PHYSICAL REVIEW D VOLUME 20, NUMBER 12 15 DECEMBER 1979

Axial-vector current and dimensional regularization

Steven Gottlieb and J. T. Donohue*
High Energy Physics Division, Argonne National Laboratory, Argonne, Illinois 60439
(Received 22 May 1979)

The properties of the axial-vector current are investigated using the dimensional-regularization scheme. The problem of defining an appropriate generalization of γ_5 in n dimensions is discussed, and previous work is briefly reviewed. For the *VVA* triangle, in QED, we find that the dimensional scheme provides for vector current conservation, with the divergence of the axial-vector current anomalous. This is shown unambiguously without specifying the anticommuting nature of γ_5 in n dimensions. If one arranges to have two species of fermions with different masses and equal but opposite couplings to the axial-vector current, the *VVA* anomaly is proportional to $n-4$, being fully canceled only at $n=4$. However, the behavior of the triangle amplitude for large external momenta is reduced by two powers, and the resulting softened triangle does not give rise to any finite (as $n \to 4$) anomalies when inserted in higher-order diagrams. Finally, the appropriate generalization of γ_5 for even-parity fermion loops is shown to be totally anticommuting, and the validity of Ward identities for two-point functions is demonstrated.

I. INTRODUCTION

Ever since the development of dimensional regularization by 't Hooft and Veltman[1] it has been recognized that γ^5 presents special problems. At the root of the difficulty is the fact that in four dimensions γ^5 has two properties which are incompatible for general n. In four dimensions, γ^5 anticommutes with all the matrices γ_μ. It is also the antisymmetric product of four Dirac γ matrices. 't Hooft and Veltman chose to drop the former property in their derivation of the triangle anomaly. One great advantage of dimensional regularization is that it yields amplitudes consistent with gauge invariance. The formal derivation of gauge invariance from the perturbation series requires that shifts of loop integration variables be allowed. Dimensional regularization provides for that. The derivation of the axial-vector Ward identities require, in addition, that γ^5 anticommute with all γ^μ. Bardeen, Gastmans, and Lautrup[2] have done some calculations in which they prefer to retain this property.

The competition between vector and axial-vector Ward identities was studied several years before the advent of dimensional regularization. Adler[3] showed that no regularization scheme can make the two consistent in the case of the *VVA* triangle (i.e., there is an anomaly). Later, Bouchiat, Iliopoulos, and Meyer,[4] and Gross and Jackiw[5] showed that in a simple Abelian theory the anomaly could destroy unitarity and renormalizability. These authors also showed that by modifying the theory so that the anomaly is canceled between two different fermions, unitarity and renormalizability are restored. Since the theories were Abelian, there was no need for dimensional regularization.

The proof that non-Abelian gauge theories are renormalizable and unitary relies heavily on the fact that dimensional regularization gives gauge-invariant amplitudes, i.e., the generalized Ward identities are true for all n. Implicit in the proofs is the assumption that there are no anomalies coming from the fermion loops, no matter how high the order of the diagram. We would now like to review the work that has been done to justify that assumption.

As briefly noted above, in the original work of 't Hooft and Veltman, it was supposed that the generalization of γ^5 to n dimensions was such that γ^5 anticommuted with the first four γ matrices, and commuted with the remaining $n-4$. Within this scheme they calculated only the anomalous part of the *VVA* triangle amplitude. Shortly thereafter, Bardeen, Gastmans, and Lautrup performed calculations using a γ^5 which anticommuted with all n γ matrices. They used this scheme because it led to amplitudes consistent with the axial-vector Ward identities for the even-parity spinor loops they calculated. We remind the reader that Bardeen[6] had shown that one could eliminate the anomalies from all even-parity loops, hence it is good to have a regularization scheme which does not introduce anomalies in those loops.

In the search for an anomaly-free regularization scheme, Bardeen[7] suggested that instead of constructing an n-dimensional analog of γ^5, one should regularize diagrams by keeping the fermion loops strictly in four dimensions, but allowing the loop integrations involving meson lines to extend to n dimensions. Such a prescription appears unwieldly in practice since some scalar products are full n-component scalar products, whereas others are products of just the first four components of a full n-component vector. Furthermore, it is not known if such a scheme is multiplicatively renormalizable.[8]

Several subsequent attempts were made to introduce an appropriate generalization of γ_5 to n dimensions. Akyeampong and Delburgo[9,10] have discussed the difficulties encountered in attempting to find a covariant object which would be the n-dimensional analog of γ_5. Breitenlohner and Maison[11] have proposed a generalization of the antisymmetric four-index tensor. Their approach is essentially identical to that of 't Hooft and Veltman, but they claim to have provided a more consistent formalism. The Breitenlohner and Maison scheme has been adopted by Marinucci and Tonin[12] for a discussion of the anomaly in Abelian theories, and by Costa, Marinucci, Tonin, and Julve[13] for non-Abelian theories. The latter paper makes the claim that if the one-loop anomalies are made to cancel, then there will be no anomalies in higher orders. This would appear to be a definitive treatment of the γ^5 problem, but since the paper is fairly formal and uses the Becchi-Rouet-Stora (BRS) approach to renormalization,[14] it is not very accessible. Furthermore, calculations take on additional complications owing to the fact that projections of n-vectors onto their last $n-4$ components appear in many places, as do similar counterterms for vector currents.[15]

Quite recently, Frampton[16,17] has called attention to a possible noncancellation of the VVA anomaly. In his method of calculating the triangle, the divergence of the axial-vector current has an extra piece, explicitly proportional to $n-4$, with a coefficient containing logarithms of fermion masses. If the introduction of additional fermions cancels the standard piece of the anomaly, there still remains the mass-dependent piece unless the new masses are equal to the old. In one-loop order, the additional anomaly vanishes at $n=4$ because it is proportional to $n-4$, no matter what the new masses. However, Frampton suggested that in higher-loop order there might be a pole at $n=4$ to cancel the factor $n-4$. If that were the case, there would be a new finite violation of the axial-vector–current Ward identity. This violation would destroy the proof of unitarity of the Weinberg-Salam model, Frampton suggested, unless the quark and lepton masses were equal, thus causing the finite violation to vanish.

Even more recently, Chanowitz, Furman, and Hinchliffe[18] have discussed γ^5 in the dimensional-regularization scheme. For even-parity loops they advocate a totally anticommuting γ^5, as in Ref. 2. For odd-parity loops, they note that if γ_5 anticommutes with all γ matrices, then the traces of γ_5 with four and six γ matrices will be consistent only if $n=4$. They try to resolve this dilemma by assuming that the trace of γ_5 with four γ matrices has an ambiguous piece of order $n-4$. In their calculation of the VVA triangle this $n-4$ encounters a factor $(n-4)^{-1}$ coming from the n-dimension loop integration, and they obtain a finite ambiguous result as $n \to 4$. In this limit they may impose either vector current or axial-vector current conservation (or neither). If they choose the ambiguous piece such that the vector current is conserved they recover the usual axial-vector anomaly. We disagree with their approach and find that the dimensional-regularization scheme yields a VVA amplitude which automatically satisfies vector current conservation, leaving the anomaly in the axial-vector divergence. The essential point, shown in the appendix, is that without specifying the anticommuting nature of γ_5 in n dimensions, one may express the triangle amplitude as a sum of traces of products of four γ matrices with γ_5. The coefficients of these traces are smooth functions of n (near $n=4$), and are such that the amplitude satisfies vector current conservation, as a function of n.

Chanowitz *et al.*, like Frampton, would find that the cancellation of the axial-vector anomaly between fermions having opposite signs is not total, since there would be a remainder proportional to $n-4$. Thus our discussion of the Frampton anomaly is relevent to their case. They did not, however, consider higher-loop order, and do not make note of the problem brought up by Frampton.

In a previous paper[19] we discussed Frampton's calculation of the triangle amplitude extensively and explained why there should not be an n-dependent anomaly. In this paper we will show that even if one insists on using this n-dependent amplitude, the extra term of the anomaly will not become finite when the triangle is embedded in other diagrams. We will concentrate on an Abelian theory for simplicity, hoping to fill in the details for the non-Abelian case in a subsequent publication. The theory contains two unequal-mass fermions which couple to the axial-vector current with opposite sign. We will see that the cancellation of the mass-independent anomaly at one-loop order so softens the triangle that there are no divergence problems in higher-loop order. We expect that the same mechanism works in non-Abelian theories.

The plan of this paper is as follows. In Sec. II we will discuss the triangle diagram. We review calculations done in Refs. 3, 16, 17, and 18, but we try to emphasize what we feel are the essential points of physics. Section III contains a discussion of the *VVA* triangle contribution to the axial-vector vertex correction (two-loop order), and to the axial-vector–current two-point function (three-loop order). Section IV deals with the question of what to do with γ^5 in even-parity loops.

We agree with the methods of Refs. 2 and 18 for even-parity loops, and significantly extend the discussion by Chanowitz *et al.* of the Ward identities for two-point functions. In Sec. V we summarize our results and establish a basis for discussing higher-order corrections. We are confident that in theories which are anomaly free at one-loop order all divergences involving γ^5 come from normal-parity spinor loops and corrections to axial-vector and pseudoscalar vertices. If that is the case, γ^5 may be anticommuted to the end of the external spinor line and then dimensional regularization may be employed in order to calculate vertex corrections. In even-parity loops, use of $\gamma_5^2=1$ and anticommutivity is sufficient to eliminate γ^5. Odd-parity loops will be convergent when the contributions of all species of fermion are added together, so one may do the integration in four dimensions, or do it in n dimensions and encounter no pole.

II. THE TRIANGLE AMPLITUDE

We wish to remind the reader that anomalies in the spinor loop were understood well before the advent of dimensional regularization. In fact, if one requires vector current conservation, the anomalous terms of all the axial-vector–current Ward identities are well defined even in the non-Abelian case.[6] The usual demonstration of Ward identities involves some identities for the γ matrices and a shift of the fermion loop momentum integration variable. If the diagram involved is divergent, the shift of integration variable may not be allowed. The great advantage of the dimensional-regularization scheme is that for arbitrary n the shift of integration variable is allowed, hence the amplitude satisfies the Ward identities for all n. If one is interested in the axial-vector current, however, there is a catch. The usual proof of that Ward identity assumes γ^5 anticommutes with every γ_μ. Unfortunately, a fully anticommuting γ_5 is inconsistent with the usual Dirac algebra unless $n=4$ for odd-parity loops. For even-parity loops, no such inconsistency arises as one may use the anticommutivity together with $\gamma_5^2=1$ to completely eliminate γ^5. Assume for the moment that γ^5 anticommutes with all γ_μ. Then

$$\mathrm{Tr}(\{\gamma_\mu,\gamma^5\}\gamma^\mu\gamma_\alpha\gamma_\beta\gamma_\gamma\gamma_\delta)=0\,.$$

If one assumes that the trace of γ^5 with fewer than four γ matrices vanishes, as it does in four dimensions, and as it does for the 't Hooft and Veltman choice, then one easily derives

$$(n-4)\mathrm{Tr}(\gamma^5\gamma_\alpha\gamma_\beta\gamma_\gamma\gamma_\delta)=0\,.$$

There are two solutions to this equation, $n=4$ or $\mathrm{Tr}(\gamma^5\gamma_\alpha\gamma_\beta\gamma_\gamma\gamma_\delta)=0$. We clearly do not want to make the latter choice, for then the odd-parity loop would vanish identically. The other solution $n=4$ prevents us from considering n to be a variable. We note, however, that if the fermion loop integral has no pole at $n=4$, that is, if it is convergent, there is no danger from the inconsistency of the Dirac algebra with an anticommuting γ^5, since in the physical limit ($n=4$), the result is unchanged by the inconsistency. However, if there is a pole, there will be a difference between the expression $n\,\mathrm{Tr}(\gamma^5\gamma_\alpha\gamma_\beta\gamma_\gamma\gamma_\delta)$ and $4\mathrm{Tr}(\gamma^5\gamma_\alpha\gamma_\beta\gamma_\gamma\gamma_\delta)$ in the physical limit.

Now we will review the formal proof of vector current conservation in the case of a three-point function with an arbitrary third current.[3] The relevant Feynman diagram is in Fig. 1. We denote the amplitude by $A^\Gamma_{\mu\nu}(p_1,p_2)$ and consider $p_{1\mu}A^\Gamma_{\mu\nu}(p_1,p_2)$ which should be zero if vector current conservation holds. (If $\Gamma=\gamma_\lambda\gamma_5$, our amplitude is related to Adler's by $A^\Gamma_{\mu\nu}=[1/(2\pi)^n]R_{\mu\nu\lambda}$.) We see that

$$A^\Gamma_{\mu\nu}(p_1,p_2)=\int\frac{d^nk}{(2\pi)^n}\mathrm{Tr}\left(\Gamma\frac{1}{\not k+\not p_2-m}\gamma_\nu\frac{1}{\not k-m}\times\gamma_\mu\frac{1}{\not k-\not p_1-m}\right)+\int\frac{d^nk}{(2\pi)^n}\mathrm{Tr}\left(\Gamma\frac{1}{\not k+\not p_1-m}\gamma_\mu\frac{1}{\not k-m}\times\gamma_\nu\frac{1}{\not k-\not p_2-m}\right),\quad(2.1)$$

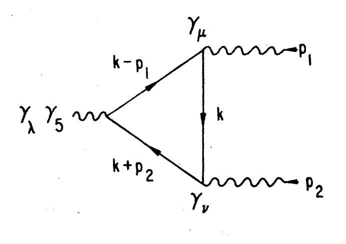

FIG. 1. The *VVA* triangle diagram. There is also a crossed diagram.

where the second term comes from the graph with vector vertices exchanged. Contracting this expression with $p_{1\mu}$, we use

$$\frac{1}{\not k - m}\not p_1 \frac{1}{\not k - \not p_1 - m} = \frac{1}{\not k - m}(-\not k + \not p_1 + m + \not k - m) \times \frac{1}{\not k - \not p_1 - m} = -\frac{1}{\not k - m} + \frac{1}{\not k - \not p_1 - m}, \quad (2.2)$$

$$\frac{1}{\not k + \not p_1 - m}\not p_1 \frac{1}{\not k - m} = \frac{1}{\not k - m} - \frac{1}{\not k + \not p_1 - m} \quad (2.3)$$

so

$$p_{1\mu} A^{\Gamma}_{\mu\nu}(p_1, p_2) = \int \frac{d^n k}{(2\pi)^n} \mathrm{Tr}\left[\Gamma \frac{1}{\not k + \not p_2 - m} \times \gamma_\nu \left(\frac{1}{\not k - \not p_1 - m} - \frac{1}{\not k - m}\right)\right] + \int \frac{d^n k}{(2\pi)^n} \mathrm{Tr}\left[\Gamma\left(\frac{1}{\not k - m} - \frac{1}{\not k + \not p_1 - m}\right) \times \gamma_\nu \frac{1}{\not k - \not p_2 - m}\right]. \quad (2.4)$$

Up to this point we have not used any properties but the n-dimensional γ-matrix rules. We have not used any property of Γ. We will perform a shift of integration variables on the second integral. The two terms in large parentheses require different shifts. For the first term we make the replacement $k \to k + p_2$, for the second, $k \to k + p_2 - p_1$:

$$p_{1\mu} A^{\Gamma}_{\mu\nu}(p_1, p_2) = \int \frac{d^n k}{(2\pi)^n} \mathrm{Tr}\left[\Gamma \frac{1}{\not k + p_2 - m} \times \gamma_\nu \left(\frac{1}{\not k - p_1 - m} - \frac{1}{\not k - m}\right)\right] + \int \frac{d^n k}{(2\pi)^n} \mathrm{Tr}\left(\Gamma \frac{1}{\not k + p_2 - m}\gamma_\nu \frac{1}{\not k - m}\right) - \int \frac{d^n k}{(2\pi)^n} \mathrm{Tr}\left(\Gamma \frac{1}{\not k + p_2 - m}\gamma_\nu \frac{1}{\not k - \not p_1 - m}\right) = 0. \quad (2.5)$$

We note that when using dimensional regularization shifts of integration variables are permitted, so the formal proof of gauge invariance for the triangle is on quite a sound basis. In carrying out the calculation of the amplitude, one should make no attempt to manipulate the factor Γ. The cancellation is completely independent of the properties of Γ. In an appendix we present the actual calculation of the amplitude for $\Gamma = \gamma_\lambda \gamma_5$. Here we present only results. Define

$$t_{\mu\nu\lambda}(p_1, p_2) = A^{\gamma_\lambda\gamma^5}_{\mu\nu}(p_1, p_2) = \left(\frac{-i}{4(2\pi)^n}\right) \times [\mathrm{Tr}(\gamma_\sigma\gamma_\mu\gamma_\nu\gamma_\lambda\gamma_5)(A_1 p_1^\sigma + A_2 p_2^\sigma) + \mathrm{Tr}(\gamma_\mu\gamma_\lambda \not p_1 \not p_2 \gamma_5)(A_3 p_{1\nu} + A_4 p_{2\nu}) + \mathrm{Tr}(\gamma_\nu\gamma_\lambda \not p_1 \not p_2 \gamma_5)(A_5 p_{1\mu} + A_6 p_{2\mu})]. \quad (2.6)$$

Note that we have not specified the trace of four γ matrices with γ_5, but we have assumed that the trace of γ^5 with fewer than four γ matrices vanishes. The quantities A_i may be expressed in terms of integrals over Feynman parameters, and there are no singularities as $n \to 4$. In contrast to Chanowitz *et al.* and to Frampton, we find that the A_i are unambiguous as functions of n and that the expression satisfies vector current conservation without the addition of any "contact term."

Of course our amplitude obeys Bose symmetry for the vector current vertices, so we have

$$\begin{aligned} A_1(p_1, p_2) &= -A_2(p_2, p_1), \\ A_3(p_1, p_2) &= -A_6(p_2, p_1), \\ A_4(p_1, p_2) &= -A_5(p_2, p_1). \end{aligned} \quad (2.7)$$

Since our A_i do satisfy vector current conservation, A_1 and A_2, the potentially divergent integrals, are finite and are given by

$$\begin{aligned} A_1 &= p_1 \cdot p_2 A_3 + p_2{}^2 A_4, \\ A_2 &= p_1{}^2 A_5 + p_1 \cdot p_2 A_6, \end{aligned} \quad (2.8)$$

where

$$A_3 = 16\pi^{n/2}\Gamma(3 - n/2) \times \int_0^1 dx \int_0^{1-x} dy\, xy(M^2 + Q^2)^{n/2-3},$$

$$A_4 = 16\pi^{n/2}\Gamma(3 - n/2) \times \int_0^1 dx \int_0^{1-x} dy\, x(1-x)(M^2 + Q^2)^{n/2-3}$$

with

$$M^2 = m^2 - x p_2{}^2 - y p_1{}^2$$

and

$$Q_\mu = xp_{2\mu} - yp_{1\mu} .$$

The only difference from Adler's result is the replacement of 1 by $3-n/2$ in the argument of the Γ function and the exponent of M^2+Q^2. If we are interested in the limit as $n\to 4$ of $t_{\mu\nu\lambda}$, the result will depend only on $\mathrm{Tr}(\gamma_\alpha\gamma_\beta\gamma_\gamma\gamma_\delta\gamma_5)$ at $n=4$, the usual $\epsilon_{\alpha\beta\gamma\delta}$ factor. Since our amplitude satisfies vector current conservation for all n, we recover at $n=4$ the result Adler obtained by imposing vector current conservation. The usual anomaly of the divergence of the axial-vector current then follows. However, if one uses the definition of the A_i in n dimensions, one finds

$$-(p_1+p_2)^\lambda t_{\mu\nu\lambda} = 2mt^5_{\mu\nu} + B\,\mathrm{Tr}(\not p_1 \not p_2 \gamma_\mu\gamma_\nu\gamma_5) ,$$

where

$$B = \frac{16}{(4\pi)^{n/2}}\Gamma(3-n/2) \times \int_0^1 dx \int_0^{1-x} dy (M^2+Q^2)^{n/2-2} .$$

Frampton brought up the point that when one allows B to have its full n dependence, rather than taking the limit $n=4$, the anomaly depends upon the fermion masses. Hence, an attempt to cancel the anomaly by adding another fermion with opposite coupling to the axial-vector current will not result in an exact cancellation unless the fermion masses are equal. At the one-loop level, the new piece of the anomaly is proportional to $n-4$; however, Frampton suggested that in higher-loop order there might be a pole to make the new piece finite. In Sec. III we will show that the new piece of the anomaly remains infinitesimal even in higher-loop order. For the moment, we will concentrate on a theory in which we attempt to cancel the one-loop anomaly between two different mass fermions. We find

$$t'_{\mu\nu\lambda} = \int \frac{d^nk}{(2\pi)^n}\left\{\mathrm{Tr}\left[\gamma_\lambda\gamma_5\left(\frac{1}{\not k+p_2-m_1}\gamma_\nu\frac{1}{\not k-m_1}\gamma_\mu\frac{1}{\not k-p_1-m_1}+\frac{1}{\not k+p_1-m_1}\gamma_\mu\frac{1}{\not k-m_1}\gamma_\nu\frac{1}{\not k-p_1-m_2}\right)\right]\right.$$
$$\left. -\mathrm{Tr}\left[\gamma_\lambda\gamma_5\left(\frac{1}{\not k+p_2-m_2}\gamma_\nu\frac{1}{\not k-m_2}\gamma_\mu\frac{1}{\not k-p_1-m_2}+\frac{1}{\not k+p_1-m_2}\gamma_\mu\frac{1}{\not k-m_2}\gamma_\nu\frac{1}{\not k-\not p_2-m_2}\right)\right]\right\} .$$

Note that if the two contributions are added, the leading behavior as $k\to\infty$ cancels, leaving a convergent integral even for $n=4$. Thus dimensional regularization is not needed to obtain a well-defined expression at $n=4$ which satisfies both vector and axial-vector Ward identities.

If we write the difference of the two integrands as

$$I(m_1)-I(m_2) = -\int_{m_1}^{m_2} dm \frac{d}{dm} I(m) ,$$

it is easy to see that the loop integrations are finite. The sum of the two triangles is now a finite integral over a mass parameter of a triangle with a mass insertion. The triangle was linearly divergent, but the triangle with a mass insertion is linearly convergent. It is easy to see that in this theory

$$A'_3 = \int_{m_1}^{m_2} dm \left[-16\pi^{n/2}\Gamma(4-n/2) \times \int_0^1 dx \int_0^{1-x} dy\, 2xym(Q^2+M^2)^{n/2-4}\right], \tag{2.9}$$

$$A'_4 = \int_{m_1}^{m_2} dm \left[16\pi^{n/2}\Gamma(4-n/2) \times \int_0^1 dx \int_0^{1-x} dy\, 2x(x-1) \times m(Q^2+M^2)^{n/2-4}\right]. \tag{2.10}$$

It is also easy to see that it was not necessary that there be only one mass parameter for the cancellation to be written as an integral over a mass insertion. Suppose the legs of the diagram correspond to particles of mass m_1, m_2, m_3 in one triangle and m'_1, m'_2, m'_3 in the other. Then

$$\begin{aligned} T(m_1,m_2,m_3) - T(m'_1,m'_2,m'_3) &= T(m_1,m_2,m_3) - T(m'_1,m_2,m_3) \\ &+ T(m'_1,m_2,m_3) - T(m'_1,m'_2,m_3) \\ &+ T(m'_1,m'_2,m_3) - T(m'_1,m'_2,m'_3) . \end{aligned}$$

Each line may be written as an integral between two masses of a mass insertion (but now the mass insertion is on a particular leg).

III. WHY THE AXIAL ANOMALY REMAINS INFINITESIMAL

In Sec. II we saw that when we try to cancel the anomaly between two fermion species, we

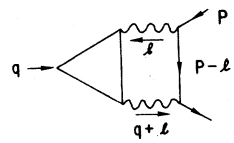

FIG. 2. The triangle contribution to the axial-vector–fermion vertex.

have actually eliminated the divergence in the fermion loop momentum integration. The integrals are convergent if they are done in four dimensions, but they may also be done via dimensional regularization. When Frampton calculated the divergence of the axial-vector current for a single fermion he found that what was simply a number in Adler's calculation had become a complicated function of Feynman parameters, external momenta, and masses raised to the power $\frac{1}{2}n-2$. Expanding for n near four, he found an infinitesimal (i.e., proportional to $n-4$) term which depends upon the fermion mass. He worried that these new terms of order $n-4$ might find a pole from a subsequent integration and become finite. This does not happen.

Having two fermions does more than make the fermion loop integrals finite, it also makes the behavior of the triangle for large external momenta softer. The single triangle grows linearly with the external momentum; however, as can be seen by comparing A_3 with A_3', when there are two fermions trying to cancel each other, the triangle decreases linearly with the external momentum. Thus, the contribution of the odd-parity triangle to the axial-vector vertex correction in Fig. 2 is in fact finite, although it looks like it has a logarithmic divergence. In fact, if the anomaly were

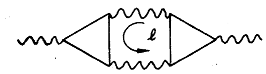

FIG. 3. The two-triangle contribution to the axial-vector two-point function.

not canceled at the one-loop level, there would be a divergence. This should be compared with Ref. 3. To make the discussion more concrete, we consider the new mass-dependent anomaly of Frampton and calculate its contribution to Fig. 2.

The mass-dependent part of Frampton's anomaly is proportional to

$$(n-4)\epsilon_{\mu\nu\sigma\tau}p_1^\sigma p_2^\tau \int_0^1 dx \int_0^{1-x} dy \ln(Q^2+M^2) \quad (3.1)$$

for a single fermion. If there are two fermions with opposite coupling to the axial-vector vertex, we have

$$(n-4)\epsilon_{\mu\nu\sigma\tau}p_1^\sigma p_2^\tau \int_0^1 dx \int_0^{1-x} dy \ln\left(\frac{Q^2+M_1^2}{Q^2+M_2^2}\right), \quad (3.2)$$

where $M_i^2=m_i^2-xp_2^2-yp_1^2$. Inserting this into Fig. 2, we substitute $p_1=l$, $p_2=-(q+l)$, to get

$$-\int \frac{d^n l}{(2\pi)^n} \frac{\gamma^\mu(\not p-\not l+m)\gamma^\nu}{l^2(q+l)^2[(p-l)^2-m^2]}(n-4)\epsilon_{\mu\nu\sigma\tau}l^\sigma q^\tau \times \int_0^1 dx \int_0^{1-x} dy \ln\frac{(Q^2+M_1^2)}{(Q^2+M_2^2)}. \quad (3.3)$$

Except for the logarithm, we would have a logarithmically divergent integral in accord with naive power counting for a vertex correction. However, as l approaches infinity, the logarithm goes to zero, thus cutting off the integral. As usual, it is easiest to see what happens by writing

$$\begin{aligned}\ln\left(\frac{Q^2+M_1^2}{Q^2+M_2^2}\right) &= -\int_{m_1}^{m_2} dm \frac{d}{dm}\ln(Q^2+M^2) \\ &= -\int_{m_1}^{m_2} dm \frac{2m}{Q^2+M^2} \\ &= -\int_{m_1}^{m_2} dm \frac{2m}{l^2[(x+y)^2-x-y]+2l\cdot q[x(x+y-1)]+x(x-1)q^2+m^2}\,. \end{aligned} \quad (3.4)$$

Written in this way, we see that the logarithm provides two extra powers of convergence so that the integral over l is finite (i.e., there is no pole at $n=4$). There is an explicit power of $n-4$ since we were considering the new mass-dependent anomaly. The new term makes no contribution in the limit $n=4$.

Next we consider the two-point function with two odd-parity triangles, as shown in Fig. 3. Although this diagram seems to have overlapping divergences, the triangle is so softened by the cancellation mechanism that it has no divergences at all. Each triangle vanishes like $1/l$ as the center loop momentum l goes to infinity, so we have four extra powers of convergence over naive power counting. The naive quadratic divergence turns into a quadratic convergence. There is no pole at $n=4$ to promote unwanted infinitesimal terms to finite status. So we see that we could have taken the limit $n=4$ for the triangle itself before inserting it into other diagrams. This is precisely what we suggested doing in a previous paper.[19]

We have seen that adding fermions to a theory does two things. First, adding the contributions of the two fermions together before the triangle loop integral is evaluated renders the integral convergent. Second, the resulting triangle amplitude is sufficiently softened that its contribution to other diagrams is finite. So far we have concentrated on an Abelian theory. We have also not considered radiative corrections to the triangle, which are covered by the Adler-Bardeen theorem.[20]

IV. WARD IDENTITIES FOR TWO-POINT FUNCTIONS

We will consider the case of the axial-vector current two-point function in this section. We have calculated the two-point function in detail assuming that γ^5 anticommutes with all other γ matrices and that $\gamma_5^{\,2}=1$. In this way γ^5 may be eliminated from the trace. The amplitude thus constructed is a tensor which has meaning in n dimensions. It is to be contrasted with the amplitudes calculated for abnormal-parity loops. These amplitudes involve the tensor $\epsilon_{\mu\nu\lambda\sigma}$ and do not have any consistent meaning outside of four dimensions. Our prescription for calculation is identical to that of Ref. 18. However, we demonstrate the Ward identity for all values of momentum, not just for $p=0$.

For the moment we restrict our attention to a theory with one fermion of mass m and consider two-point functions of vector, axial-vector and pseudoscalar currents. We denote the currents by J_μ, J^5_μ, and J^5 respectively. The two-point functions which correspond to even-parity loops are $\langle 0|\, TJ_\mu J_\nu|\, 0\rangle$, $\langle 0|\, TJ^5_\mu J^5_\nu|\, 0\rangle$, $\langle 0|\, TJ^5_\mu J^5|\, 0\rangle$, and $\langle 0|\, TJ^5 J^5|\, 0\rangle$. The abnormal-parity functions $\langle 0|\, TJ_\mu J^5_\nu|\, 0\rangle$ and $\langle 0|\, TJ_\mu J^5|\, 0\rangle$ must vanish because they depend only on one momentum but must involve the ϵ tensor. We see that

$$\begin{aligned}\langle 0|\, TJ_\mu J_\nu|\, 0\rangle &= (-1)\int \frac{d^nk}{(2\pi)^n}(i)^2\frac{\mathrm{Tr}[(\not p+\not k+m)\gamma^\mu(\not k+m)\gamma^\nu]}{[(p+k)^2-m^2](k^2-m^2)}\\ &=\int_0^1 dx\int\frac{d^nk}{(2\pi)^n}\mathrm{Tr}[(\not p+\not k+m)\gamma^\mu(\not k+m)\gamma^\nu]\,\frac{1}{(k^2+2xp\cdot k+xp^2-m^2)^2}\\ &=\int_0^1 dx\,\frac{i\pi^{n/2}}{(2\pi)^n\Gamma(2)}\,\frac{1}{(-xp^2+m^2+x^2p^2)^{2-n/2}}\big(\mathrm{Tr}\{[(1-x)\not p+m]\gamma^\mu(-x\not p+m)\gamma^\nu\}\Gamma(2-n/2)\\ &\qquad -\tfrac{1}{2}[-x(1-x)p^2+m^2]\,\mathrm{Tr}(\gamma^\lambda\gamma_\mu\gamma_\lambda\gamma_\nu)\Gamma(1-n/2)\big).\end{aligned}\tag{4.1}$$

But using the n-dimensional rules for γ matrix algebra, the factor in heavy parentheses is just

$8x(1-x)\Gamma(2-n/2)(p^2g_{\mu\nu}-p_\mu p_\nu)$

so that

$$\langle 0|\, TJ_\mu J_\nu|\, 0\rangle=\frac{i}{(4\pi)^{n/2}}\int_0^1 dx\frac{8x(1-x)}{[m^2-x(1-x)p^2]^{2-n/2}}\Gamma(2-n/2)(p^2g_{\mu\nu}-p_\mu p_\nu)\,.\tag{4.2}$$

Next we consider $\langle 0|\, TJ^5_\mu J^5_\nu|0\rangle$. $\mathrm{Tr}[(\not p+\not k+m)\gamma_\mu(\not k+m)\gamma_\nu]$ is replaced by

$\mathrm{Tr}[(\not p+\not k+m)\gamma_\mu\gamma_5(\not k+m)\gamma_\nu\gamma_5]$.

Letting γ_5 anticommute with γ_μ and setting $\gamma_5^{\,2}=1$, we may eliminate it from the trace:

$$\mathrm{Tr}[(\not p+\not k+m)\gamma_\mu\gamma^5(\not k+m)\gamma_\nu\gamma^5]=\mathrm{Tr}[(\not p+\not k+m)\gamma_\mu(\not k+m)\gamma_\nu]-2m\,\mathrm{Tr}[(\not p+\not k+m)\gamma_\mu\gamma_\nu]\,.$$

We see immediately that

$$\begin{aligned}\langle 0|\, TJ^5_\mu J^5_\nu|0\rangle &=\langle 0|\, TJ_\mu J_\nu|\, 0\rangle-8m^2g_{\mu\nu}\int\frac{d^nk}{(2\pi)^n}\frac{1}{[(p+k)^2-m^2](k^2-m^2)}\\ &=\langle 0|\, TJ_\mu J_\nu|\, 0\rangle-\frac{8i}{(4\pi)^{n/2}}\int_0^1 dx\frac{1}{[m^2-x(1-x)p^2]^{2-n/2}}\Gamma(2-n/2)m^2g_{\mu\nu}\,.\end{aligned}\tag{4.3}$$

Similarly we calculate $\langle 0|\, TJ^5_\mu J^5|\, 0\rangle$ and $\langle 0|\, TJ^5J^5|\, 0\rangle$:

$$\langle 0|\, TJ^5_\mu J^5|\, 0\rangle=\frac{i}{(4\pi)^{n/2}}\int_0^1 dx\frac{4}{[m^2-x(1-x)p^2]^{2-n/2}}\Gamma(2-n/2)mp_\mu\,,\tag{4.4}$$

$$\langle 0|\, TJ^5J^5|\, 0\rangle=\frac{i}{(4\pi)^{n/2}}\int_0^1 dx\frac{4}{[m^2-x(1-x)p^2]^{2-n/2}}\{\Gamma(2-n/2)[m^2+x(1-x)p^2]+\tfrac{1}{2}n\Gamma(1-\tfrac{1}{2}n)[m^2-x(1-x)p^2]\}\,.\tag{4.5}$$

Now that we have the two-point functions we ask whether they are consistent with the canonical equations of motion. It is easy to derive the equations of motion from the Lagrangian of QED. For the fermion, $(i\not\partial-e\not A-m)\Psi=0$. Letting $J^5_\mu=\bar\Psi\gamma_\mu\gamma^5\Psi$, $J_\mu=\bar\Psi\gamma_\mu\Psi$, and $J^5=\bar\Psi\gamma^5\Psi$,

$$\partial_\mu J_\mu=0\,,\tag{4.6}$$

$$\partial_\mu J^5_\mu = 2imJ^5 . \tag{4.7}$$

From Eq. (4.2) we see that it is consistent with (4.6) since $p_\mu(p^2 g_{\mu\nu} - p_\mu p_\nu) = 0$. This is well known. For the axial-vector–current two-point function we must have

$$-ip_\mu\langle 0|\, TJ^5_\nu J^5_\mu |\, 0\rangle = 2im\langle 0|\, TJ^5_\nu J^5 |\, 0\rangle \tag{4.8}$$

to be consistent with (4.7). Comparing (4.3) and (4.4) it is quite easy to verify this. We would also like to see if the equation of motion gives the correct relation between $\langle 0|\, TJ^5_\mu J^5 |\, 0\rangle$ and $\langle 0|\, TJ^5 J^5 |\, 0\rangle$. This is not as obvious as the last two relations. First, the equation of motion is not quite as easy to apply to this case because $[J^5_0, J^5] \neq 0$. We find

$$\begin{aligned}\partial^\mu_x\langle 0|\, TJ^5_\mu(x)J^5(y)|\, 0\rangle &= \langle 0|\, T\partial^\mu J^5_\mu(x)J^5(y)|\, 0\rangle \\ &\quad + \delta(x_0 - y_0)\langle 0|\, [J^5_0(x), J^5(y)]|\, 0\rangle ,\end{aligned} \tag{4.9}$$

$$\begin{aligned}\delta(x_0 - y_0)[J^5_0(x), J^5(y)] &= -\delta^4(x-y) \\ &\quad \times [\bar\Psi(y)\Psi(x) + \bar\Psi(x)\Psi(y)] .\end{aligned} \tag{4.10}$$

We choose to write the right-hand side in this way because it makes a later equation more covariant looking. Our Fourier transform conventions are

$$\begin{aligned}\langle 0|\, TJ^5_\mu(x)J^5(y)|\, 0\rangle &= \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot(x-y)}\Delta^5_\mu(k) , \\ \langle 0|\, TJ^5(x)J^5(y)|\, 0\rangle &= \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot(x-y)}\Delta^5(k) , \\ \langle 0|\, \bar\Psi(y)\Psi(x) + \bar\Psi(x)\Psi(y)|\, 0\rangle &= \int \frac{d^4k}{(2\pi)^4} \times e^{ik\cdot(x-y)}C(k) .\end{aligned} \tag{4.11}$$

Thus, $\Delta^5_\mu(k)$ $[\Delta^5(k)]$ corresponds to $\langle 0|\, TJ^5_\mu J^5 |\, 0\rangle$ $[\langle 0|\, TJ^5 J^5 |\, 0\rangle]$ in Eq. (4.4) [(4.5)]. Note that we have kept the term from the commutator in the form of a vacuum expectation value rather than as the trace of a propagator. In momentum space, (4.9) becomes

$$ik^\mu\Delta^5_\mu(k) = 2im\Delta^5(k) - \int \frac{d^4l}{(2\pi)^4} C(l) . \tag{4.12}$$

Evaluation of C is quite easy using the free-field decomposition of $\Psi, \bar\Psi$ as is appropriate to this order:

$$C(k) = -8\pi m\,\delta(k^2 - m^2) . \tag{4.13}$$

The evaluation of the last term of (4.12) involves a divergent integral. We will use dimensional regularization to evaluate it:

$$\begin{aligned}\int \frac{d^nk}{(2\pi)^n} C(k) &= -8\pi m \int \frac{d^nk}{(2\pi)^n}\delta(k^2 - m^2) \\ &= \frac{-8\pi m}{(2\pi)^n}\int dk_0 \int d^{n-1}\vec{k}\,\delta(k_0{}^2 - \vec{k}^2 - m^2) \\ &= \frac{-4m}{(2\pi)^{n-1}}\int d^{n-1}\vec{k}\,\frac{1}{(\vec{k}^2 + m^2)^{1/2}} \\ &= \frac{-4m}{(2\pi)^{n-1}}\,\frac{2\pi^{(n-1)/2}}{\Gamma\left(\frac{n-1}{2}\right)}\int_0^\infty \frac{k^{n-2}dk}{(k^2 + m^2)^{1/2}} \\ &= \frac{-8m}{(4\pi)^{n/2}}\Gamma(1 - n/2)m^{n-2} .\end{aligned}$$

Now we are ready to verify (4.12). The left- and right-hand sides are given by (4.14a) and (4.14b), respectively:

$$-4mk^2\frac{1}{(4\pi)^{n/2}}\int_0^1 dx\frac{1}{[m^2 - x(1-x)k^2]^{2-n/2}}\Gamma(2 - n/2) , \tag{4.14a}$$

$$\begin{aligned}-8m\frac{1}{(4\pi)^{n/2}}\int_0^1 dx\frac{1}{[m^2 - x(1-x)k^2]^{2-n/2}}\{&[m^2 + x(1-x)k^2]\Gamma(2 - n/2) \\ &+ \tfrac{1}{2}n[m^2 - x(1-x)k^2]\Gamma(1 - n/2)\} + \frac{8m}{(4\pi)^{n/2}}m^{n-2}\Gamma(1 - n/2) .\end{aligned} \tag{4.14b}$$

It is trivial to show that at $k^2 = 0$ both sides are zero. For $k^2 \neq 0$, to verify (4.14a) = (4.14b) it suffices to show

$$\int_0^1 \frac{dx}{[m^2 - x(1-x)k^2]^{2-n/2}}\{[m^2 + x(1-x)k^2 - \tfrac{1}{2}k^2]\Gamma(2 - \tfrac{1}{2}n) + \tfrac{1}{2}n\Gamma(1 - \tfrac{1}{2}n)[m^2 - x(1-x)k^2]\} = m^{n-2}\Gamma(1 - n/2) . \tag{4.15}$$

Fortunately it is easy to evaluate the integral over the Feynman parameter x (Ref. 21):

$$\int_0^1 dx\frac{[x(1-x)]^\alpha}{[m^2 - x(1-x)k^2]^{2-n/2}} = (m^2)^{n/2-2}(\tfrac{1}{4})^{\alpha+1/2}B(\tfrac{1}{2}, \alpha+1)\,{}_2F_1\left(2 - \frac{n}{2}, \alpha+1; \alpha+\tfrac{3}{2}; \frac{k^2}{4m^2}\right) .$$

Using this integral we express the left-hand side of (4.15) in terms of two hypergeometric functions, eliminating k^2 via $k^2=4m^2x$:

$$m^{n-2}\Gamma\left(1-\frac{n}{2}\right)\left\{\left[1-2x\left(1-\frac{n}{2}\right)\right]{}_2F_1\left(2-\frac{n}{2},1;\tfrac{3}{2};x\right)+\tfrac{2}{3}x(1-n)\,{}_2F_1\left(2-\frac{n}{2},2;\tfrac{5}{2};x\right)\right\}. \qquad (4.16)$$

We eliminate the second hypergeometric function by use of the identity[22]

$$\frac{3}{2}\left(1-\frac{n}{2}\right){}_2F_1\left(2-\frac{n}{2},1;\tfrac{3}{2};x\right)-\frac{1}{2}\left(2-\frac{n}{2}\right){}_2F_1\left(3-\frac{n}{2},1;\tfrac{5}{2};x\right)=\frac{(1-n)}{2}\,{}_2F_1\left(2-\frac{n}{2},2;\tfrac{5}{2};x\right).$$

Then the expression in curly brackets reduces to[23]

$${}_2F_1\left(2-\frac{n}{2},1;\tfrac{3}{2};x\right)-\tfrac{2}{3}x\left(2-\frac{n}{2}\right){}_2F_1\left(3-\frac{n}{2},1;\tfrac{5}{2};x\right)=F\left(3-\frac{n}{2},0;\tfrac{3}{2};x\right)=1. \qquad (4.17)$$

Replacing the curly brackets in (4.16) by 1 in accord with (4.17), we see that the left-hand side of (4.15) reduces to $m^{n-2}\Gamma(1-n/2)$, thus verifying that identity. The verification of (4.15) is sufficient to prove the Ward identity (4.9). We have gone through all this analysis to show that the two-point functions may be calculated using dimensional regularization if one simply uses the anticommutivity of γ^5 (with all γ matrices) and $(\gamma^5)^2=1$. The resulting amplitudes obey the Ward identities for all n without any anomalous terms. Verification of two of the identities was trivial, the third was not.

Using a γ^5 which anticommutes with all γ matrices is consistent for even-parity loops. It most closely imitates the action of γ^5 in four dimensions. The two-point functions are the most divergent even-parity loops. We have shown how nicely this procedure works in that case. It should not be difficult to verify that this procedure works for all even-parity loops even for a non-Abelian theory. Calculations of certain even-parity three-point functions were previously done by Bardeen *et al.* as stated above. Using an anticommuting γ^5, they found that the Ward identities hold.

V. SUMMARY AND CONCLUSIONS

We have been considering several questions in this paper. Although these questions generally come up in the context of non-Abelian theories, we have concentrated on Abelian theories for simplicity and because we feel that the basic physics may be seen in such theories.

We first discussed the *VVA* triangle. Dimensional regularization may be used to see that there is an anomaly; however, if one deals with a theory designed to cancel the anomaly, the integration over the fermion loop momentum is finite. That is, the amplitude requires no regularization dimensional or otherwise if one adds all triangle contributions together before doing the integration.

If one insists on using the form of the amplitude for arbitrary n derived from dimensional regularization, there appears to be an infinitesimal mass-dependent anomaly. We have shown that it remains infinitesimal even when the triangle is embedded in a larger diagram. The diagram in Fig. 2 was considered in detail in Sec. III. We showed that although it appears to be divergent, the integration over l is actually convergent because the cancellation of the anomaly softens the triangle.

Let us consider radiative corrections to Fig. 2. In Fig. 4(a) there is a divergence from the photon self-energy. There is also a diagram with the self-energy counterterm. The divergence of Fig. 4(a) is just in the photon propagator. If we consider Fig. 2 to be the first term of a skeleton expansion and replace the bare propagators and vertices by renormalized ones as in Fig. 4(b), we see that the two remaining loop integrations are finite once we take both fermions into account. The fact that the leading behavior of the renormalized propagators and vertices is identical to that of the bare ones, up to logarithms, is essential. In the non-Abelian theories, the situation is a bit more subtle since there may be anomalies in four- and five-point functions unless the fermion representations are properly arranged. We expect that a careful analysis would show that if all the one-loop anomalies are can-

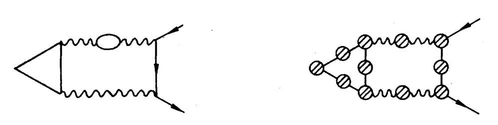

FIG. 4. (a) A photon self-energy insertion from the radiative corrections to Fig. 2. (b) The full set of corrections. The shaded blobs are renormalized propagators and vertices.

celed there is no problem with radiative corrections. If that is the case, it would be very convenient to use dimensional regularization and the minimal-subtraction scheme to calculate the renormalized propagators and vertices. Then, the only divergences involving γ^5 would come from even-parity loops and axial-vector or pseudoscalar vertex corrections, in which γ^5 is on an external fermion line. We have seen that an even-parity loop may be calculated via dimensional regularization and that the answer is consistent with all Ward identities. When γ^5 is on an external line, we may use its anticommutivity to move it to the end of the line and then use dimensional regularization on the diagram as if it were absent.

It would be useful to show explicitly that all even-parity loops obey the Ward identities when calculated in the way indicated. It is also necessary to carefully consider the non-Abelian case to show that the cancellation of all the one-loop anomalies is sufficient to assure that the radiative corrections to the triangle do not contribute to the anomaly. We feel, however, that we have demonstrated the basic physics of the anomaly cancellation mechanism, how one may calculate with γ^5 in the context of dimensional regularization, and why Frampton's new anomaly remains infinitesimal. After completing this work we received a report by Ishikawa and Milton[24] in which they advance arguments similar to those of Sec. III.

ACKNOWLEDGMENTS

We are happy to acknowledge several interesting discussions with William Bardeen. This work was performed under the auspices of the United States Department of Energy.

APPENDIX

The evaluation of the VVA triangle in the dimensional scheme can be carried out without making explicit the nature of γ_5 for general n except that its trace with odd numbers of γ matrices or with two γ matrices vanishes. Let us write the amplitude, following the approach of Adler, as

$$t_{\mu\nu\lambda} = \int \frac{d^n k}{(2\pi)^n} \mathrm{Tr}\left[\left(\frac{1}{\not k + \not p_1 - m}\gamma_\mu \frac{1}{\not k - m}\gamma_\nu \frac{1}{\not k - \not p_2 - m} + \frac{1}{(-\not k + \not p_2 - m)}\gamma_\nu \frac{1}{(-\not k - m)}\gamma_\mu \frac{1}{(-\not k - \not p_1 - m)}\right)\gamma_\lambda\gamma_5\right]. \tag{A1}$$

After rationalizing and combining denominators, one finds

$$t_{\mu\nu\lambda} = \frac{-2}{(2\pi)^n}\int_0^1 dx \int_0^{1-x} dy \int d^n k [m^2 - xp_2{}^2 - yp_1{}^2 + 2(xp_2 - yp_1)\cdot k - k^2]^{-3} \times [m^2 \mathrm{Tr}(\gamma_\mu\gamma_\nu\gamma_\tau\gamma_\lambda\gamma_5) 2(k + p_1 - p_2)^\tau + \mathrm{Tr}(\gamma^\alpha\gamma_\mu\gamma^\beta\gamma_\nu\gamma^\tau\gamma_\lambda\gamma_5)(k+p_1)_\alpha k_\beta (k-p_2)_\tau - \mathrm{Tr}(\gamma^\alpha\gamma_\nu\gamma^\beta\gamma_\mu\gamma^\tau\gamma_\lambda\gamma_5)(k-p_2)_\alpha k_\beta (k+p_1)_\tau]. \tag{A2}$$

Although this expression involves the trace of products of six γ matrices with γ_5, one may reduce the expression to sums of traces of products of four γ matrices with γ_5 by performing the k integration and using the n-dimensional Dirac algebra. It is not necessary to specify the anticommuting nature of γ_5, and one obtains, unambiguously,

$$t_{\mu\nu\lambda} = \frac{-i}{4(2\pi)^n}[\mathrm{Tr}(\gamma_\mu\gamma_\nu\gamma_\tau\gamma_\lambda\gamma_5)(\tilde A_1 p_1^\tau + \tilde A_2 p_2^\tau) + \mathrm{Tr}(\gamma_\mu\gamma_\xi\gamma_\tau\gamma_\lambda\gamma_5)p_1^\xi p_2^\tau(\tilde A_3 p_{1\nu} + \tilde A_4 p_{2\nu}) + \mathrm{Tr}(\gamma_\nu\gamma_\xi\gamma_\tau\gamma_\lambda\gamma_5)p_1^\xi p_2^\tau(\tilde A_5 p_{1\mu} + \tilde A_6 p_{2\mu})], \tag{A3}$$

where the invariant quantities $\tilde A_i$ may be written

$$\tilde A_i = 16\pi^{n/2}\Gamma(3 - n/2)\int_0^1 dx \int_0^{1-x} dy\, \tilde a_i D^{n/2-3}, \tag{A4}$$

where

$$D = m^2 - x(1-x)p_2{}^2 - y(1-y)p_2{}^2 - 2xyp_1\cdot p_2, \tag{A5}$$

and for $3 \le i \le 6$,

$$\tilde{a}_3 = xy\,, \tag{A6}$$

$$\tilde{a}_4 = x(1-x)\,, \tag{A7}$$

$$\tilde{a}_5 = -y(1-y)\,, \tag{A8}$$

$$\tilde{a}_6 = -xy\,. \tag{A9}$$

In the limit $n=4$, these $\tilde{A}_i$ become the A_3–A_6 of Adler. For A_1 and A_2, which are represented by formally divergent integrals when $n=4$, Adler chose to define them by current conservation, which rendered these quantities unambiguous and well defined. The dimensional-regularization method yields expressions for $\tilde{a}_1$ and $\tilde{a}_2$ which may be written as

$$\tilde{a}_1 = (1-3y)D/(4-n) + \tfrac{1}{2}\{y(1-y)(1-2y)p_1^2 + 4xy(1-y)p_1\cdot p_2 + x[(1-y)(1-2x)+1]p_2^2\}\,, \tag{A10}$$

$$\tilde{a}_2 = -(1-3x)D/(4-n) + \tfrac{1}{2}\{p_1^2 y[(2y-1)(1-x)-1] - 4xy(1-x)p_1\cdot p_2 + x(1-x)(2x-1)p_2^2\}\,. \tag{A11}$$

These expressions contain a pole at $n=4$, but the residue of this pole is zero after integration over x and y. Motivated by Adler's imposition of current conservation, we write

$$\tilde{a}_1 = p_1\cdot p_2 a_3 + p_2^2 a_4 + \overline{a}_1 \tag{A12}$$

and we now prove that the integral over x and y of $\overline{a}_1 D^{n/2-3}$ is zero. First we note that the coefficient of $(n-4)^{-1}$ in Eq. (A10) may be integrated by parts:

$$\frac{1}{4-n}\int_0^{1-x} dy(1-3y)D^{n/2-2} = \frac{1}{(4-n)}[1-x-\tfrac{3}{2}(1-x)^2][m^2-(p_1+p_2)^2x(1-x)]^{n/2-2} + \tfrac{1}{2}\int_0^{1-x} dy\, y(1-\tfrac{3}{2}y)[(2y-1)p_1^2 - 2xp_1\cdot p_2]D^{n/2-3}\,. \tag{A13}$$

We may then write

$$\int_0^1 dx\int_0^{1-x} dy\,\overline{a}_1 D^{n/2-3} = \frac{1}{4-n}\int_0^1 dx(1-x)(\tfrac{3}{2}x-\tfrac{1}{2})[m^2-(p_1+p_2)^2x(1-x)]^{n/2-2} + \tfrac{1}{2}\int_0^1 dx\int_0^{1-x} dy\, D^{n/2-3}N\,, \tag{A14}$$

where

$$N = \tfrac{1}{2}y^2(1-2y)p_1^2 - xyp_1\cdot p_2 + xy(2x-1)p_2^2\,. \tag{A15}$$

It may then be noticed that

$$N = -\tfrac{1}{2}y^2\frac{\partial D}{\partial y} + xy\frac{\partial D}{\partial x}\,, \tag{A16}$$

from which it follows that

$$\tfrac{1}{2}\int_0^1 dx\int_0^{1-x} dy\, D^{n/2-3}N = \frac{1}{(n-4)}\int_0^1 dx[-\tfrac{1}{2}(1-x)^2 + x(1-x)][m^2-(p_1+p_2)^2x(1-x)]^{n/2-2}\,, \tag{A17}$$

which exactly cancels the first integral on the right-hand side of (A14). Similar manipulations may be carried out for $\overline{a}_2$. We have proved that the dimensional scheme provides unambiguous answers for the coefficients of the products of traces of products of four γ matrices with γ_5, and that these coefficients do satisfy the vector Ward identities for all n. The anomaly then appears in the axial-vector Ward identity. The reason the anomaly naturally appears in the axial-vector current is clear. The "proof" of the usual axial-vector Ward identity requires both that γ_5 anticommute with all γ matrices and that momentum shifts be allowed. For the strictly anticommuting γ_5, the condition

$$(n-4)\,\mathrm{Tr}(\gamma_\alpha\gamma_\beta\gamma_\gamma\gamma_\delta\gamma_5) = 0$$

means that the trace cannot have a smooth value as n is varied, hence the dimensional scheme may not be used to justify the necessary shifts of loop momentum.

It should be noted that if instead of the VVA triangle we had considered a VVS triangle, where S denotes a scalar vertex, there is no difficulty associated with defining the amplitude in n dimensions. One expects vector current conservation to hold, and it does, provided one goes through manipulations similar to those needed to prove that $\int dx \int dy\, \overline{a}_1 D^{n/2-3}$ was zero. The point is that straightforward application of dimensional integration techniques will always produce amplitudes consistent with vector current conservation, but it may take some algebraic gymnastics to prove it.

*On leave from Laboratoire de Physique Théorique, Université de Bordeaux I, Chemin du Solarium, 33170 Gradignan, France.

[1]G. 't Hooft and M. Veltman, Nucl. Phys. B44, 189 (1972).

[2]W. A. Bardeen, R. Gastmans, and B. Lautrup, Nucl. Phys. B46, 319 (1972).

[3]S. L. Adler, in *Lectures on Elementary Particles and Quantum Field Theory*, edited by S. Deser, M. Grisaru, and H. Pendleton (M.I.T. Press, Cambridge, 1970).

[4]C. Bouchiat, J. Iliopoulos, and Ph. Meyer, Phys. Lett. 38B, 519 (1972).

[5]D. J. Gross and R. Jackiw, Phys. Rev. D 6, 477 (1972).

[6]W. A. Bardeen, Phys. Rev. 184, 1848 (1969).

[7]W. A. Bardeen, in *Proceedings of the XVI International Conference on High Energy Physics*, edited by J. D. Jackson and A. Roberts (Fermi National Accelerator Laboratory, Batavia, 1972).

[8]W. A. Bardeen, private communication.

[9]D. Akyeampong and R. Delbourgo, Nuovo Cimento 17A, 578 (1973).

[10]D. Akyeampong and R. Delbourgo, Nuovo Cimento 19A, 219 (1974).

[11]P. Breitenlohner and D. Maison, Commun. Math. Phys. 52, 11 (1977).

[12]T. Marinucci and M. Tonin, Nuovo Cimento 31A, 381 (1976).

[13]G. Costa, T. Marinucci, M. Tonin, and J. Julve, Nuovo Cimento 38A, 373 (1977).

[14]C. Becchi, A. Rouet, and R. Stora, Commun. Math. Phys. 42, 127 (1975).

[15]In order to illustrate the difficulties inherent in using the schemes of Breitenlohner and Maison and of Akyeampong and Delbourgo, we note that in O. Nachtmann and W. Wetzel [Phys. Lett. 81B, 211 (1979)] it is claimed that a spontaneous breakdown of chiral symmetry occurs in a theory with two massless quarks. They calculated the VV and AA two-point functions using these schemes, and found them to be unequal. It is pointed out by Chanowitz, Furman, and Hinchliffe (Ref. 18), that if one uses a fully anticommuting γ_5, chiral symmetry is preserved in the massless case.

[16]P. H. Frampton, Ohio State University Report No. C00-1545-248 (unpublished).

[17]P. H. Frampton, preceding paper, Phys. Rev. D 20, 3372 (1979).

[18]M. Chanowitz, M. Furman, and I. Hinchliffe, LBL Report No. 8855 (unpublished).

[19]S. Gottlieb and J. T. Donohue, Phys. Rev. D (to be published).

[20]S. L. Adler and W. A. Bardeen, Phys. Rev. 182, 1517 (1969).

[21]G. Leibbrandt, Rev. Mod. Phys. 47, 849 (1975).

[22]I. S. Gradsteyn and I. W. Ryzhik, *Table of Integrals, Series, and Products* (Academic, New York, 1965), p. 1044, identity 9.137 No. 5.

[23]Reference 22 , previous identity.

[24]K. Ishikawa and K. A. Milton, Phys. Rev. D (to be published).