Mechanism of stimulated radiation by charged particles

W. W. Zachary
Naval Research Laboratory, Washington, D. C.
(Received 30 April 1979)

We have studied the mechanism for radiation by charged particles called stimulated electromagnetic shock radiation (SESR) by Schneider and Spitzer caused by the interaction between a relativistic charged particle and an externally applied electromagnetic plane wave in a dielectric. The present theory predicts that the SESR effect is large when the frequency of the plane wave lies in the microwave region but is small at higher frequencies for plane-wave field strengths smaller than the breakdown field of the dielectric.

I. INTRODUCTION

There has been considerable interest in recent years in the study of stimulated radiation resulting from the interaction of relativistic electron beams with coherent electromagnetic fields in polarizable media. In a series of interesting papers,[1-5] Schneider and Spitzer have discussed a mechanism for the generation of "stimulated electromagnetic shock radiation" (SESR). As of this writing, the SESR process has not been experimentally observed, although active experimental efforts in this direction are under way at present.[6] In order to aid in the interpretation of these experiments, it is important to have theoretical estimates of the amount of SESR radiation produced by a relativistic electron compared, for example, with Čerenkov radiation.

Such estimates have been given by Schneider and Spitzer[1,4,5] based upon exact solutions for the SESR fields in dispersionless media. In this case the potentials are discontinuous along a conical surface (shock front) which travels with the electron and are, in fact, infinite on this surface. The trouble with this approach is that these singular expressions must be differentiated in order to determine the radiated energy, a procedure that is mathematically ambiguous owing to the singularity on the shock front. In fact, Zin[7] pointed out that the Čerenkov fields for a dispersionless medium have the wrong direction owing to the fact that one is differentiating a discontinuous function. In view of this result, one might expect that the fields corresponding to stimulated radiation in a dispersionless medium may be subject to question.

In the case of Čerenkov radiation it is possible to derive a relatively simple expression for the radiated energy for dispersive media in which no divergence appears, as shown by Frank and Tamm[8] and Tamm.[9]

In the present paper we will discuss the results of calculations for stimulated radiation of the same type as that of Tamm for Čerenkov radiation. In summary, we find that the SESR contribution to the total radiated energy is large for plane-wave frequencies in the microwave region but is small in the optical region for reasonable external field strengths (i.e., for those smaller than the breakdown field of the dielectric). These results are in direct conflict with the assertions of Schneider and Spitzer. The discrepancy is accounted for by the fact that these authors neglected part of the SESR contribution.

The power spectrum of Čerenkov plus stimulated radiation has been investigated by Šoln[10] by a different method, but he does not give a quantitative discussion of the energy radiated in dispersive media. In more recent work[11] Šoln and Williams have given some numerical results based on this approach. However, because of the occurrence of a divergence that we discuss in Sec. III, they do not calculate the total radiated power above threshold. We will give results of this type.

In Sec. II we give integral expressions for the fields for the Čerenkov and stimulated-radiation mechanisms for a large class of real-valued dielectric functions describing dispersive media. Although the Čerenkov case is well known, we include a brief discussion of it for purposes of easy comparison with the stimulated terms.

In Sec. III we utilize these fields to derive integral expressions for the energy lost per unit path by a relativistic electron traveling in a dielectric. It is pointed out that our analog of the method of Tamm[9] is a good approximation in the microwave region but, because of the SESR mechanism, this appears not to be the case in the optical region. We point out that it can still be concluded that the stimulated terms are small in the latter region on the basis of a "general" scaling argument. The quotation marks are used here because this argument is general in the sense that it is independent of our method of calculation of the energy loss, but it does depend upon the use of first-order perturbation theory. Actually, it can be shown that the result (3.21) agrees with the corresponding quantity of Ref. 10 if the differences in units used are taken into account so that our approximation turns out to give the "exact" answer. The reason for this is not completely understood. We have put the word "exact" in quotation marks because both methods, Šoln's and ours, depend upon the use of first-order perturbation theory.

In Sec. IV we use the equations derived in Sec. III to obtain numerical results for the energy lost by an electron in radiation for the case of a dielectric function which is a modification of the well-known Lorentz susceptibility model.

In the Appendix we state conditions which define the class of dielectric functions for which our considerations in Secs. II and III apply, and then we give a justification for taking derivatives with respect to space and time coordinates inside certain integrals over frequency which appear in the field expressions given in Sec. II. This procedure is essential in the calculations of energy loss discussed in Sec. III.

II. FIELDS

In this section we give expressions for the electromagnetic fields of the radiative mechanisms of interest for a nonmagnetic homogeneous dielectric medium of infinite extent for a class of dielectric "constants" which are real-valued functions of frequency ω. We use Gaussian units throughout the paper.

Consider a cylindrically symmetric coordinate system in which a charged particle with charge e is traveling along the z axis with constant velocity $\vec{u}=\beta c\hat{z}$, where c denotes the velocity of light in a vacuum. In the presence of an external electromagnetic wave with amplitude $\vec{E}_0$ the current density reads, to first order in $\vec{E}_0$,

$$\vec{j}(\vec{x},t)\cong\vec{j}_0(\vec{x},t)+\vec{j}_1(\vec{x},t)+\vec{j}_2(\vec{x},t)\,,$$

where

$$\vec{j}_0(\vec{x},t)=e\vec{u}\,\delta(\vec{x}-\vec{u}t)\,, \tag{2.1}$$

$$\vec{j}_1(\vec{x},t)=e\vec{v}(t)\,\delta(\vec{x}-\vec{u}t)\,, \tag{2.2}$$

$$\vec{j}_2(\vec{x},t)=-e\vec{u}[\vec{r}(t)\cdot\vec{\nabla}_x]\delta(\vec{x}-\vec{u}t)\,. \tag{2.3}$$

Here $\vec{j}_0$ represents the current density corresponding to the Čerenkov mechanism whereas the remaining terms, $\vec{j}_1$ and $\vec{j}_2$, represent a decomposition of the SESR current density into transverse and longitudinal parts, respectively. It will be convenient for calculational purposes to consider this decomposition even though the total, $\vec{j}_1+\vec{j}_2$, is the only meaningful object in a physical sense.[12] There are charge densities ρ_0 and ρ_2 corresponding to $\vec{j}_0$ and $\vec{j}_2$, respectively, which differ from the latter only by the factor $\vec{u}$.

The quantities $\vec{v}(t)$ and $\vec{r}(t)$ represent the velocity and position perturbations, respectively, caused by the electron's interaction with the incident wave. These quantities are obtained by solving the following approximation to the Lorentz force equation:

$$\gamma\frac{d}{dt}\,\vec{v}(t)=\frac{e}{m}\left(\vec{E}+\frac{\vec{u}}{c}\times\vec{H}\right), \tag{2.4}$$

where $\gamma=(1-\beta^2)^{-1/2}$ and the incident wave is represented as

$$\vec{E}=\vec{E}_0\sin(\omega_0-\vec{k}_0\cdot\vec{u})t\,,$$
$$\vec{H}=\vec{H}_0\sin(\omega_0-\vec{k}_0\cdot\vec{u})t$$

where $\vec{H}_0=[\epsilon(\omega_0)]^{1/2}\hat{k}_0\times\vec{E}_0$ with $\hat{k}_0$ a unit vector and $\epsilon(\omega_0)$ denotes the dielectric function evaluated at the angular frequency of the incident wave. We will evaluate the field at the instantaneous position of the electron, $\vec{u}t$, and will consider the case in which the electron and the plane wave are counterstreaming so that

$$\vec{k}_0=-\frac{\omega_0}{c}\,[\epsilon(\omega_0)]^{1/2}\hat{z}\,.$$

One finds, from (2.4),[5]

$$\vec{v}(t)=\frac{-e\vec{E}_0}{\gamma m\omega_0}\cos\Omega t\,,$$
$$\vec{r}(t)=\frac{-e\vec{E}_0}{\gamma m\omega_0\Omega}\sin\Omega t\,, \tag{2.5}$$

where $\Omega=\omega_0\{1+\beta[\epsilon(\omega_0)]^{1/2}\}$.

We now consider the fields. These are obtained by solving Maxwell's equations subject to the current densities (2.1)–(2.3) and the corresponding charge densities [in the cases (2.1) and (2.3)].

The vector potential for the Čerenkov fields is[9]

$$\vec{A}^c(\vec{x},t)=\hat{z}\,\frac{e}{2c}\int_{-\infty}^{\infty}d\omega\exp[i\omega(t-z/u)]a(r,\omega)\,,$$

where we have introduced cylindrical coordinates (r,ϕ,z) with the cylinder axis along the direction of the electron's velocity and have defined

$$a(r,\omega)=\begin{cases}\dfrac{2}{\pi}K_0\!\left(r\,\dfrac{|\omega|}{u}\,[1-\beta^2\epsilon(\omega)]^{1/2}\right), & \beta^2\epsilon(\omega)<1\\[2ex] -iH_0^{(2)}\!\left(r\,\dfrac{\omega}{u}\,[\beta^2\epsilon(\omega)-1]^{1/2}\right), & \beta^2\epsilon(\omega)>1,\quad \omega>0\\[2ex] iH_0^{(1)}\!\left(r\,\dfrac{|\omega|}{u}\,[\beta^2\epsilon(\omega)-1]^{1/2}\right), & \beta^2\epsilon(\omega)>1,\quad \omega<0\,.\end{cases} \tag{2.6}$$

Here K_0 is a modified Bessel function and $H_0^{(1)}, H_0^{(2)}$ are Hankel functions corresponding to outgoing and incoming cylindrical waves, respectively. The Čerenkov electric field has nonvanishing radial and axial components, but we shall need only the latter.

$$E_z^c(\vec{x},t) = \frac{-ie}{2c^2} \int_{-\infty}^{\infty} d\omega \exp[i\omega(t - z/u)] \times \omega[1 - 1/\beta^2\epsilon(\omega)]a(r,\omega)\,. \tag{2.7}$$

The Čerenkov magnetic field has only one nonvanishing component:

$$H_\phi^c(\vec{x},t) = -\frac{e}{2c}\frac{\partial}{\partial r}\int_{-\infty}^{\infty} d\omega \exp[i\omega(t - z/u)]a(r,\omega)\,. \tag{2.8}$$

The vector potential for the transverse SESR fields is

$$\vec{A}_1(\vec{x},t) = -\frac{r_0\vec{E}_0}{4\gamma\omega_0\beta}\int_{-\infty}^{\infty} d\omega \exp[i\omega(t - z/u)][\exp(i\Omega z/u)b_-(r,\omega) + \exp(-i\Omega z/u)b_+(r,\omega)]\,, \tag{2.9}$$

where $r_0 = e^2/mc^2$ denotes the classical electron radius and we have defined

$$b_\pm(r,\omega) = \begin{cases} \frac{2}{\pi} K_0\left(\frac{r}{u} a_\pm'(\omega)\right), & [a_\pm(\omega)]^2 < 0 \\ -iH_0^{(2)}\left(\frac{r}{u} a_\pm(\omega)\right), & [a_\pm(\omega)]^2 > 0, \quad \omega > 0 \\ iH_0^{(1)}\left(\frac{r}{u} a_\pm(\omega)\right), & [a_\pm(\omega)]^2 > 0, \quad \omega < 0 \end{cases} \tag{2.10}$$

with

$$[a_\pm(\omega)]^2 = -[a_\pm'(\omega)]^2 = \beta^2\omega^2\epsilon(\omega) - (\omega \pm \Omega)^2\,. \tag{2.11}$$

Since $\epsilon(\omega)$ is real valued and an even function of frequency, as noted in Eq. (A1), we obtain the following symmetry properties for the functions defined by (2.10):

$$[b_\pm(r,-\omega)]^* = b_\mp(r,\omega)\,. \tag{2.12}$$

Since there is no charge density associated with the transverse SESR fields,[5] these are obtained from (2.9) by means of the formulas

$$\vec{E}_1(\vec{x},t) = -\frac{1}{c}\frac{\partial}{\partial t}\vec{A}_1(\vec{x},t), \quad \vec{H}_1(\vec{x},t) = \vec{\nabla}\times\vec{A}_1(\vec{x},t)\,. \tag{2.13}$$

The fields associated with the current density (2.3) and the associated charge density are polarized in the same direction as the Čerenkov fields. Thus, there will be nonvanishing radial and axial longitudinal SESR electric fields and an azimuthal longitudinal SESR magnetic field. We will need only the latter two of these, which are

$$E_{2z}(\vec{x},t) = -\frac{r_0 E_{0r}\partial/\partial r}{4\gamma\omega_0\Omega}\int_{-\infty}^{\infty} d\omega \exp[i\omega(t - z/u)] \times\left[\left(\omega - \frac{\omega - \Omega}{\beta^2\epsilon(\omega)}\right)\exp\left(\frac{i\Omega z}{u}\right)b_-(r,\omega) - \left(\omega - \frac{\omega + \Omega}{\beta^2\epsilon(\omega)}\right)\exp\left(\frac{-i\Omega z}{u}\right)b_+(r,\omega)\right], \tag{2.14}$$

$$H_{2\phi}(\vec{x},t) = \frac{ir_0 cE_{0r}\partial^2/\partial r^2}{4\gamma\omega_0\Omega}\int_{-\infty}^{\infty} d\omega \exp[i\omega(t - z/u)]\left[\exp\left(\frac{i\Omega z}{u}\right)b_-(r,\omega) - \exp\left(-i\frac{\Omega z}{u}\right)b_+(r,\omega)\right]. \tag{2.15}$$

In order to calculate the energy loss of the electron due to the Čerenkov and stimulated-radiation mechanisms, which we will do in the following section, it is essential that we be able to take the indicated differentiations in (2.8) and (2.13)–(2.15) inside the integrals over frequency. It is not obvious *a priori* that such a procedure is valid. Indeed, it is *not* valid for a dispersionless medium because the integrals in question are discontinuous in this case. This can be seen explicitly in the expressions given in the Čerenkov[9] and transverse SESR[5] cases when ϵ is a constant. We prove in the

Appendix that the procedure in question is justified for the dielectric functions that we consider.

For dispersive media the integration over ω in any of the preceding formulas is limited to a finite frequency range, say $0 \leq \omega_1 \leq \omega \leq \omega_u < \infty$. We now want to mention briefly some problems that arise in the application of the results proved in the Appendix. The first thing is that $\epsilon(\omega_u)$ may be infinite when medium absorption is neglected. This happens, for example, in the case of the simple Lorentz susceptibility model,

$$\epsilon_1(\omega) = 1 + \frac{\epsilon_1(0) - 1}{1 - (\omega/\omega_r)^2} , \tag{2.16}$$

where ω_r denotes the resonance frequency. This difficulty is really not very serious and can be remedied by giving the medium nonzero absorption. All real materials have this property anyhow. We will consider only dielectric functions for which $\epsilon(\omega_u) < \infty$. The second difficulty is more serious. Because of the different behavior of the functions (2.6) and (2.10) when $\beta^2\epsilon(\omega) <$ or >1, respectively, $[a_\pm(\omega)]^2 < 0$ or >0 it will be seen in Sec. III that it is desirable to divide the integration into portions corresponding to these different frequency regions. We are then left with Bessel and Hankel functions with arguments that vanish at an end point of the integration region. Moreover, this situation occurs irrespective of whether the medium is absorptive or not. It presents a problem because of the singularity of the K_0 and Hankel functions at vanishing argument.

It will be shown in Sec. III that the expressions for energy loss can be given in a relatively simple form (as compared with the expressions for the fields) which do not involve Bessel or Hankel functions even though the fields depend upon them. Our procedure, therefore, will be to replace the limit ω_l at which the arguments of the Bessel and Hankel functions vanish by a different limit $\omega_l + \nu$, $\nu > 0$, for which the proof in the Appendix is valid. The limit $\nu \to 0$, will then be taken at the end of the energy-loss calculation where it can be shown to be harmless. The estimates of the Appendix show that the transverse SESR fields are the only ones for which this procedure is necessary.

III. ENERGY LOSS–GENERAL CASE

It was indicated in Sec. II that both the Čerenkov and longitudinal SESR fields have nonvanishing components E_r, E_z, and H_ϕ but all other components vanish. Thus, the Poynting vector of the fields $\vec{E} = \vec{E}^C + \vec{E}_2$, $\vec{H} = \vec{H}^C + \vec{H}_2$ has the same directionality character as Čerenkov alone. We will take advantage of this by calculating the radiated energy by a method similar to that of Tamm[9] and Fermi[13] in the Čerenkov case. That is, we calculate the energy loss per unit time, Φ, by integrating the radial component of the Poynting vector over the surface of a cylinder of radius r and infinite length whose axis lies along the z axis. The energy loss per unit path length, W, is obtained by dividing Φ by the magnitude of the electron's velocity $|\vec{u} + \vec{v}(t)| \cong |\vec{u}|$.

The presence of the transverse SESR contribution disrupts the symmetry of the above picture because, in general, this component of SESR has nonvanishing E_ϕ, H_r, and H_z components in addition to those components exhibited by the Čerenkov and longitudinal SESR fields. Nevertheless, the main features of the above picture will be preserved for incident frequencies ω_0 in the microwave region because then the transverse SESR contribution to the energy loss is only a small perturbation to the energy losses of the other terms. With ω_0 in the optical region the above procedure of calculating only the radial component of the Poynting vector appears not to be a good approximation because the two types of SESR contributions are comparable. Nevertheless, our conclusion that the transverse SESR component is small at all frequencies can be derived from the present approach. The reason for this is that the predominant dependence on ω_0 comes from the velocity and position perturbations (2.5). Since these are squared in calculating the Poynting vector, the transverse SESR energy loss scales as ω_0^{-2} and that due to the longitudinal part of SESR has two types of contributions, one which scales as ω_0^{-2} and the other as ω_0^{-4}. This frequency dependence will be present for whatever component of the Poynting vector one calculates. It is a curious fact that, as already noted in the Introduction, the present method gives the same expression for energy loss as the apparently more general method of Ref. 10.

According to the prescription in the first paragraph of this section, the energy loss per unit path length W is

$$\begin{aligned} W &= \frac{2\pi r}{u} \int_{-\infty}^{\infty} S_r \, dz \\ &= \frac{rc}{2u} \int_{-\infty}^{\infty} (\vec{E} \times \vec{H})_r \, dz , \end{aligned} \tag{3.1}$$

where $\vec{E}$ and $\vec{H}$ denote the total fields. In writing out $\vec{E}$ and $\vec{H}$ in terms of the Čerenkov and SESR contributions it is convenient to separate W into the following naturally occurring combinations:

$$W = W_{00} + W_{01} + W_{02} + W_{11} + W_{12} + W_{22} , \tag{3.2}$$

where

$$W_{00} = -\frac{rc}{2u} \int_{-\infty}^{\infty} E_z^C H_\phi^C \, dz , \tag{3.3}$$

$$W_{01} = -\frac{rc}{2u}\int_{-\infty}^{\infty} E_z^C H_{1\phi}\,dz\,, \tag{3.4}$$

$$W_{02} = -\frac{rc}{2u}\int_{-\infty}^{\infty} (E_z^C H_{2\phi} + E_{2z}H_\phi^C)\,dz\,, \tag{3.5}$$

$$W_{11} = \frac{rc}{2u}\int_{-\infty}^{\infty} E_{1\phi} H_{1z}\,dz\,, \tag{3.6}$$

$$W_{12} = -\frac{rc}{2u}\int_{-\infty}^{\infty} E_{2z} H_{1\phi}\,dz\,, \tag{3.7}$$

$$W_{22} = -\frac{rc}{2u}\int_{-\infty}^{\infty} E_{2z} H_{2\phi}\,dz\,. \tag{3.8}$$

Here W_{00} denotes the Čerenkov contribution, W_{01} and W_{02} are cross terms between Čerenkov and the SESR terms, W_{11} and W_{22} are the SESR contributions, and W_{12} is the cross term between the transverse and longitudinal SESR contributions. We shall examine each of these terms in turn.

Substitution of (2.7) and (2.8) into (3.3) gives

$$W_{00} = -\frac{i\pi e^2 r}{4c^2}\int_0^{\infty} d\omega\,\omega\left(1-\frac{1}{\beta^2\epsilon(\omega)}\right)\times\left[a(r,\omega)\frac{\partial}{\partial r}a(r,-\omega) - a(r,-\omega)\frac{\partial}{\partial r}a(r,\omega)\right], \tag{3.9}$$

where we have taken the differentiation $\partial/\partial r$ inside the integration in (2.8) which is justified with the aid of estimate (A6) if $\epsilon(\omega)$ satisfies the conditions stated in the Appendix. Since this implies in particular that $\epsilon(\omega)$ is real valued and satisfies (A1), we see from (2.6) that $a(r,-\omega)=a(r,\omega)$ when $\beta^2\epsilon(\omega)<1$. Thus, for media with such dielectric functions this frequency range gives a zero contribution to the integral in (3.9), as already noted by Tamm.[9] Using (2.6), we write (3.9) in the form

$$W_{00} = \frac{e^2}{c^2}\int_{\substack{\omega\geq 0\\ \beta^2\epsilon(\omega)\geq 1}} d\omega\,\omega\left(1-\frac{1}{\beta^2\epsilon(\omega)}\right), \tag{3.10}$$

which is the well-known result obtained by Frank and Tamm[8] and by Tamm.[9] It will be noted that no Bessel or Hankel functions occur in the integrand of (3.10). This is because we have used the Wronskian relation

$$Y_0(z)J_1(z) - J_0(z)Y_1(z) = \frac{2}{\pi z} \tag{3.11}$$

in the manner of Ref. 9.

In our discussion of Čerenkov radiation thus far, we have neglected the so-called "polarization loss" which arises from frequencies for which the dielectric function vanishes.[13,14] Since hereafter we will only be comparing the radiation mechanisms in the "superluminal region", $\beta^2\epsilon(\omega)>1$, such frequencies will lie outside the range of integration.

Using (2.7), (2.9), and (2.13) we find from (3.4)

$$W_{01} = \frac{\pi e r_0 E_{0r} r}{8\gamma\omega_0^2 u^2}\int_{-\infty}^{\infty} d\omega\,\omega^2\left(1-\frac{1}{\beta^2\epsilon(\omega)}\right)a(r,\omega)[b_-(r,-(\omega-\Omega))\exp(i\Omega t) + b_+(r,-(\omega+\Omega))\exp(-i\Omega t)]\,,$$

where we have justified the passage of a z differentiation through the integration in (2.9) by the considerations of the Appendix, the estimates (A8) in particular. It can now be seen that W_{01} vanishes when averaged over a period of the frequency $\Omega/2\pi$. Thus, there is no net radiation yield from the term W_{01}. This could have been anticipated since W_{01} is the cross term between the Čerenkov fields which do not depend upon Ω and the transverse SESR fields that do. See also the remarks concerning this point in Ref. 10.

A similar result holds for W_{02}. This is easily verified by using (2.7), (2.8), (2.14), (2.15), and the estimates (A6), (A10), (A12), and (A13) to derive an integral expression for W_{02} from (3.5).

Now consider the transverse SESR contribution (3.6). Using (2.9), (2.13), (A9), and (A10) we find

$$W_{11} = -\frac{i\pi r_0^2 E_{0\phi}^2}{16\omega_0^2\beta^2\gamma^2}\int_{-\infty}^{\infty} d\omega\,\omega\Big[b_-(r,\omega)\frac{\partial}{\partial r}rb_+(r,-\omega) + b_+(r,\omega)\frac{\partial}{\partial r}rb_-(r,-\omega) + \exp(2i\Omega t)b_-(r,\omega)\frac{\partial}{\partial r}rb_-(r,-(\omega-2\Omega)) + \exp(-2i\Omega t)b_+(r,\omega)\frac{\partial}{\partial r}rb_+(r,-(\omega+2\Omega))\Big], \tag{3.12}$$

and we note that the last two terms in (3.12) vanish if we time-average over a period of the frequency $2\Omega/2\pi$. Denoting such time-averaging by $\langle\cdot\rangle$ we find

$$\langle W_{11}\rangle = \frac{-i\pi r_0^2 E_{0\phi}^2}{16\omega_0^2\beta^2\gamma^2}\int_0^{\infty} d\omega\,\omega\Big[b_-(r,\omega)\frac{\partial}{\partial r}rb_+(r,-\omega) - b_-(r,-\omega)\frac{\partial}{\partial r}rb_+(r,\omega) + b_+(r,\omega)\frac{\partial}{\partial r}rb_-(r,-\omega) - b_+(r,-\omega)\frac{\partial}{\partial r}rb_-(r,\omega)\Big]. \tag{3.13}$$

It will be convenient to consider (3.13) separately for different frequency ranges. Referring to (2.11), we denote the contributions to (3.13) in the ranges $[a_-(\omega)]^2<0$, $[a_+(\omega)]^2<0<[a_-(\omega)]^2$, and $[a_+(\omega)]^2>0$ by $\langle W_{11}\rangle_0$, $\langle W_{11}\rangle_1$, and $\langle W_{11}\rangle_2$, respectively. We will use similar notation when we consider W_{12} and W_{22} later in this section.

First consider the region $[a_-(\omega)]^2<0$. From (A1) and (2.10)–(2.12) we have $b_\pm(r,-\omega)=b_\mp(r,\omega)$ so that $\langle W_{11}\rangle_0=0$.

Next consider the frequency range $[a_+(\omega)]^2>0$. We find from (3.13)

$$\langle W_{11}\rangle_2=\frac{r_0{}^2E_{0\phi}{}^2}{2\omega_0{}^2\beta^2\gamma^2}\int_{\substack{\omega\geq 0\\ [a_+(\omega)]^2>0}} d\omega\omega\,, \tag{3.14}$$

where we have used the fact that $a_\pm(-\omega)=a_\mp(\omega)$ for real-valued $\epsilon(\omega)$ and have eliminated the Bessel and Hankel functions from the final result by the use of (3.11).

For the final range of frequencies, (3.13) becomes

$$\langle W_{11}\rangle_1=\frac{r_0{}^2E_{0\phi}{}^2r}{8\omega_0{}^2\beta^2\gamma^2u}\int_{\substack{\omega\geq 0\\ [a_+(\omega)]^2<0<[a_-(\omega)]^2}} d\omega\omega\left[H_0^{(2)}\left(\frac{r}{u}a_-(\omega)\right)K_1\left(\frac{r}{u}a'_+(-\omega)\right)a'_+(-\omega)+H_0^{(1)}\left(\frac{r}{u}a_-(-\omega)\right)K_1\left(\frac{r}{u}a'_+(\omega)\right)a'_+(\omega)\right.$$
$$\left.-K_0\left(\frac{r}{u}a'_+(\omega)\right)H_1^{(1)}\left(\frac{r}{u}a_-(-\omega)\right)a_-(-\omega)-K_0\left(\frac{r}{u}a'_+(-\omega)\right)H_1^{(2)}\left(\frac{r}{u}a_-(\omega)\right)a_-(\omega)\right]. \tag{3.15}$$

Since (3.15) vanishes in the limit $r\to\infty$, we conclude that $\langle W_{11}\rangle_1$ does not represent a contribution to the energy radiated by the electron.

We next consider the cross term (3.7) between the transverse and longitudinal SESR contributions. From (2.9), (2.13), (2.14), (A8), and (A12) we obtain after time averaging over a period of the frequency $2\Omega/2\pi$,

$$\langle W_{12}\rangle=\frac{i\pi r_0{}^2E_{0r}{}^2r}{16\omega_0{}^2\beta^2\gamma^2\Omega}\int_0^\infty d\omega\left[(\omega+\Omega)\left(\omega-\frac{\omega+\Omega}{\beta^2\epsilon(\omega)}\right)b_-(r,-\omega)\frac{\partial}{\partial r}b_+(r,\omega)+(\omega-\Omega)\left(\omega-\frac{\omega-\Omega}{\beta^2\epsilon(\omega)}\right)\right.$$
$$\times b_-(r,\omega)\frac{\partial}{\partial r}b_+(r,-\omega)-(\omega-\Omega)\left(\omega-\frac{\omega-\Omega}{\beta^2\epsilon(\omega)}\right)b_+(r,-\omega)\frac{\partial}{\partial r}b_-(r,\omega)$$
$$\left.-(\omega+\Omega)\left(\omega-\frac{\omega+\Omega}{\beta^2\epsilon(\omega)}\right)b_+(r,\omega)\frac{\partial}{\partial r}b_-(r,-\omega)\right]. \tag{3.16}$$

We now split up the consideration of (3.16) into the three frequency ranges previously defined in connection with (3.13). In a similar fashion to the treatment of $\langle W_{11}\rangle$, we find that $\langle W_{12}\rangle_0=0$ and $\lim_{r\to\infty}\langle W_{12}\rangle_1=0$. Finally, for $\langle W_{12}\rangle_2$ we find

$$\langle W_{12}\rangle_2=\frac{r_0{}^2E_{0r}{}^2}{2\omega_0{}^2\beta^2\gamma^2}\int_{\substack{\omega\geq 0\\ [a_+(\omega)]^2>0}} d\omega\omega\left(1-\frac{2}{\beta^2\epsilon(\omega)}\right), \tag{3.17}$$

where we have again used the fact that $a_\pm(-\omega)=a_\mp(\omega)$ for real $\epsilon(\omega)$ and have eliminated the Bessel and Hankel functions by again using (3.11).

Finally, we consider the contribution of the longitudinal SESR term (3.8). Using (2.14), (2.15), (A12), and (A13) we obtain after time averaging over a period of the frequency $2\Omega/2\pi$,

$$\langle W_{22}\rangle=\frac{-i\pi r_0{}^2E_{0r}{}^2c^2r}{16\omega_0{}^2\gamma^2\Omega^2}\int_0^\infty d\omega\left[\left(\omega-\frac{\omega-\Omega}{\beta^2\epsilon(\omega)}\right)\frac{\partial}{\partial r}b_-(r,\omega)\frac{\partial^2}{\partial r^2}b_+(r,-\omega)-\left(\omega-\frac{\omega+\Omega}{\beta^2\epsilon(\omega)}\right)\frac{\partial}{\partial r}b_-(r,-\omega)\frac{\partial^2}{\partial r^2}b_+(r,\omega)\right.$$
$$+\left(\omega-\frac{\omega+\Omega}{\beta^2\epsilon(\omega)}\right)\frac{\partial}{\partial r}b_+(r,\omega)\frac{\partial^2}{\partial r^2}b_-(r,-\omega)$$
$$\left.-\left(\omega-\frac{\omega-\Omega}{\beta^2\epsilon(\omega)}\right)\frac{\partial}{\partial r}b_+(r,-\omega)\frac{\partial^2}{\partial r^2}b_-(r,\omega)\right]. \tag{3.18}$$

We now follow an analogous procedure to that used above for (3.13) and (3.16) to deduce that $\langle W_{22}\rangle_0=0$ and $\lim_{r\to\infty}\langle W_{22}\rangle_1=0$. For $\langle W_{22}\rangle_2$ we find

$$\langle W_{22}\rangle_2=\frac{r_0{}^2E_{0r}{}^2}{4\omega_0{}^2\beta^2\gamma^2\Omega^2}\int_{\substack{\omega\geq 0\\ [a_+(\omega)]^2>0}} d\omega\left[\left(\omega-\frac{\omega-\Omega}{\beta^2\epsilon(\omega)}\right)[a_-(\omega)]^2+\left(\omega-\frac{\omega+\Omega}{\beta^2\epsilon(\omega)}\right)[a_+(\omega)]^2\right] \tag{3.19a}$$

$$=\frac{r_0{}^2E_{0r}{}^2}{2\omega_0{}^2\beta^2\gamma^2\Omega^2}\int_{\substack{\omega\geq 0\\ [a_+(\omega)]^2>0}} d\omega\omega\left[\omega^2(\beta^2\epsilon(\omega)-2)-\Omega^2+\frac{\omega^2+3\Omega^2}{\beta^2\epsilon(\omega)}\right], \tag{3.19b}$$

and we have again used the fact that $a_\pm(-\omega)=a_\mp(\omega)$ for real $\epsilon(\omega)$. The Bessel and Hankel functions have been eliminated by the use of the following Wronskian relation[15]:

$$J_0'(z)Y_0''(z)-Y_0'(z)J_0''(z)=\frac{2}{\pi z}\,, \tag{3.20}$$

where the primes denote differentiation with respect to the argument. In the present case, in contradistinction to the cases W_{00}, $\langle W_{11}\rangle_2$, and $\langle W_{12}\rangle_2$, the integration is not independent of the functions $a_\pm(\omega)$ defined in (2.11).

The total energy loss per unit length for stimulated radiation is obtained by adding (3.14), (3.17), and (3.19b):

$$W_S=\langle W_{11}\rangle_2+\langle W_{12}\rangle_2+\langle W_{22}\rangle_2$$
$$=\frac{r_0^2}{2\omega_0^2\beta^2\gamma^2}\int_{\substack{\omega\geq 0\\ [a_+(\omega)]^2>0}} d\omega\,\omega\left[E_{0\phi}^2+\frac{E_{0r}^2}{\beta^2\epsilon(\omega)}\left(1+\frac{\omega^2}{\Omega^2}\,[\beta^2\epsilon(\omega)-1]^2\right)\right]. \tag{3.21}$$

As a consequence of the fact that $\epsilon(\omega)$ occurs in the numerator of the integrands in (3.19b) and (3.21), one easily shows that $\langle W_{22}\rangle_2$ and W_s are logarithmically divergent for the simple Lorentz model (2.16). This appearance of $\epsilon(\omega)$ in the numerator is, in turn, due to the occurrence of the functions $[a_\pm(\omega)]^2$ in (3.19a) which is brought about by the fact that the fields occurring in the expression (3.8) for W_{22} have higher-order radial derivatives than is the case for the expressions (3.3)–(3.7) for the other W_{ij}. This is, of course, related to the fact that $\vec{A}_1$ lies in the direction of $\vec{E}_0$, whereas $\vec{A}_2$ lies in the direction of $\vec{u}$, i.e., along the z axis.

Because of the divergence mentioned above, Šoln and Williams[11] do not calculate W_{22} or W_s. In the following section we will give results for these quantities on the basis of a simple model which includes the effects of some (small) medium absorption thereby removing the divergence noted above.

IV. ENERGY LOSS–AN EXAMPLE

In the present section we will discuss a simple model for a dielectric function $\epsilon(\omega)$ which contains some effects of medium absorption.

We shall consider the dielectric function[16]

$$\epsilon_2(\omega)=1+\frac{\omega_r^2(\omega_r^2-\omega^2)[\epsilon_2(0)-1]}{(\omega_r^2-\omega^2)^2+\omega^2\Gamma^2}\,, \tag{4.1}$$

where $\Gamma>0$ is a parameter (with the dimensions of frequency) which characterizes the absorption properties of the medium. We will only consider (4.1) in the limit of very small absorption for which Γ is much smaller than the characteristic frequencies of the problem. In particular, $\Gamma\ll\omega_r$. In this case (4.1) is seen to be a small correction to (2.16) except for frequencies in the neighborhood of the resonance frequency ω_r. Note that $\epsilon_2(\omega_r)$ is finite, whereas $\epsilon_1(\omega_r)$ is not. Thus, the introduction of absorption has the mathematical effect of providing a cutoff to eliminate the divergence.

Evaluation of (3.10), (3.14), (3.17), and (3.19b) for the dielectric function (4.1) gives

$$W_{00}=\frac{e^2\omega_r^2}{2u^2}\left[-(1-\beta^2)\left(\frac{\omega_u}{\omega_r}\right)^2+\left(\frac{\epsilon_2(0)-1}{2}\right)\ln\left(\frac{\epsilon_2(0)}{C_u(\epsilon_2(0))}\right)+\frac{D}{2A}\ln\left(\frac{[B(\epsilon_2(0))-2(\omega_u/\omega_r)^2-A][B(\epsilon_2(0))+A]}{[B(\epsilon_2(0))-2(\omega_u/\omega_r)^2+A][B(\epsilon_2(0))-A]}\right)\right], \tag{4.2}$$

$$\langle W_{11}\rangle_2=\frac{r_0^2E_{0\phi}^2}{2\omega_0^2\beta^2\gamma^2}\lim_{\nu\to 0}\int_{\omega_l+\nu}^{\omega_u}d\omega\,\omega=\frac{r_0^2E_{0\phi}^2\omega_r^2}{4\omega_0^2\beta^2\gamma^2}\left[\left(\frac{\omega_u}{\omega_r}\right)^2-\left(\frac{\omega_l}{\omega_r}\right)^2\right], \tag{4.3}$$

$$\langle W_{12}\rangle_2=\frac{r_0^2E_{0r}^2}{2\omega_0^2\beta^2\gamma^2}\lim_{\nu\to 0}\int_{\omega_l+\nu}^{\omega_u}d\omega\,\omega\left(1-\frac{2}{\beta^2\epsilon_2(\omega)}\right)$$
$$=\frac{r_0^2E_{0r}^2\omega_r^2}{4\omega_0^2\beta^4\gamma^2}\left\{-2\left(1-\frac{\beta^2}{2}\right)\left[\left(\frac{\omega_u}{\omega_r}\right)^2-\left(\frac{\omega_l}{\omega_r}\right)^2\right]+[\epsilon_2(0)-1]\ln\left(\frac{C_l(\epsilon_2(0))}{C_u(\epsilon_2(0))}\right)\right.$$
$$\left.-\frac{D}{A}\ln\left(\frac{[B(\epsilon_2(0))-2(\omega_u/\omega_r)^2+A][B(\epsilon_2(0))-2(\omega_l/\omega_r)^2-A]}{[B(\epsilon_2(0))-2(\omega_u/\omega_r)^2-A][B(\epsilon_2(0))-2(\omega_l/\omega_r)^2+A]}\right)\right\}, \tag{4.4}$$

$$\langle W_{22}\rangle_2=\frac{r_0^2E_{0r}^2\omega_r^4}{4\omega_0^2\Omega^2\beta^2\gamma^2}\left\{\left[[\epsilon_2(0)-1](\beta^{-2}-\beta^2)+(3\beta^{-2}-1)\left(\frac{\Omega}{\omega_r}\right)^2\right]\left[\left(\frac{\omega_u}{\omega_r}\right)^2-\left(\frac{\omega_l}{\omega_r}\right)^2\right]\right.$$
$$+\left[\tfrac{1}{2}(\beta^{-2}+\beta^2)-1\right]\left[\left(\frac{\omega_u}{\omega_r}\right)^4-\left(\frac{\omega_l}{\omega_r}\right)^4\right]+\frac{\beta^2}{2}\,[\epsilon_2(0)-1](1-\Gamma'^2)\ln\left(\frac{C_l(1)}{C_u(1)}\right)$$
$$+\frac{\beta^2[\epsilon_2(0)-1]\Gamma'(3-\Gamma'^2)}{2(1-\Gamma'^2/4)^{1/2}}$$
$$\times\tan^{-1}\left(\frac{4\Gamma'(1-\Gamma'^2/4)^{1/2}[(\omega_u/\omega_r)^2-(\omega_l/\omega_r)^2]}{4\Gamma'^2(1-\Gamma'^2/4)+\{-2[1-(\omega_u/\omega_r)^2]+\Gamma'^2\}\{-2[1-(\omega_l/\omega_r)^2]+\Gamma'^2\}}\right)$$
$$-\frac{\beta^{-2}}{2}\,[\epsilon_2(0)-1]\left[\epsilon_2(0)+3\left(\frac{\Omega}{\omega_r}\right)^2-\Gamma'^2\right]\ln\left(\frac{C_l(\epsilon_2(0))}{C_u(\epsilon_2(0))}\right)$$
$$\left.+\frac{\beta^{-2}[\epsilon_2(0)-1]G}{2A}\,\ln\left(\frac{[B(\epsilon_2(0))-2(\omega_u/\omega_r)^2+A][B(\epsilon_2(0))-2(\omega_l/\omega_r)^2-A]}{[B(\epsilon_2(0))-2(\omega_u/\omega_r)^2-A][B(\epsilon_2(0))-2(\omega_l/\omega_r)^2+A]}\right)\right\}, \qquad (4.5)$$

with $\Gamma'=\Gamma/\omega_r$,

$$A=\{[\epsilon_2(0)-1]^2-2\Gamma'^2[\epsilon_2(0)+1]+\Gamma'^4\}^{1/2}, \quad B(x)=x+1-\Gamma'^2,$$
$$C_{u,l}(x)=\left(\frac{\omega_{u,l}}{\omega_r}\right)^4-B(x)\left(\frac{\omega_{u,l}}{\omega_r}\right)^2+x, \quad D=[\epsilon_2(0)-1]^2-\Gamma'^2[\epsilon_2(0)+1],$$
$$G=3\left(\frac{\Omega}{\omega_r}\right)^2[B(\epsilon_2(0))-2]+\epsilon_2(0)[\epsilon_2(0)-1]-\Gamma'^2[2\epsilon_2(0)+1]+\Gamma'^4.$$

The upper limit of integration ω_u is determined from the equation

$$\beta^2\epsilon_2(\omega_u)=1 \qquad (4.6)$$

and the lower limit, ω_l, is a zero of $[a_+(\omega)]^2$.

Solution of the biquadratic equation resulting from (4.1) and (4.6) yields

$$\omega_u^2=\frac{1}{2}\left[-\Gamma^2-\omega_r^2(\alpha-2)+\alpha\omega_r^2\left(1+\frac{2\Gamma^2(\alpha-2)}{\alpha^2\omega_r^2}+\frac{\Gamma^4}{\alpha^2\omega_r^4}\right)^{1/2}\right] \qquad (4.7)$$

with

$$\alpha=\beta^2\gamma^2[\epsilon_2(0)-1],$$

which reduces to

$$\left(\frac{\omega_u}{\omega_r}\right)^2=1-\frac{\Gamma'^2}{\alpha} \qquad (4.8)$$

for small Γ. We have discarded the other root of the biquadratic equation because it does not reduce to ω_r^2 when $\Gamma\to 0$.

Now consider the zeros of $[a_+(\omega)]^2$ for the dielectric function (4.1). One finds that these are determined by a sixth-degree algebraic equation in ω. From Descarte's rule of signs one deduces that there are either two or zero positive real roots. In the limit $\Gamma=0$ it is easily shown that two of the roots are $\pm\omega_r$ so that for $\Gamma\neq 0$ there are in fact two real positive zeros. We are interested in the one that does not approach ω_r in the limit $\Gamma\to 0$. Call it ω_l. To simplify matters, we will consider approximate results for ω_l by evaluating it in the limit $\Gamma=0$. Then it is equal to the unique positive real root, ω_+, of $[a_+(\omega)]^2=0$ for the dielectric function (2.16). Since we are taking $\Gamma>0$ and $\beta^2\epsilon_1(0)-1\geq 0$ it can be shown that $0<\omega_+\leq\omega_r$ with equality obtaining if and only if $\epsilon_1(0)=1$. ω_+ is determined by solving a quartic equation. However, this equation can be simplified by considering the limit $\beta\to 1$ in which case it reduces to a cubic equation. Taking this limit, we solve the resulting equation by the method of Cardan.[17]

Using the above facts concerning ω_u and ω_l, we find that (4.2)–(4.4) are finite in the limit $\Gamma\to 0$, whereas (4.5) is logarithmically divergent in this limit, as we have already mentioned at the end of Sec. III. In the case of (4.2) we recover the result given by Frank and Tamm[8] in the limit $\beta\to 1$.

We will now discuss some estimates for the ratio of stimulated radiation to the Čerenkov radiation emitted by the electron based on the formulas (4.2)–(4.5), i.e., for a model dielectric described by (4.1) with $\Gamma'=\Gamma/\omega_r$ very small. Thus, we shall consider numerical results for the quantity

$$R\equiv\frac{1}{W_{00}}\left(\frac{\langle W_{11}\rangle_2}{E_{0\phi}^2}+\frac{\langle W_{12}\rangle_2}{E_{0r}^2}+\frac{\langle W_{22}\rangle_2}{E_{0r}^2}\right). \qquad (4.9)$$

It is evident from (4.3)–(4.5) that the ratio of stimulated to Čerenkov radiation can be made large if ${E_{0r}}^2$ and/or ${E_{0\phi}}^2$ are given sufficiently large values. In a practical sense, however, $|\vec{E}_0|$ should be much smaller than the breakdown field of the dielectric in order that the medium not be disturbed by the passage of the electromagnetic wave and that our perturbation analysis be valid.

Some typical values for R with ω_0 in the microwave region are shown in Table I. The blank space in this table arises because at that low value of $\epsilon_2(0)$ (1.0005), the value $\Gamma' = 10^{-3}$ does not correspond to a small absorption frequency. This is because the correct smallness parameter is not Γ', but is instead Γ' divided by a quantity proportional to $\epsilon_2(0) - 1$, as is evident in the passage from (4.7) to (4.8).

In Table I the first three values of $\epsilon_2(0)$ correspond to gases, the fourth to a liquid, and the last to a solid (a glass).

If we take $E_{0r} \approx E_{0\phi} \approx 100$ statvolts/cm in Table I, which corresponds to fields approximately $\frac{1}{10}$ that of the breakdown field strength in typical materials, we see that the ratio of stimulated to Čerenkov radiation ranges from approximately 100 to 4000 for gases and solids, respectively, for ω_0 in the microwave region. For larger frequencies these estimates are reduced because of the inverse powers of ω_0 occurring in (3.21). For typical optical frequencies[18] for ω_0, the above estimates for the ratio of stimulated to Čerenkov radiation should be multiplied by a factor of the order of 10^{-16}. This factor is obtained in the following manner. Optical frequencies are larger than microwave frequencies by a factor of the order 10^4, so that the first two items in (3.21) are smaller by factors of the order 10^{-8} due to the factor ${\omega_0}^{-2}$. The remaining term in (3.21) is smaller by approximately 10^{-16} due to the factor ${\omega_0}^{-4}$ [see the definition of Ω after (2.5)]. But, in the microwave region the first two terms in (3.21) are smaller than the remaining one by factors of the order 10^{-8} so that all three terms are of the same order of magnitude for ω_0 in the optical region, which is approximately 10^{-16} times the value of the last term in (3.21) in the microwave region where it is the dominant term.

TABLE I. Values of R for ω_0 in the microwave region. E_{0r}, $E_{0\phi}$ are in Gaussian units. $\omega_r = 6\times10^{15}$ sec^{-1}, $\omega_0 = 2\pi\times10^{10}$ sec^{-1}, $\beta^2 = (1.0005)^{-1}$.

Γ' \ $\epsilon_2(0)$	1.0005	1.005	1.05	1.33	2.3
10^{-3}		0.00969	0.053	0.140	0.277
10^{-4}	0.0114	0.033	0.086	0.203	0.401

The first and second terms in (4.9) are opposite in sign and of the same order of magnitude which is small in comparison with the third term for ω_0 in the microwave range (by approximately a factor of 10^{-8} as noted above). Thus, our conclusions of the present study are as follows.

For external field strengths smaller than the dielectric breakdown field, the SESR mechanism gives a large contribution to the total radiated energy at incident frequencies in the microwave region but only a small contribution in the optical region. Most of the radiated energy loss is due to the SESR mechanism for microwave frequencies, and to Čerenkov radiation for sufficiently large frequencies and sufficiently high electron velocities.

The last remark refers to the fact that the value of R is reduced if the electron velocity is greater than the threshold value $\beta = [\epsilon_2(0)]^{-1/2}$. In Table I the result for $\epsilon_2(0) = 1.0005$ is for β equal to the threshold value so that the results for this case are reduced for larger β. Conversely, the results for the remaining values of $\epsilon_2(0)$ in Table I are for β not at the threshold value so that the corresponding threshold results in these cases are larger.

The results discussed above are for total energy losses in which one integrates over all frequencies ω allowed by the relation $\beta^2\epsilon_2(\omega) \geq 1$. However, we can also obtain the spectral distribution of radiation from the respective integrands in (3.10) and (3.21). It is evident that the ratio of $dW_s/d\omega$ to $dW_{00}/d\omega$ can be increased relative to the ratio of stimulated to Čerenkov total energy losses for frequencies ω such that $\beta^2\epsilon(\omega) \geq 1$. This increase is due to the decrease in the Čerenkov contribution at such frequencies rather than to the increase of the stimulated radiation term.

Some values of the ratio of these quantities in the case of the dielectric function (4.1) for ω_0 in the microwave range are shown in Table II. In compiling this table we have chosen the electron velocity so that the threshold condition, $\beta^2\epsilon_2(0) = 1$, is satisfied for each value of $\epsilon_2(0)$. Then the quantity $\beta^2\epsilon_2(\omega) - 1$ can be made as small as desired by choosing ω sufficiently small. It will be noted that the results given in the last row of Table II corresponding to $\omega/\omega_r = 10^{-4}$ are larger than the corresponding values of R in Table I. The entries ∞ in this row arise because, for this frequency, one has $\epsilon_2(\omega) = \epsilon_2(0)$ to the accuracy at which the results are given.

ACKNOWLEDGMENTS

I wish to thank A. W. Sáenz, J. B. Aviles, L. Cohen, N. Seeman, S. Schneider, and R. Spitzer for helpful discussions concerning this work.

TABLE II. Values of $[(d/d\omega)(\langle W_{11}\rangle_2/E_{0\phi}^2 + \langle W_{12}\rangle_2/E_{0r}^2 + \langle W_{22}\rangle_2/E_{0r}^2)]/(dW_{00}/d\omega)$ for ω_0 in the microwave region. $E_{0r}, E_{0\phi}$ are in Gaussian units. $\omega = 6\times 10^{15}$ sec^{-1}, $\omega_0 = 2\pi\times 10^{10}$ sec^{-1}, $\beta^2 = [\epsilon_2(0)]^{-1}$, $\Gamma' = 10^{-4}$. The numbers in parentheses denote the powers of 10 by which the accompanying number is to be multiplied.

ω/ω_r \ $\epsilon_2(0)$	1.0005	1.005	1.05	1.33	2.3
10^{-1}	7.79(−6)	8.05(−6)	2.97(−5)	0.000 749	0.006 65
10^{-2}	0.000 784	0.000 788	0.000 823	0.001 04	0.001 80
10^{-3}	0.039	0.078	0.0823	0.104	0.180
10^{-4}	∞	∞	3.92	11.2	17.9

APPENDIX

In this appendix we will demonstrate the validity of passing the space and time derivatives through the integration over ω for the fields occurring in Eqs. (3.3)–(3.8) for a large class of real-valued dielectric functions. We will use the following criterion.[19]

Theorem. Let $f(x,\alpha)$ *be a real-valued function of a real variable* x *belonging to a finite interval* $-\infty < a \le x \le b < \infty$ *and dependent upon a real parameter* α. *Then the equation*

$$\frac{d}{d\alpha}\int_a^b f(x,\alpha)dx = \int_a^b \frac{\partial}{\partial\alpha} f(x,\alpha)dx$$

is true if both $f(x,\alpha)$ *and* $\partial f(x,\alpha)/\partial\alpha$ *are Riemann integrable with respect to* x *and* $\partial f(x,\alpha)/\partial\alpha$ *is a continuous function of* α *uniformly in* x.

We will have need of an extended version of the above theorem in which f is replaced by a complex-valued function g of the real variable x. This extended theorem can easily be established by separating g into its real and imaginary parts and then invoking the above result.

For the functions that we consider, the conditions involving Riemann integrability are easily verified so that we will only discuss the condition involving uniform continuity.

As noted in Sec. II, we suppose that the dielectric function ϵ is a real-valued function of frequency. It then follows from well-known general considerations[20] that ϵ is an even function of ω,

$$\epsilon(-\omega) = \epsilon(\omega). \tag{A1}$$

For the cases of interest to us, the frequency ω will be limited to a finite interval $0 \le \omega_l \le \omega \le \omega_u < \infty$. We will assume that $\epsilon(\omega_u) < \infty$.

In the estimates given below we shall restrict ourselves to the range of frequency for which $[a_+(\omega)]^2 > 0$. The other cases discussed in Sec. III, i.e., $[a_-(\omega)]^2 < 0$ and $[a_+(\omega)]^2 < 0 < [a_-(\omega)]^2$, can be handled in a similar fashion by replacing the estimates for appropriate Hankel functions occurring below by corresponding estimates for the modified Bessel function K_0.

From (2.6) and (2.8) we have for the Čerenkov magnetic field (with $\tau = t - z/u$)

$$H_\phi^C(\vec{x},t) = \frac{ie}{c}\frac{\partial}{\partial r}\int_0^{\omega_u} d\omega\left[\exp(i\omega\tau)H_0^{(2)}\left(r\frac{\omega}{u}[\beta^2\epsilon(\omega)-1]^{1/2}\right) - \exp(-i\omega\tau)H_0^{(1)}\left(r\frac{\omega}{u}[\beta^2\epsilon(\omega)-1]^{1/2}\right)\right], \tag{A2}$$

where ω_u is determined by the equation $\beta^2\epsilon(\omega_u) = 1$ as we have already noted in (4.6) in the special case $\epsilon = \epsilon_2$. Considering the function

$$f(r,\omega) = \exp(-i\omega\tau)H_0^{(1)}((r\omega/u)[\beta^2\epsilon(\omega)-1]^{1/2}), \tag{A3}$$

we find

$$\left|\frac{\partial}{\partial r'} f(r',\omega) - \frac{\partial}{\partial r''} f(r'',\omega)\right| = \left|\int_{r''}^{r'} dr\,\frac{\partial^2}{\partial r^2} f(r,\omega)\right|$$

$$\le \frac{\omega^2}{u^2}\,|\beta^2\epsilon(\omega)-1|\left|\int_{r''}^{r'} dr\left|\frac{d^2}{dy^2}H_0^{(1)}(y)\right|\right|, \tag{A4}$$

where y denotes the argument of the Hankel function in (A3). Now, using some well-known relations among the Hankel functions and the estimates[21]

$$|H_0^{(1)}(y)| \le \left(\frac{2}{\pi y}\right)^{1/2}, \quad |H_2^{(1)}(y)| \le \text{const}\times(y^{-1/2}+y^{-2}), \quad y \ge 0, \tag{A5}$$

we obtain from (A4)

$$\left|\frac{\partial}{\partial r'}f(r',\omega)-\frac{\partial}{\partial r''}f(r'',\omega)\right|\leq\frac{\omega^2}{u^2}\,|\beta^2\epsilon(\omega)-1|\,\tfrac{1}{2}\int_{r''}^{r'}dr\,|H_0^{(1)}(y)-H_2^{(1)}(y)|$$

$$\leq C\left[\left(\frac{\omega_u}{u}\right)^{3/2}[\beta^2\epsilon(\omega_m)-1]^{3/4}(r'^{1/2}-r''^{1/2})+(r''^{-1}-r'^{-1})\right], \tag{A6}$$

showing that $\partial f(r,\omega)/\partial r$ is a continuous function of r for $r>0$ uniformly in ω for $\omega\in[0,\omega_u]$. In (A6) C denotes a positive constant which is, in general, different from the one appearing in the estimate (A5). Throughout this appendix we will use the letter C to stand for a generic constant independent of r, z, t, and ω which does not take the same value in all the estimates to follow. The symbol ω_m denotes that frequency in the interval $[0,\omega_u]$ at which $\epsilon(\omega)$ obtains its maximum value. We also require an estimate for $(\partial/\partial r)[f(r,\omega)]^*$ with f defined by (A3). This is easily obtained in the same manner as the above derivation of (A6).

We now consider the transverse SESR fields. From (2.9) and (2.10) the appropriate vector potential is

$$\vec{A}_1(\vec{x},\tau)=\frac{ir_0\vec{E}_0}{4\omega_0\beta\gamma}\int_{\omega_l}^{\omega_u}d\omega\left[\exp\left(i\omega\tau-i(\omega-\Omega)\frac{z}{u}\right)H_0^{(2)}\left(\frac{r}{u}a_-(\omega)\right)+\exp\left(i\omega\tau-i(\omega+\Omega)\frac{z}{u}\right)H_0^{(2)}\left(\frac{r}{u}a_+(\omega)\right)\right.$$

$$\left.-\exp\left(-i\omega\tau+i(\omega+\Omega)\frac{z}{u}\right)H_0^{(1)}\left(\frac{r}{u}a_+(\omega)\right)-\exp\left(-i\omega\tau+i(\omega-\Omega)\frac{z}{u}\right)H_0^{(1)}\left(\frac{r}{u}a_-(\omega)\right)\right],$$

where ω_l denotes the lower limit of ω for which $[a_+(\omega)]^2\geq 0$.

The transverse SESR fields are given by (2.13) so that we define the functions

$$g_\pm(r,z,t;\omega)=\exp\left(-i\omega\tau+i(\omega\pm\Omega)\frac{z}{u}\right)H_0^{(1)}\left(\frac{r}{u}a_\pm(\omega)\right), \tag{A7}$$

and show that $\partial g_\pm/\partial r$, $\partial g_\pm/\partial z$, and $\partial g_\pm/\partial t$ are continuous functions of r, z, and t, respectively, uniformly in ω for $\omega\in(\omega_l,\omega_u]$, as well as the corresponding results when $g_\pm$ are replaced by the respective complex conjugates $g_\pm^*$. We will only gives estimates in the case of the derivatives of $g_\pm$, since those for the derivatives of $g_\pm^*$ are analogous.

The relevant estimates for $H_{1\phi}$ are, using the first bound in (A5),

$$\left|\frac{\partial}{\partial z'}g_\pm(r,z',t;\omega)-\frac{\partial}{\partial z''}g_\pm(r,z'',t;\omega)\right|\leq|z'-z''|\,(\omega_u+\Omega)^2u^{-3/2}\left(\frac{2}{\pi r}\right)^{1/2}[a_\pm(\omega_l+\nu)]^{-1} \tag{A8}$$

showing that $\partial g_\pm/\partial z$ are continuous functions of z for $r>0$ uniformly in ω for $\omega\in[\omega_l+\nu,\omega_u]$ for $\nu>0$. The quantity ν is not needed for the lower sign, but it is necessary for the upper one and its presence is the reason for the procedure discussed at the end of Sec. II.

In a similar manner, we find the following estimates appropriate for $E_{1\phi}$:

$$\left|\frac{\partial}{\partial t'}g_\pm(r,z,t';\omega)-\frac{\partial}{\partial t''}g_\pm(r,z,t'';\omega)\right|\leq|t'-t''|\,\omega_u^2\left(\frac{2u}{\pi r a_\pm(\omega_l+\nu)}\right)^{1/2}. \tag{A9}$$

The estimates appropriate for H_{1z} are similar to those given in (A4) for the Čerenkov magnetic field. The only important difference lies in the respective arguments of the Hankel functions. We have

$$\left|\frac{\partial}{\partial r'}g_\pm(r',z,t;\omega)-\frac{\partial}{\partial r''}g_\pm(r'',z,t;\omega)\right|\leq C\{u^{-3/2}[\beta^2\omega_u^2\epsilon(\omega_m)+(\omega_u+\Omega)^2]^{3/4}(r'^{1/2}-r''^{1/2})+(r''^{-1}-r'^{-1})\}, \tag{A10}$$

where we have used the bound

$$|a_\pm(\omega)|\leq[\beta^2\omega_u^2\epsilon(\omega_m)+(\omega_u+\Omega)^2]^{1/2} \tag{A11}$$

which is easily derived from (2.11).

We now consider the longitudinal SESR fields. From (2.10) and (2.14) we have,

$$E_{2z}(\vec{x},t)=\frac{ir_0E_{0r}\partial/\partial r}{4\omega_0\gamma\Omega}\int_{\omega_l}^{\omega_u}d\omega\left[\exp\left(i\omega t-i(\omega-\Omega)\frac{z}{u}\right)\left(\omega-\frac{\omega-\Omega}{\beta^2\epsilon(\omega)}\right)H_0^{(2)}\left(\frac{r}{u}a_-(\omega)\right)\right.$$

$$-\exp\left(i\omega t-i(\omega+\Omega)\frac{z}{u}\right)\left(\omega-\frac{\omega+\Omega}{\beta^2\epsilon(\omega)}\right)H_0^{(2)}\left(\frac{r}{u}a_+(\omega)\right)$$

$$+\exp\left(-i\omega t+i(\omega+\Omega)\frac{z}{u}\right)\left(\omega-\frac{\omega+\Omega}{\beta^2\epsilon(\omega)}\right)H_0^{(1)}\left(\frac{r}{u}a_+(\omega)\right)$$

$$-\exp\left(-i\omega t+i(\omega-\Omega)\frac{z}{u}\right)\left(\omega-\frac{\omega-\Omega}{\beta^2\epsilon(\omega)}\right)H_0^{(1)}\left(\frac{r}{u}a_-(\omega)\right)\Big].$$

We now define the functions

$$h_\pm(r,z,t;\omega)=\exp\left(-i\omega t+i(\omega\pm\Omega)\frac{z}{u}\right)\left(\omega-\frac{\omega\pm\Omega}{\beta^2\epsilon(\omega)}\right)H_0^{(1)}\left(\frac{r}{u}a_\pm(\omega)\right)$$

and show that $\partial h_\pm/\partial r$ are continuous functions of r for $r>0$ uniformly in ω for $\omega\in[\omega_l,\omega_u]$. As will be evident from the estimates given below, the introduction of the quantity ν is not necessary in the present case. Corresponding results for $\partial h_\pm^*/\partial r$, can be proved in an analogous fashion. We find

$$\left|\frac{\partial}{\partial r'}h_\pm(r',z,t;\omega)-\frac{\partial}{\partial r''}h_\pm(r'',z,t;\omega)\right|$$
$$\leq C\left(\omega_u+\frac{\omega_u+\Omega}{\beta^2}\right)\left\{u^{-3/2}[\beta^2\omega_u^2\epsilon(\omega_m)+(\omega_u+\Omega)^2]^{3/4}(r'^{1/2}-r''^{1/2})+(r''^{-1}-r'^{-1})\right\}, \quad (A12)$$

which establishes the stated assertion. In (A12) we have again used (A11) and also the fact that $\epsilon(\omega)\geq 1$ for all $\omega\in[\omega_l,\omega_u]$.

The final field component to consider is

$$H_{2\phi}(\vec{x},t)=\frac{r_0cE_{0r}\partial^2/\partial r^2}{4\omega_0\gamma\Omega}\int_{\omega_l}^{\omega_u}d\omega\Big[\exp\left(i\omega t-i(\omega-\Omega)\frac{z}{u}\right)H_0^{(2)}\left(\frac{r}{u}a_-(\omega)\right)$$
$$-\exp\left(i\omega t-i(\omega+\Omega)\frac{z}{u}\right)H_0^{(2)}\left(\frac{r}{u}a_+(\omega)\right)-\exp\left(-i\omega t+i(\omega+\Omega)\frac{z}{u}\right)H_0^{(1)}\left(\frac{r}{u}a_+(\omega)\right)$$
$$+\exp\left(-i\omega t+i(\omega-\Omega)\frac{z}{u}\right)H_0^{(1)}\left(\frac{r}{u}a_-(\omega)\right)\Big],$$

where we have used (2.10) and (2.15). Thus, we are led to consider once again the functions defined in (A7). In particular, we must show that $\partial^2 g_\pm/\partial r^2$ and $\partial^2 g_\pm^*/\partial r^2$ are continuous functions of r for $r>0$ uniformly in ω for $\omega\in[\omega_l,\omega_u]$. As before, we will only give the proof for the cases $\partial^2 g_\pm/\partial r^2$. Note that the corresponding results for $\partial g_\pm/\partial r$ are implied by the estimates (A10). The estimates that do the trick are

$$\left|\frac{\partial^2}{\partial r'^2}g_\pm(r',z,t;\omega)-\frac{\partial^2}{\partial r''^2}g_\pm(r'',z,t;\omega)\right|\leq C\Big\{u^{-7/2}[\beta^2\omega_u^2\epsilon(\omega_m)+(\omega_u+\Omega)^2]^{7/4}(r'^{3/2}-r''^{3/2})$$
$$+u^{-2}[\beta^2\omega_u^2\epsilon(\omega_m)+(\omega_u+\Omega)^2]\ln\frac{r'}{r''}$$
$$+u^{-3/2}[\beta^2\omega_u^2\epsilon(\omega_m)+(\omega_u+\Omega)^2]^{3/4}(r''^{-1/2}-r'^{-1/2})+(r''^{-2}-r'^{-2})\Big\}. \quad (A13)$$

In obtaining (A13) we have used (A5), (A11), and the relation

$$\frac{d^3}{dy^3}H_0^{(1)}(y)=\tfrac{1}{2}yH_0^{(1)}(y)+(\tfrac{1}{2}y-y^{-1})H_2^{(1)}(y),$$

which is easily derived from the standard recurrence relations for the Hankel functions.

[1]S. Schneider and R. Spitzer, Nature (London) 250, 643 (1974).

[2]S. Schneider and R. Spitzer, Can. J. Phys. 55, 1499 (1977).

[3]S. Schneider and R. Spitzer, Appl. Phys. 13, 197 (1977).

[4]S. Schneider and R. Spitzer, IEEE Trans. Microwave Theory Tech. MTT-25, 551 (1977).

[5]S. Schneider and R. Spitzer, in *Novel Sources of Coherent Radiation*, edited by S. F. Jacobs, M. Sargent, III, and M. O. Scully (Addison-Wesley, Reading, Mass., 1978).

[6]L. Cohen, private communication.

[7]G. Zin, Nuovo Cimento 22, 706 (1961). See also, V. P. Zrelov, *Čerenkov Radiation in High-Energy Physics* (Israel Program for Scientific Translations, Jerusa-

lem, 1970), Part I, pp. 88–92.

[8]I. Frank and I. Tamm, C. R. Acad. Sci USSR 16, 109 (1937).

[9]I. Tamm, J. Phys. USSR 1, 439 (1939).

[10]J. Šoln, Phys. Rev. D 18, 2140 (1978).

[11]J. Šoln and R. Williams, Harry Diamond Laboratories, Adelphi, Maryland, report (unpublished).

[12]In the present paper we define this sum of (2.2) and (2.3) as the SESR current density. This differs from the definition of Schneider and Spitzer in which only (2.2) is considered.

[13]E. Fermi, Phys. Rev. 57, 485 (1940).

[14]B. M. Bolotovskii, Usp. Fiz. Nauk 62, 201 (1957).

[15]G. N. Watson, *Theory of Bessel Functions* (Cambridge Univ. Press, Cambridge, England, 1958), 2nd edition, p. 76.

[16]One easily verifies that this function satisfies the requirements imposed on our class of dielectric functions in the Appendix.

[17]See, for example, N. B. Conkwright, *Introduction to the Theory of Equations* (Ginn, New York, 1957), Chap. V.

[18]In Sec. III we mentioned the fact that our approximation of computing only the radial component of the Poynting vector appears not to be valid for optical frequencies. However, the results stated here concerning the optical region are based on the presence of the scaling factors in ω_0 which occur, as we have shown in Sec. III, independently of our Poynting vector approximation. Recall that we have previously mentioned in Secs. I and III that our apparently approximate procedure gives results which agree with those in Ref. 10.

[19]E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (Cambridge Univ. Press, Cambridge, England, 1963), 4th edition, p. 67.

[20]L. D. Landau and E. M. Lifschitz, *Electrodynamics of Continuous Media* (Pergamon, London, 1960), p. 250.

[21]For the estimate involving $H_2^{(1)}$ see R. G. Newton, J. Math. Phys. 12, 1552 (1971).