lu, Wald, and Young.[3]

n	$\alpha=0.2$	$\alpha=0.4$	$\alpha=0.6$	$\alpha=0.8$	$\alpha=1.0$
0	2.039 46	2.145 74	2.233 48	2.315 66	2.397 99
1	6.510 73	7.341 73	8.145 73	8.896 26	9.596 51
2	11.629 19	13.691 82	15.556 64	17.242 33	18.786 13
3	17.233 16	20.838 88	23.989 99	26.795 53	29.343 00
4	23.239 06	28.624 27	33.234 45	37.302 23	40.977 24
5	29.593 02	36.951 92	43.163 90	48.611 94	53.517 35
6	36.256 68	45.755 86	53.692 61	60.622 82	66.847 56
7	43.200 97	54.987 39	64.757 72	73.260 43	80.883 40
8	50.402 92	64.608 72	76.310 73	86.467 49	95.559 98
9	57.843 84	74.589 53	88.312 88	100.198 30	110.825 57

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columns (b) are correct (except for one unit in the last figure at two places), while the values in columns (c) are wrong. The reason for this may be due to some of the errors in the formulas pointed out above or to errors in the numerical calculations, but we have not investigated this question further. We only give the correct values which should replace the values in columns (c) of Table I in the paper by Lu, Wald, and Young.[3]

Finally we remark that the difference between the results to second order in $\hbar$, given by Lu, Wald, and Young[3] in columns (b) of their Table I, and the JWKB results obtained by Handelsman and Lew[5] to second order in $\hbar$, quoted by Lu, Wald, and Young[3] in columns (d) of their Table I, is only due to the fact that Handelsman and Lew,[5] starting from the JWKB-quantization condition, obtain the eigenvalue in the form of an explicit series expansion, which they truncate after a finite number of terms.

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[1]S. C. Miller Jr. and R. H. Good Jr., Phys. Rev. 91, 174 (1953).
[2]P. Lu, J. Chem. Phys. 51, 1524 (1969).
[3]P. Lu, S. S. Wald, and B.-L. Young, Phys. Rev. D 7, 1701 (1973).
[4]N. Fröman and P. O. Fröman, J. Math. Phys. 18, 96 (1977).
[5]R. A. Handelsman and J. S. Lew, J. Chem. Phys. 50, 3342 (1969).