Comments and Addenda

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Comments on an approximation scheme for strong-coupling expansions

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We examine some general properties of an approximation scheme for taking a strong-coupling lattice theory off the lattice proposed recently by Bender *et al*.

In the past few years there have been several attempts by many authors[1-3] to formulate a strong-coupling expansion for local field theories by perturbing in the kinetic terms of the action about the remaining static-ultra-local[4,5] (or independent-valued[6]) theory.

There is a major difficulty in that the diagrams in the resulting diagrammatic expansion are extremely divergent. This happens because the "propagator" in these diagrams is the *inverse* of the usual Feynman propagator, and hence is a very singular distribution. As an intermediate step, the easiest way to give numerical meaning to the diagrams is to regularize by introducing an elementary length a. Most authors put the fields on a lattice,[5] but alternative regularization schemes can be devised. This introduction of a length eliminates problems of measure,[6] and makes all diagrams finite.

The difficulty arises in taking the length a to zero. Most authors do not attempt this necessary (for a local field theory) final step. Recently in Ref. 7 a novel attempt was made to come off the lattice, and this brief paper is a comment on the methods adopted in Ref. 7 to do this.

The organization of our work is as follows: First, we summarize the ansatz of Ref. 7 for setting the length a equal to zero. Second, we attempt to determine general circumstances for which the ansatz is true. Third, we consider examples, including a simplified theory [$O(N)$-invariant $\lambda(\varphi^2)^2$ theory in the large-N limit], in which our conclusion can be tested.

We begin by reiterating the tactics of Ref. 7, to which the reader is referred for greater detail. Restricting ourselves to $\lambda\varphi^4$ theory in $d < 4$ dimensions, we define the dimensionless parameter

$$x = \lambda^{-1/2} a^{d/2-2} , \tag{1}$$

where a is the lattice length. It is found that all relevant quantities (e.g., masses) are expressible in terms of the dimensional bare parameters of the theory multiplied by series of the type

$$s(x) = x^\beta \sum_{n=0}^{\infty} A_n x^n , \quad \beta > 0, \quad A_0 \neq 0 \tag{2}$$

where each A_n is a fixed combinatoric factor obtained from the relevant regularized diagrams. As in any perturbation theory, only the first few A_n's are readily computable.

The new step made in Ref. 7 was to postulate a sequence of approximants s_p ($p = 1, 2, \ldots$) for the $x \to \infty$ limit of $s(x)$ that converge sufficiently rapidly for a useful result to be obtainable from knowledge of a very few A_n's. First, the power series for

$$s(x)^{1/\beta} = x \sum_{n=0}^{\infty} \alpha_n x^n \tag{3}$$

is determined. This is then inverted as

$$s(x)^{1/\beta} = \frac{x}{\sum_{n=0}^{\infty} a_n x^n} . \tag{4}$$

Then, if this denominator

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \tag{5}$$

has, for its pth power, the series

$$f(x)^p = \sum_{n=0}^{\infty} a_n^{(p)} x^n \ , \tag{6}$$

we compute the sequence $s_1, s_2, s_3, \ldots$, where

$$s_p = [a_p^{(p)}]^{-\beta/p} \ . \tag{7}$$

If the s_p's are *real*,[8] and the limit

$$\bar{s} = \lim_{p\to\infty} s_p \tag{8}$$

exists, it is tempting to assume that

$$s_\infty = \lim_{x\to\infty} s(x) \tag{9}$$

also exists, and that

$$\bar{s} = s_\infty \ . \tag{10}$$

There is no doubt that for the examples considered in Ref. 7 the approximants s_p seem to converge rapidly. Moreover, for the anharmonic oscillator the s_p's converge rapidly to the known solutions (e.g., 0.2% error by s_5). For this reason the approximant scheme must be taken seriously, and the whole merit of Ref. 7 depends upon its usefulness. However, on applying this approximant scheme to random series $f(x)$ it is not difficult to find functions $s(x)$ with finite s_∞, such that (10) is *not* true. This possibility was not discussed in Ref. 7. It follows that the approximant scheme is worthless unless its success can be anticipated by invoking further information than merely the existence of series $s(x)$.

Our first aim is to determine necessary and sufficient conditions for the equality of $\bar{s}$ and s_∞, assuming that s_∞ exists. We have no particular expectations for the $f(x)$'s arising in $\lambda\varphi^4$ theory, but we assume that any series $f(x)$ has nonzero radius of convergence (and can be defined outside this radius by analytic continuation in the cut x plane). It follows from the definition (7) that s_p can be written as

$$s_p = \left[\frac{1}{2\pi i}\oint_C \frac{dx}{x}\, s(x)^{-p/\beta}\right]^{-\beta/p} \ , \tag{11}$$

where the contour C of integration is a counterclockwise loop around $x=0$ within the circle of convergence of f. The integrand vanishes at the zeros of f and, in addition to $x=0$, has singularities at the singularities at f. Moreover, the saddle points of $-\ln s$ occur whenever

$$s'(x) = 0 = f(x) - xf'(x) \ . \tag{12}$$

In order to evaluate s_p we try to deform C in such a way that it passes through zeros of $f(x)$, negotiating the nearest (of any) saddle points on the way. For large p any arc of the contour beginning and ending on zeros and traversing a saddle point will give a contribution dominated by it.

Suppose, for the sake of simplicity, we have a single *finite* saddle-point solution to (12) at $x = x_s$. Furthermore, suppose that C can be split into two arcs C_s and C_b (terminating at zeros of f) such that C_s traverses the saddle point. Then

$$s_p = \{B(p) + Ap^{-1/2} s(x_s)^{-p/\beta}[1 + O(p^{-1})]\}^{-\beta/p} , \tag{13}$$

where $B(p)$ is the contribution from the "background" arc C_b, and the second term (with A independent of p) is the saddle-point contribution from C_s.

We see that, if the saddle-point term dominates $B(p)$, we have

$$\bar{s} = \lim_{p\to\infty} s_p = s(x_s) \neq s_\infty \ \text{(in general).} \tag{14}$$

If we further assume that, if the saddle point does not dominate, $\bar{s} = s_\infty$, we must identify

$$\bar{B} = \lim_{p\to\infty} B(p)^{-\beta/p}$$
$$= \lim_{p\to\infty} \left[\frac{1}{2\pi i}\int_{C_b} \frac{dx}{x}\, s(x)^{-p/\beta}\right]^{-\beta/p} , \tag{15}$$

with s_∞. Thus if a single saddle point exists at $x = x_s$, a plausible rule of thumb for the approximants s_p to converge to a limit $\bar{s}$ *different* from s_∞ is that

$$s'(x_s) = 0, \quad \left|\frac{s(x_s)}{s_\infty}\right| < 1 \ . \tag{16}$$

If there is more than one saddle point, we expect $\bar{s} \neq s_\infty$ if (16) is satisfied for *any* of the saddle points. A corollary to these assumptions is that, if there are *no* finite saddle-point solutions to (12), we expect $\bar{s} = s_\infty$. (If s_∞ does not exist, we would expect s_p to diverge if there are no saddle points.)

To show how this works in practice we consider the following simple example:

Example I. A simple (unphysical) series with one saddle point. Consider $s(x) = x/f(x)$ where

$$f(x) = bx + a\sum_{r=0}^{\infty} \binom{-N}{r}(-x)^r , \quad b > 0 \tag{17}$$

(integer N). For $|x| < 1$, f converges to

$$f(x) = bx + a(1-x)^{-N} \tag{18}$$

and is defined by this elsewhere.

Evaluating s_p directly we find that

$$(s_p)^{-p} = b^p \left[\frac{2\pi(N+1)}{N}\right]^{1/2} \prod_{k=1}^{N} \left[\frac{\Gamma(k/(N+1))}{\Gamma(k/N)}\right] {}_{N+1}F_N\left(-p, \frac{1}{N+1}, \frac{2}{N+1}, \ldots, \frac{N}{N+1}; \frac{1}{N}, \frac{2}{N}, \ldots, 1; -\frac{(N+1)^{N+1}}{N^N}\,\frac{a}{b}\right), \tag{19}$$

where ${}_{N+1}F_N$ is a generalized hypergeometric function. The integral representation for ${}_{N+1}F_N$ enables us to rewrite s_p as

$$(s_p)^{-p} = \left[\frac{2\pi(N+1)}{N}\right]^{1/2} \prod_{k=1}^{N} \left(\frac{k}{N(N+1)}\right)^{-1} \int_0^1 \prod_1^N \left[dt_k\, t_k^{-1+k/(N+1)} (1-t_k)^{-1+k/N(N+1)}\right] \times \left(b + \frac{(N+1)^{N+1}}{N^N}\, a t_1 t_2 \cdots t_N\right)^p . \tag{20}$$

The large-p behavior of s_p is thus given by the end-point behavior of the integrand. We have several cases to consider:

$$\text{(i) } b \neq 0, \quad \left|1 + \frac{(N+1)^{N+1}}{N^N}\,\frac{a}{b}\right| > 1. \tag{21}$$

The dominant contribution comes from $t_i \approx 1$ to give

$$\bar{s} = \left(b + \frac{(N+1)^{N+1}}{N^N}\, a\right)^{-1} = s(x_s) , \tag{22}$$

where $x_s = (N+1)^{-1}$ is the position of the saddle point. That is, the saddle point dominates the contour integral [Fig. 1(a)] and, as expected,

$$\bar{s} = s(x_s) < s_\infty = b^{-1} , \tag{23}$$

satisfying (16).

$$\text{(ii) } b \neq 0 , \quad \left|1 + \frac{(N+1)^{N+1}}{N^N}\,\frac{a}{b}\right| = \left|\frac{s_\infty}{s(x_s)}\right| < 1 . \tag{24}$$

The dominant contribution comes from $t_i \approx 0$ to give

$$\bar{s} = b^{-1} = s_\infty . \tag{25}$$

As expected from (16) the approximant scheme works, the saddle point not dominating [Fig. 1(b)].

Finally, we consider $b = 0$, when

$$s(x) = a^{-1} x (1-x)^N \sim x^{N+1} \tag{26}$$

and s_∞ does not exist. However, (20) reduces to a product of B functions to give

$$\bar{s} = \frac{(N+1)^{N+1}}{N^N}\, a = s(x_s) . \tag{27}$$

That is, $\bar{s}$ exists and is real (although s_p is not necessarily real) even though s_∞ does not exist, and can take any power behavior. (If however, there had been no saddle point, we would have expected s_p to diverge appropriately.)

With this example reinforcing our prejudices, our conclusions are that a necessary condition for the working of the approximant scheme is that, if $s(x)$ has any extrema in the cut x plane, $|s/s_\infty| > 1$ at them all. Further exactly soluble

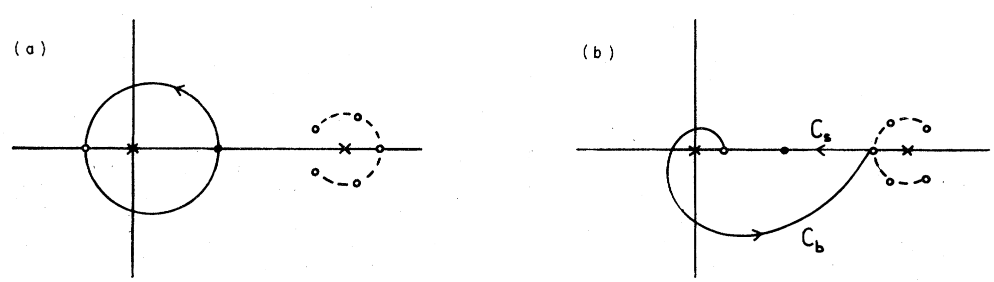

FIG. 1. Contours for the integral (11) for s_p for the example (17). Crosses denote singularities of the integrand at $x = 0$ and $x = 1$, open circles denote zeros, and the solid dot denotes the saddle point at $x_s = (N+1)^{-1}$. (a) $a/b > 0$ (and small) satisfying (21). There is no contour C_b and the saddle point dominates. (b) $a/b < 0$ (and small) satisfying (24). The contour C_s containing the saddle point is dominated by C_b. In both cases we have taken N odd.

examples can be constructed. We have not been able to find any contradictions.

Example II. The (large-N) O(N)-invariant $\lambda(\varphi^2)^2$ *theory in one dimension.* In the previous example there were variable parameters, giving different results in different ranges. In physical problems, all parameters in $f(x)$ are purely combinatoric and fixed. Let us first consider the $O(N)$-invariant anharmonic oscillator (the anharmonic oscillator being the most discussed example in Ref. 7) with Lagrangian

$$L=\tfrac{1}{2}\dot{\varphi}_a^2-\tfrac{1}{2}m_0^2\varphi_a^2-\frac{\lambda_0}{4N}(\varphi_a^2)^2 \tag{28}$$

$(a=1,2,\ldots,N)$. We take $m_0^2>0$.

As an example of a power series $s(x)$, consider the diagrammatic expansion for the propagator of the scalar fields. All diagrams have

$$\delta_{ab}K(x-y)=\delta_{ab}\partial^2\delta(x-y) \tag{29}$$

as their propagators, and the derivatives λ_n of

$$W[j_a]=a^{-1}\int dx\ln F(j_a(x)) \tag{30}$$

(at $j=0$) as their n-point vertices, where

$$F(q_a)=\int_{-\infty}^{\infty}\prod_1^N dx_a\exp\left\{-a\left[\tfrac{1}{2}m_0^2x_a^2+\frac{\lambda_0}{4N}(x_a^2)^2+q_ax_a\right]\right\}. \tag{31}$$

It is sufficient for our purposes to note[9] that the p point vertex is $O(N^{1-p/2})$. Furthermore, every closed loop gives a factor N. In the large-N limit we see that the propagator W_2 is a geometric series of the form

$$W_2^{ab}(p^2)=\frac{\delta^{ab}\Lambda}{\Lambda p^2-1}, \tag{32}$$

where Λ is the sum of all one-particle irreducible [in the sense of (29)] bubble diagrams. Figures 2(a) give examples of such diagrams.

Rather than evaluate Λ (on the lattice) directly by summing such diagrams, we know by conventional large-N tactics[10] that it obeys the equation[11]

$$\frac{1}{\lambda_0}=\frac{m_0^2\Lambda}{\lambda_0}+\Lambda^2[1+\Lambda\partial^2+\Lambda^2(\partial^2)^2+\Lambda^3(\partial^2)^3+\cdots]\,\delta(x-y)\big|_{x=y}. \tag{33}$$

The right-hand side of (33), containing progressively more singular terms, is evaluated on the lattice (lattice spacing a) by the replacement

$$\partial^{2r}\delta(x-y)\big|_{x=y}=(-1)^r a^{-2r-1}\frac{(2r)!}{(r!)^2} \tag{34}$$

whence (33) becomes

$$\frac{1}{\lambda_0}=\frac{m_0^2\Lambda}{\lambda_0}+\frac{\Lambda^2}{a}\left[1-2\left(\frac{\Lambda}{a^2}\right)+6\left(\frac{\Lambda}{a^2}\right)^2+\cdots\right] \tag{35}$$

$$=\frac{m_0^2\Lambda}{\lambda_0}+\frac{\Lambda^2}{(4\Lambda+a^2)^{1/2}}. \tag{36}$$

Rewriting (36) as

$$\Lambda=\left(\frac{a}{\lambda_0}\right)^{1/2}(1-m_0^2\Lambda)^{1/2}\left(1+\frac{4\Lambda}{a^2}\right)^{1/4} \tag{37}$$

and expanding the right-hand side in powers of Λ, (37) can be solved iteratively for Λ as a power series in $\lambda_0^{-1/2}$, the lattice strong-coupling expansion. Defining x by (1), Λ can be further expressed in terms of series of the type (2). For example, if $m_0^2=0$ for simplicity, we have

$$\chi=\Lambda^{-1}=\frac{\lambda_0^{2/3}x^{1/3}}{h(x)}, \tag{38}$$

$$h(x)=1+x-\tfrac{1}{2}x^2+\tfrac{7}{8}x^4-2x^5+\tfrac{39}{16}x^6+\cdots. \tag{39}$$

In fact, by exact calculation $h(x)$ is given by

$$2^{1/2}3^{1/4}h(x)=[2(s_+^2+s_-^2+1)^{1/2}-(s_+-s_-)]^{1/2}+[s_+-s_-]^{1/2}, \tag{40}$$

where

$$s_\pm=[(1+27x^4)^{1/2}\pm3\sqrt{3}\,x^2]^{1/3}, \tag{41}$$

showing that it has a finite radius of convergence in the x plane (and further that it is a power series over the rational numbers).

From (39) the approximants

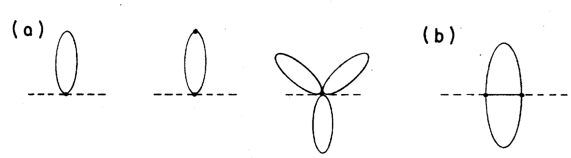

FIG. 2. Diagrams contributing to Λ. (a) Typical diagrams contributing to leading $O(1)$ in the N^{-1} expansion, of order λ_0^{-1}, $\lambda_0^{-3/2}$, λ_0^{-2}, respectively (from left to right) (b) The diagram of order λ_0^{-2} contributing to $O(N^{-1})$.

$$s_p = \lambda_0^{-2/3}\Lambda_p \tag{42}$$

to Λ are

$$s_1 = 1.442,$$
$$s_2 = 1.513,$$
$$s_3 = 1.537, \tag{43}$$
$$s_4 = 1.550,$$
$$s_5 = 1.557,$$

converging monotonically to the known limit

$$s_\infty = 2^{2/3} = 1.587\ . \tag{44}$$

No doubt improved schemes would give more rapid convergence.

Do our previous considerations explain the success of the approximant scheme in this case? From (37) we see that

$$\frac{d\Lambda}{dx}\left[\lambda_0^2\Lambda^3 - (1-m_0^2\Lambda)^2 + 2\Lambda m_0^2(1-m_0^2\Lambda)\right] = -\frac{(1-m_0^2\Lambda)}{3\lambda_0^{2/3}x^{7/3}}\left[(1-m_0^2\Lambda) + \tfrac{1}{2}m_0^2 x\frac{d\Lambda}{dx}\right] \tag{45}$$

with *no* finite x solutions to $\Lambda' = 0$ and hence *no* saddle points. We would thus expect the success of the approximant scheme.

This is very gratifying, but the large-N behavior is really a classical field theory to be treated in its own right, and it is difficult to estimate its relevance to the $N=1$ anharmonic oscillator of Ref. 7. However, we note that, of the *totality* of diagrams of order $\lambda_0^{-1/2}$, λ_0^{-1}, $\lambda_0^{-3/2}$, λ_0^{-2} contributing to W_2, only the *single* diagram of Fig. 2(b) (order λ_0^{-2}) would be omitted to leading order in an N^{-1} expansion. This leads us to the next example.

Example III. The Hartree-Fock approximation to $\lambda\varphi^4$ theory in one dimension. The previous example suggests that, insofar as the first few terms of (for the $N=1$ anharmonic oscillator) are any guide, it may be a good approximation to retain only the one-particle irreducible bubble diagrams. This is the Hartree-Fock approximation to Λ, in which W_2 is still a pure pole term with Λ now satisfying

$$\frac{1}{\lambda_0} = \frac{m_0^2\Lambda}{\lambda_0} + \frac{3\Lambda^2}{(4\Lambda + a^2)^{1/2}} \tag{46}$$

obtained trivially from (36) by the replacement $\lambda_0 \to 3\lambda_0$. The approximant scheme will therefore be valid for this approximation which, as anticipated, is numerically good, the errors in $\Lambda^{-1/2}$ being 5% or less over the whole range of λ_0 and m_0^2.[12] It has been shown[13] for $a=0$ that there are yet better polynomial functional relationships than (46) (which presumably would be mimicked by nonleading terms in the N^{-1} expansion). We have not attempted to construct these $a \neq 0$, but they would enable us to assess the validity of the approximant scheme for the full theory.

Alternatively, if there were a general theorem to the effect that, for fixed m_0^2 and λ_0, the energy levels of the anharmonic oscillator were monotonic functions of the lattice length a, this would provide a sufficient condition for the validity of the approximant scheme.

So far we have not considered infinite renormalization.

Example IV. The (large-N) $O(N)$ $\lambda(\varphi^2)^2$ theory in three dimensions. We very briefly consider the O(N) scalar theory in three dimensions in the large-N limit (or equivalently, the Hartree-Fock approximation to the $N=1$ theory). In Ref. 7 it was shown that the s_p diverged, which was interpreted as indicating the need for infinite renormalization.

Let us again consider Λ of (32). As before, Λ satisfies (33) [with $\delta(x)$ replaced by $\delta^3(x)$]. On the lattice we would expect this to become

$$\frac{1}{\lambda_0} = \left(\frac{m_0^2}{\lambda_0} + \frac{1}{2\pi a}\right)\Lambda - \frac{1}{2\pi}\frac{\Lambda}{(4\Lambda + a^2)^{1/2}} \tag{47}$$

or, in terms of x,

$$\frac{1}{\lambda_0} = \left(\frac{m_0^2}{\lambda_0} + \frac{\lambda_0 x^2}{2\pi}\right)\Lambda - \frac{1}{2\pi}\frac{\Lambda\lambda_0 x^2}{(1+4\Lambda\lambda_0^2 x^4)^{1/2}}\ . \tag{48}$$

We know that the correct renormalization procedure (in general) is to make m_0^2 (which needs infinite renormalization) a *function* of x^2. That is, for this case we take[10]

$$m_0^2(x) = m^2 - \frac{\lambda_0^2 x^2}{2\pi}\ , \tag{49}$$

where m^2 is x independent.

We are unsure of the merits of keeping m_0^2 fixed and developing series in x, as suggested in Ref. 7. If we wish to do this, however, and *a priori* neglect the fact that Λ^{-1} diverges as x^2, it follows from (48) that there are no solutions to $\Lambda' = 0$ for finite x. From our early comments, this suggests that all sequences of approximants s_p diverge. Still proceeding as in Ref. 7, and evaluating the most singular part χ_s of Λ^{-1} (by setting $m_0^2 = 0$), we can incorporate this divergence by writing

$$\chi_s = x^2\chi_f = x^2\Lambda_f^{-1}\ , \tag{50}$$

where [cf. (48)]

$$\frac{2\pi}{\lambda_0^{\ 2}} = \Lambda_f - \frac{\Lambda_f}{(1+4\Lambda_f\lambda_0^{\ 2}x^2)^{1/2}} \quad . \tag{51}$$

As $x \to \infty$, Λ_f remains finite and, since $\Lambda_f' = 0$ has no finite-x solutions, the approximants for Λ_f (or equivalently, χ_s/x^2) converge to the correct limit.

We conclude with an observation on the stability of the approximant scheme which will be important for divergent series such as the above. This is that if $s(x)$, $h_1(x)$, $h_2(x)$ are series in x of the form (2) such that

$$\lim_{x\to\infty} (s/h_1) = \lim_{x\to\infty} (s/h_2) \ , \tag{52}$$

in general the approximants will have limits

$$\overline{(s/h_1)} = \lim_{p\to\infty} (s/h_1)_p \neq \lim_{p\to\infty} (s/h_2)_p = \overline{(s/h_2)} \ . \tag{53}$$

This is because these latter limits depend on the finite-x behavior (e.g., saddle points) which can be markedly different. Thus choosing trial functions h such that $\overline{(s/h)}$ exists via real s_p is an *ambiguous* process.

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$$\chi = m_0^{\ 2} + \lambda_0(2\pi)^{-1}\int dk\,(k^2+\chi)^{-1},$$

where $\chi = K^{-1}$ is the position of the pole (in p^2) of W_2.
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