PHYSICAL REVIEW D VOLUME 20, NUMBER 12 15 DECEMBER 1979

Complete integration of U(N) lattice gauge theory in a large-N limit

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(Received 9 May 1979)

We present an approximation for the integration over a link in the Feynman path integral for U(N) lattice quantum chromodynamics (QCD). The approximation is valid when $N \to \infty$ and g^2N is fixed to a large value. The result is such that subsequent link integrations have the same form, allowing a complete evaluation of the path integral in any dimension over the entire lattice by repeated application of the approximation. Our technique can be applied to a variety of problems. As sources for the Yang-Mills field we use quarks traveling on world lines. We present a complete action formulation of Dirac particles on world lines with spin, color, flavor, and mass interacting with the gluon field. Evaluating the meson propagator in such a theory in our approximation we arrive at a previously proposed string model with quarks at the ends, thus demonstrating the dynamical equivalence of strings and QCD in our limit, on the lattice.

I. INTRODUCTION

Quantum chromodynamics (QCD) is generally regarded as the most likely theory of strong interactions. The perturbative predictions of the theory based on asymptotic freedom are in good agreement with experiments whenever such tests were possible. Much remains to be understood about the hadronic world, and the real tests of QCD are still in the future. They await calculations on the strong-interaction aspects of QCD which are outside the realm of ordinary perturbation theory.

Several methods have been proposed to investigate the strong forces in QCD. These include the $1/N$ expansion,[1] the strong-coupling expansion,[2,3] and semiclassical methods via instantons.[4,5] The primary purpose of our paper is to introduce a new approximation which is a special case of the $1/N$ expansion and is valid when g^2N is fixed to a large value. The virtue of our technique is that the Feynman path integral on the lattice can be completely evaluated. Our method is applicable to a variety of problems.

One of the aims of all these attempts, including ours, has been to derive the string picture of hadrons which is successful in describing many qualitative features of strong interactions, including confinement. The results presented in this paper go the furthest so far in establishing a string picture in QCD in four space-time dimensions. Other recent attempts also suggest a string theory.[6] Earlier work establishing certain exact relations between QCD and a string theory in two dimensions was presented in Refs. 7 and 9.

It was pointed out some time ago by 't Hooft[1] that in the large-N expansion QCD exhibits a diagrammatic structure with topologies reminiscent of the string theory of hadrons. This suggestion by itself was not sufficient to establish any definite relation with the *dynamics* of the string theory. It must be understood that 't Hooft's $1/N$ expansion yields the topology of a surface in U(N) *index space*, as opposed to actual spacetime surfaces. Indeed, according to perturbation theory, a gluon that leaves an interaction vertex in a Feynman diagram travels through space time via all possible paths until it reaches another interaction vertex. The space-time structure of such a Feynman diagram is not in any obvious way related to a two-dimensional space-time surface, which is essential in the string model.

Wilson[2] in his strong-coupling approach to QCD on the lattice has shown how confinement can occur, by establishing the area law for the Wilson loop. While this, again, suggests a relationship to the string theory, it does not imply it by itself, since only the *smallest area* bounded by the Wilson loop is shown to arise. A string theory would require *all* possible simply connected areas bounded by the loop. A strong-coupling expansion involves at the same order of $1/g^2$ many other topological structures in addition to areas with trivial topology, and it does not appear to be a string expansion.

In two dimensions a definite relationship[7] exists between the $1/N$ expansion of QCD and a specific interacting string theory[8,9] with quarks at the ends. These two theories have been shown to be equivalent[9] in detail up to and including interactions of order $1/N$ for all values of the coupling g^2N. In the zeroth order of the $1/N$ expansion a free meson consists of a quark and an antiquark interacting via a string. Interactions between mesons are of order $1/N$. These interactions are reproduced in detail in the string theory only by a very special and natural treatment of the end points of the string. The interactions of order $1/N$ necessarily correspond to string-string *end-point* interactions and depend crucially on the details of the end-point actions. The resulting interaction[9] differs from the standard dual-model interaction[10] and does not exhibit the same $d=26$ problem. Indeed, this interacting string model is completely consistent for $d=2$. It also includes longitudinal modes[7] which are absent in the standard treatment of the string model.

These developments taken together provide strong clues that QCD in higher dimensions might behave like a string theory in an appropriate large-N strong-coupling limit. While this premise has been speculated on,[11] little tangible progress has been reported.[12] It is important to establish a QCD-string connection as firmly as possible since the ability of QCD to explain high-energy phenomena such as linearly rising trajectories, Regge behavior, duality, etc., in addition to confinement could be demonstrated by an equivalent string theory. Furthermore, in the process of establishing such a result new calculational techniques emerge that can be used to extract further information from QCD.

Our paper contains two essential parts: the first builds a case for a simplified quark action functional, the second presents the $1/N$ approximation and applies it to the calculation of the meson propagator. These two parts establish independent results and could be studied separately, but are being presented here together for completeness. Taken together they lead to a more complete string picture as derived from QCD in our approximation.

Many take the point of view that the quark part of the action in QCD is inessential in understanding the strong forces and thus they concentrate mainly on the Yang-Mills part of the total action. In a complicated problem such as QCD this is indeed a welcome simplification even without a complete justification. Section II, which forms the first part of our paper, is aimed at developing a simplified quark action which despite its simplicity remains very close to that of QCD. Instead of using Dirac fields to describe quarks, we give a description in terms of point particles that carry spin, color, flavor, and mass and travel on world lines. Our purpose is to present a *complete action* including sources (quarks) that interact via the Yang-Mills field. We supply motivation based on arguments developed from two dimensions for proposing this quark action. We argue that results derived from our action will closely resemble those of QCD if compared for $N\to\infty$ on the light-cone frame. Using this formalism we derive the general form of the meson propagator that includes the Wilson loop of an arbitrary shape describing the interaction of quarks in *motion*.

In Sec. III, which forms the second part of our paper, we treat several topics which are geared toward the evaluation of the Wilson loop. We rewrite the continuum Yang-Mills action in terms of only line integrals.[3,13] This provides a gauge-invariant formalism in which one can go on the lattice naively by simply replacing each derivative by a difference.[14] We then present our approximation which consists of doing the integral over one link on the lattice in the $N\to\infty$ limit with g^2N fixed to a large value. The structure of our result shows that further integrations over the remaining links can be done by using the same form as the original integral. With this observation we can then evaluate all the integrals over the entire lattice in any dimension. For clarity of presentation we first treat two-dimensional QCD, then a three-dimensional world consisting of a single lattice cube, and finally the infinite lattice. We show that all contributions come in the form of areas reminiscent of a moving string. We also explicitly show that space-time structures which are not simply connected areas, such as handles, are suppressed in our limit. Our final result is identical to the expression for the *functional integral of the string theory*, in latticized form. This remarkable result is discussed and summarized in Sec. IV.

II. A SIMPLIFIED ACTION FOR QUARKS AS SOURCES FOR THE YANG-MILLS FIELD

This section is independent from Sec. III which describes our approximations. The reader interested in the next topic can directly skip to Sec. III.

To simplify the investigation of QCD it is helpful to treat the quarks as structureless point particles traveling on world lines. Our purpose in this section is to provide a complete Lorentz and gauge-invariant model of spinning, massive quarks with color and flavor, interacting with gluons via the Yang-Mills action. We build a case for our proposed action by basing our arguments on results from two-dimensional QCD. We argue that calculations with our action should, for most purposes, resemble those of the full four-dimensional QCD action in the $N\to\infty$ limit, only in the infinite momentum frame.

Two-dimensional quantum chromodynamics (QCD_2) fortunately provides us with a concrete example to follow in finding a connection between QCD and strings. QCD_2 has been investigated by a number of authors, in particular, on the light cone for large N.[15] In this gauge and in this $N\to\infty$ limit the meson sector of QCD_2 has been demonstrated to be equivalent to a string theory on the light cone with massive and spinning quarks at the end points.[7,8,9] The crucial effects of the spin degrees of freedom become apparent only in interactions. More precisely, the light-cone Hamiltonian of such a "free" string reduces to

$$p^- = m_1{}^2/2p_1^+ + m_2{}^2/2p_2^+ + \gamma|x_1^- - x_2^-| , \tag{2.1}$$

where x_1^- and x_2^- are the positions of the quark end points, $p_{1,2}^+$ is the momenta, and $m_{1,2}{}^2$ are the renormalized *effective* masses. The Schrödinger equation for the string wave function $\phi(p_1, p_2)$ using the above Hamiltonian is identical to the Bethe-Salpeter equation, derived from QCD_2 in the $N\to\infty$ limit, for the wave function $\phi(p_1, p_2)$ of a meson of mass M and total momentum $p^+ = p_1^+ + p_2^+$:

$$\left(\frac{M^2}{2p^+} - \frac{m_1{}^2}{2p_1^+} - \frac{m_2{}^2}{2p_2^+}\right)\phi(p_1, p_2) = -\frac{\gamma}{\pi} \fint \frac{dk^+}{(k^+)^2}\, \phi(p_1^+ + k^+,\; p_2^+ - k^+) . \tag{2.2}$$

This result was generalized to show the equivalence of QCD_2 with an *interacting* string theory. In QCD_2 interactions are of order $1/N$. The exact details of these interactions are in complete agreement with a string theory with massive and spinning quarks at the ends, which also carry flavor. These results follow from the following action[9] for interacting strings (written in d dimensions):

$$S = \sum S_{\text{strings}} + \sum_I S_{\text{end points}}(x_I) + \sum_{I,J} S_{\text{int}}(x_I, x_J) , \tag{2.3}$$

where the Nambu action for the string with end points x_1^μ, x_2^μ is

$$S_{\text{string}} = \int_1^2 d\sigma\, d\tau\, (-g)^{1/2} . \tag{2.4}$$

The end points are described by the action

$$S_{\text{end point}} = \int d\tau\, \overline{\psi}\left(\frac{i\not{x}_\tau \partial_\tau}{(-x_\tau{}^2)^{1/2}} - m(-x_\tau{}^2)^{1/2}\right)\psi \tag{2.5}$$

and the end-point interactions are given by

$$S_{\text{int}} = -\frac{(2\pi)^d}{4\pi} \sum_{I\neq J} \int d\tau\, d\tau'\, \delta^{(d)}[x_I^\mu(\tau) - x_J^\mu(\tau')](-x_{I\tau}{}^2)^{-1/2}(-x_{J\tau'}{}^2)^{-1/2} \times [\overline{\psi}_I(\tau)(i\overleftarrow{\partial}_\tau \not{x}_{I\tau} + m x_{I\tau}{}^2)\psi_J(\tau') + \overline{\psi}_J(\tau')(\not{x}_{I\tau} i\partial_\tau + m x_{I\tau}{}^2)\psi_I(\tau)] , \tag{2.6}$$

where $x_{I\tau}^\mu = \partial x_I^\mu/\partial\tau$, and $\not{x}_{I\tau} = \gamma_\mu \partial x_I^\mu/\partial\tau$, and I, J label different particles. Note that the interactions only involve the end points. Therefore, the details of the interaction is completely dependent on the treatment of the end points. The end point action in Eq. (2.5) is closely related to the Dirac equation as discussed in Refs. 8 and 9. It is obtained from the Dirac Lagrangian by replacing the derivative ∂^μ by the directional derivative $x_\tau^\mu \partial_\tau / x_\tau^2$ on the world lines. It is important to note that the interactions in QCD_2 as well as its equivalent string description of Eq. (2.6) deviate significantly from the standard dual-model interactions as treated by Mandelstam.[10]

It is much more difficult, even for QCD_2, to establish its connection to strings in other gauges than the light cone. Quantization in the axial gauge $A_1 = 0$ leads to a much more complicated Bethe-Salpeter equation. The complications arise from the possibility of $q\bar{q}$ pair creation and annihilation even for $N\to\infty$. It is then necessary to use two meson wave functions $\phi_\pm$, one for forward propagation (ϕ_+) and one for backward propagation (ϕ_-). Then the QCD_2 Bethe-Salpeter equations take the form[16]

$$[E(p) + E(r-p) - r^0]\,\phi_+(r,p) = \frac{\gamma}{\pi}\fint \frac{dk}{(p-k)^2}\,[C(p,k,r)\phi_+(r,k) + S(p,k,r)\phi_-(r,k)] ,$$
$$[E(p) + E(r-p) + r^0]\,\phi_-(r,p) = \frac{\gamma}{\pi}\fint \frac{dk}{(p-k)^2}\,[C(p,k,r)\phi_-(r,k) + S(p,k,r)\phi_+(r,k)] , \tag{2.7}$$

which is not easily described as a *free* string theory. Here r is the total momentum and r^0 the total energy of the meson, p and $r-p$ are the momenta of the quarks and $E(p)$ and $E(r-p)$ their energies. $C(p,k,r)$ and $S(p,k,r)$ are complicated functions of momenta as given in Ref. 16. It is the wave function and not the spectrum which is different than in the former case. The $\phi_\pm(r,p)$ wave functions give a description of the meson in an arbitrary reference frame as opposed to the infinite momentum frame which is automatically chosen by the light-cone gauge. In fact, by taking

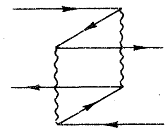


FIG. 1. Time-ordered QCD_2 graph in the axial gauge in the $N\to\infty$ limit, displaying $q\bar{q}$ creation and annihilation at end points of interacting strings.

the $r\to\infty$ limit it was demonstrated that Eq. (2.7) reduces to the light-cone Bethe-Salpeter Eq. (2.2), thus ensuring the same spectrum for the meson. The important fact is that in the infinite momentum frame pair creation and annihilation is suppressed leading to significant simplifications. Thus, it is impossible to give the string description of QCD_2 in the axial gauge without considering string-string *interactions*. This is because in an arbitrary frame the quarks at the end points get created and destroyed an infinite number of times within a free meson[16] as shown in Fig. 1. Presumably, the interacting string action of Eqs. (2.3)–(2.6) which reproduces the light-cone results of QCD_2 also reproduces this more complicated description of the meson in an arbitrary frame, since the action is Lorentz and gauge invariant. Again we expect that the details of the end points as given are crucial for such an equivalence.

Hence we see from two dimensions that a simple string theory for free mesons can result (if at all) only if we take the following limits and prescriptions for higher-dimensional QCD:

(1) N must be taken large to ensure a planar structure with no holes.

(2) We must work on the light cone to suppress pair creation so that end points follow smooth world lines.

Only then can we separate a zeroth order (in $1/N$) light-cone Hamiltonian for *free mesons* which corresponds to a *noninteracting* string theory.

Assuming that a *free* string theory can result in these limits in higher dimensions, the form of the interaction will be determined by the $1/N$ expansion. To order $1/N$ the interaction will correspond to string-string interactions at the *end points only*. Thus, the treatment of the end points in a string theory is crucial for determining the interaction and the closely related $d=26$ problem. We have seen in two dimensions that this interaction is different than that considered by Mandelstam, and depends on the details of the fermionic degrees of freedom at the end points of the string. Since there is a limit (light cone and $N\to\infty$) in which the quark in a meson can be expected to be describable approximately by a world-line picture[8] (no pairs), we will seek a formulation which is exact in two dimensions and try to apply it to four dimensions. A different attempt which is related to ours in two dimensions but which is considerably more complicated in four dimensions is given by Halpern *et al.*[17–19]

Motivated by the above picture, we will investigate a modified version of QCD_4 in the $N\to\infty$ limit in which the quark field is replaced by point particles moving on world lines. The quark will be described by the world-line action of Eq. (2.5) except for interactions with the gluon field. The pure Yang-Mills action is unmodified. Just as in two dimensions the "quark" action describes massive spin-$\frac{1}{2}$ Dirac fermions with color and flavor and an *effective* mass. The effective mass will be identified with the total dynamical mass of the quark. The quark is restricted to move on a smooth world line such that pair creation and annihilation is not allowed. Although the model thus defined is covariant and can be studied in any gauge and any Lorentz frame, we expect it to yield results resembling ordinary QCD_4 in the $N\to\infty$ limit only in the light-cone frame, based on the discussion given above. The main difference between ordinary QCD_4 and our present model is in the quark propagator: The momentum dependent full quark propagator of QCD_4 in the $N\to\infty$ limit is replaced in our model by the propagator of a spin-$\frac{1}{2}$ particle with an *effective* mass (but not static). In two dimensions for $N\to\infty$ this is *exact* on the light cone, because then the quark propagator has a single pole.[15] In four dimensions it could not be exact, since dynamical chiral-symmetry breaking could be understood only by considering the full propagator. Nevertheless, our "quark" approach provides a complete model with sources in addition to the Yang-Mills action, which can be studied independently in order to understand the nature of, say, the confining forces. With regard to such general considerations we expect our simplified model to yield similar results to *light-cone* QCD_4 as $N\to\infty$ since the details of quark propagation are not the essential ingredients in determining these forces. The study of QCD_4 with a full quark field is deferred to the future.

Thus, we define a four-dimensional effective theory by the action

$$S=S_{YM}+S_1+S_2\ , \tag{2.8}$$

where

$$S_{YM}=\int d^4x\,\mathrm{Tr}\left(-\frac{1}{2g^2}\ F_{\mu\nu}F^{\mu\nu}\right)\ , \tag{2.9}$$

and, motivated by the above discussion, we take for a quark on a world line the action

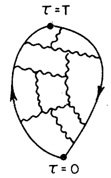


FIG. 2. Quark loop.

$$S_1 = \int d\tau\, \bar\psi_1(\tau)\left(\frac{i\not{x}_{1\tau}}{(-x_{1\tau}{}^2)^{1/2}} D_{1\tau} - m(-x_{1\tau}{}^2)^{1/2}\right)\psi_1(\tau)\,, \tag{2.10}$$

and similarly for S_2 with x_2^μ and ψ_2 describing the second quark. The covariant derivative is

$$iD_{1\tau} = i\partial_\tau + A^\mu(x_1)x_{1\tau\mu}\,. \tag{2.11}$$

Above we used

$$x_\tau^\mu = \partial x^\mu/\partial\tau\,, \quad \not{x}_\tau = \gamma_\mu \partial x^\mu/\partial\tau\,.$$

This describes a theory of two spinning, colored, flavored, massive particles $\psi_{1\alpha}^{ai}$ and $\psi_{2\alpha}^{ai}$ (α = spin, a = color, i = flavor) which live on the world lines $x_{1\mu}(\tau)$ and $x_{2\mu}(\tau)$, respectively. A relation to the Dirac action is evident.[8] The quark wave functions $\psi_{1,2}(\tau)$ interact via the gluon field $A_\mu[x(\tau)]$. Note that the world-line variables $x_1^\mu(\tau)$, $x_2^\mu(\tau)$ as well as the $\psi_1^{ai}(\tau)$, $\psi_2^{ai}(\tau)$ are dynamical degrees of freedom to be quantized along with the gauge field $A_\mu(x)$.

Our goal is this paper is to describe the propagator of a meson made of a quark-antiquark pair. Then the two world lines $x_1(\tau)$, $x_2(\tau)$ corresponding to a pair of quarks which are created at $\tau=0$ and annihilated at $\tau=T$ will form a closed loop, one branch of which represents the quark, the other the antiquark (see Fig. 2).

Thus, we consider the Green's function

$$\left\langle 0 \left| \frac{1}{\sqrt N}\,\bar\chi_1(T)\Gamma'\psi_2(T)\frac{1}{\sqrt N}\,\bar\chi_2(0)\Gamma\psi_1(0)\right| 0\right\rangle, \tag{2.12}$$

where $\chi_{1,2}(\tau)$ are the canonical conjugates to $\psi_{1,2}(\tau)$ determined by the action

$$\bar\chi(\tau) = i\bar\psi\left(\frac{\not{x}_\tau}{(-x_\tau{}^2)^{1/2}}\right)\,. \tag{2.13}$$

We note that $(1/\sqrt N)\bar\chi_2\Gamma\psi_1$ creates or annihilates a normalized color-singlet state of quark-antiquark whose spin and flavor content is described by the color-singlet matrix Γ. Following a modification[20] of the methods of Halpern *et al.*[18] the above matrix element can be put into the form of a functional integral:

$$Z^{-1}\int [dA][dx_1][dx_2][d\psi_1 d\bar\chi_1][d\psi_2 d\bar\chi_2]e^{-(S_{\rm YM}+S_1+S_2)}$$
$$\times \exp\{\tfrac12 i[\bar\chi_1(0)\psi_1(0)+\bar\chi_2(0)\psi_2(0)+\bar\chi_1(T)\psi_1(T)+\bar\chi_2(T)\psi_2(T)]\}\,\frac{1}{N}\,{\rm Tr}[\Gamma\psi_1(0)\bar\chi_1(T)\Gamma'\psi_2(T)\bar\chi_2(0)]\,. \tag{2.14}$$

The factor Z^{-1} is the usual normalization. The integrand consists of the usual action term, the initial and final wave functions as they occur in the above matrix element written in trace form over color, spin, and flavor, and finally the term $\exp[\frac12 i(\bar\chi\psi + \text{etc.})]$ which is explained by Halpern *et al.*[20] This final factor ensures that the quark number $-i\bar\chi\psi$ for each species is maintained at $-i\bar\chi\psi = 1$ throughout propagation.

Once again referring to the appendix of Ref. 18 and modifying it for time-dependent Hamiltonians we evaluate the Fermion integrations. We use the following result

$$\int [d\bar\chi][d\psi]\exp\left[\int_0^T d\tau[\bar\chi\partial_\tau\psi - i\bar\chi H(\tau)\psi]\right]\psi_A(0)\exp\{\tfrac12 i[\bar\chi(0)\psi(0)]\}\exp\{\tfrac12 i[\bar\chi(T)\psi(T)]\}\bar\chi_B(T)$$
$$= \left\{T\exp\left[-i\int_0^T H(\tau)d\tau\right]\right\}_{AB}\,, \tag{2.15}$$

where T is the time-ordered product and $H(\tau)$ is a time-dependent matrix in color-flavor-spin space. By writing the quark actions S_1 and S_2 in terms of canonical conjugates for the fermions

$$S = \int d\tau[\bar\chi\partial_\tau\psi - i\bar\chi(m\not{x}_\tau + x_\tau\cdot A)\psi] \tag{2.16}$$

and using the above integral we obtain for our functional integral the form

$$Z^{-1}\int [dA][dx_1][dx_2]\exp[-S_{\rm YM}(A)]$$
$$\times\frac{1}{N}\,{\rm Tr}\left\{\Gamma\left[P\exp\left(i\int_0^T[m\gamma_\mu+A_\mu(x_1)]\,dx_1^\mu\right)\right]\Gamma'\left[P\exp\left(i\int_0^T[m\gamma_\mu+A_\mu(x_2)]\,dx_2^\mu\right)\right]\right\}, \quad (2.17)$$

where we have used

$$\int d\tau(m\dot{x}\!\!\!/_\tau+x_\tau\cdot A)=\int (m\gamma_\mu+A_\mu)dx^\mu$$

and ordered the product over the paths instead of time. Since the Dirac matrix γ_μ is in a different space than the color matrix A_μ we can separate the exponentials and traces into the form

$${\rm Tr}\left\{\Gamma\left[P\exp\left(i\int_T^0 m\gamma\cdot dx_1\right)\right]\Gamma'\left[P\exp\left(i\int_0^T m\gamma\cdot dx_2\right)\right]\right\}\frac{1}{N}\,{\rm Tr}\left[P\exp\left(i\int A\cdot dx\right)\right], \quad (2.18)$$

where the last closed color loop is just the Wilson loop which will be indicated from now on by W:

$$W=P\exp\left(i\int A\cdot dx\right)=\left[P\exp\left(i\int_T^0 dx_1\cdot A(x_1)\right)\right]\left[P\exp\left(i\int_0^T dx_2\cdot A(x_2)\right)\right]. \quad (2.19)$$

Therefore, the integral over the gauge field now simply reduces to the evaluation of a Wilson loop of arbitrary shape and is equal just to the expectation value in the pure gluon sector:

$$\left\langle\frac{1}{N}\,{\rm Tr}\,W\right\rangle=\int (dA)\exp[-S_{\rm YM}(A)]\frac{1}{N}\,{\rm Tr}\,W\bigg/\int (dA)\exp[-S_{\rm YM}(A)]\,. \quad (2.20)$$

This quantity depends on the boundaries parametrized by the functions $x_1^\mu(\tau)$ and $x_2^\mu(\tau)$ over which further path integrals should be performed as follows:

$$\int (dx_1)(dx_2)\,{\rm Tr}\left\{\Gamma\left[P\exp\left(i\int_T^0 m\gamma\cdot dx_1\right)\right]\Gamma'\left[\exp\left(i\int_0^T m\gamma\cdot dx_2\right)\right]\right\}\exp[-S_{\rm pot}(x_1,x_2)], \quad (2.21)$$

where following Wilson[2] we have defined an effective action $S_{\rm pot}$ representing the potential between quarks in a meson by

$$\exp[-S_{\rm pot}(x_1,x_2)]\equiv\left\langle\frac{1}{N}\,{\rm Tr}\,W\right\rangle. \quad (2.22)$$

The trace in this path integral can be rewritten by introducing fermion variables without color and retracing the steps above. We will denote these fermion variables again with the same symbol ψ_α^i by omitting the color index a. Thus, we obtain a form similar to Eq. (2.14) where there is no gluon integration:

$$\int [d\psi_1 d\bar\chi_1][d\psi_2 d\bar\chi_2][dx_1][dx_2]\exp[-S_{\rm pot}(x_1,x_2)+S_1+S_2)]$$
$$\times\exp\{\tfrac{1}{2}i[\bar\chi_1(0)\psi_1(0)+\bar\chi_2(0)\psi_2(0)+\bar\chi_1(T)\psi_1(T)+\bar\chi_2(T)\psi_2(T)]\}\;{\rm Tr}[\Gamma\psi_1(0)\bar\chi_1(T)\Gamma'\psi_2(T)\bar\chi_2(0)]\,. \quad (2.23)$$

Here S_1 and S_2 are exactly of the same form as in Eq. (2.10) except for the covariant derivative D_τ replaced by the ordinary derivative ∂_τ. This shows that the meson is described by the effective action

$$S_{\rm eff}=S_1+S_2+S_{\rm pot}\,. \quad (2.24)$$

In the subsequent sections we will exhibit an approximation of QCD in four dimensions in which $S_{\rm pot}$ can be rewritten in terms of the path integral over the Nambu string action of Eq. (2.4), written in four dimensions:

$$\exp[-S_{\rm pot}(x_1,x_2)]=\int dx_{\rm string}\exp[-S_{\rm string}]\,. \quad (2.25)$$

This resulting model differs from the standard string model by the additional action at the end points. As discussed above end points play an important role in determining the string-string interaction that corresponds to QCD in the large-N expansion. This model was previously suggested by Bars and Hanson on an intuitive basis[8] based on QCD, and was shown to be equivalent[9] to QCD in two dimensions after including interaction. It was subsequently shown by Kikkawa *et al.*[21] that the Bars-Hanson quark action is also related to Wilson's action for fermions on the lattice. The model was further analyzed by Kikkawa *et al.* in four dimensions in a semiclassical WKB approach[22] and shown to reproduce certain interesting features of the known particle spectrum. We consider these preliminary results encouraging for our approach. In our opinion a fully quantum-mechanical treatment of the model on the light cone (no pairs) is necessary for comparison with QCD (for $N\to\infty$, on the light cone).

The main result of this section is Eq. (2.24) and the detailed quark actions. They describe the motions of the quarks interacting via the potential calculated through the Wilson loop. We have built a case for the validity of our quark action, in comparison to the full field theory description of quarks, in the limits indicated. Except for effects of chiral-symmetry breaking we expect our approach to work well.

III. WILSON LOOP

A. Passage to the lattice

In Sec. II we derived a Feynman path integral for the meson propagator involving the Wilson loop taken over an arbitrary closed curve parametrized by $x^\mu(\tau)$. We will evaluate this integral by latticizing space and time. We digress for a moment to discuss a method for going on the lattice which appeals to us. We first make a change of variables in the *continuum* by expressing the gauge potentials $A_\mu(x)$ in terms of path ordered line integrals. The details of this substitution has been discussed extensively in Ref. 13. Briefly, we denote by $B_\mu(x)$ the line integrals over straight lines as shown in Fig. 3(a). They are unitary matrices which satisfy

$$\left.\begin{aligned}
&B_\mu(x)=P\exp\left[-i\int_\infty^x dx'_\mu A_\mu(x')\right],\\
&A_\mu=B_\mu^\dagger i\partial_\mu B_\mu\,,\\
&F_{\mu\nu}=B_\mu^\dagger\partial_\mu(B_{\mu\nu}\partial_\nu B_{\nu\mu})B_\mu\,,\\
&\mathrm{Tr}(F_{\mu\nu})^2=\mathrm{Tr}[\partial_\mu(B_{\mu\nu}\partial_\nu B_{\nu\mu})]^2,\\
&B_{\mu\nu}=B_\mu B_\nu^\dagger
\end{aligned}\right\}\ \text{no sum on } \mu \text{ or } \nu\,. \tag{3.1}$$

Thus, the action is a function only of the gauge-invariant variables $B_{\mu\nu}$. They correspond to line integrals along paths as shown in Fig. 3(b). For gauge transformations which vanish at ∞ these variables remain invariant. The $B_{\mu\nu}$ can be treated as ordinary local variables and the quantum theory can be carried out entirely in terms of these gauge-invariant variables.[3,14] The passage to latticized space time is now straightforward since we are dealing with gauge-invariant variables. All derivatives are simply replaced by differences:

$$\partial_\mu B_{\mu\nu}(x)=\frac{1}{a}\left[B_{\mu\nu}(x+\hat{\mu}a)-B_{\mu\nu}(x)\right]. \tag{3.2}$$

As shown in Ref. 14, the new local lattice theory thus defined preserves the quantum commutator structure of the continuum theory. In this version of the lattice theory the $B_{\mu\nu}$ are corner variables as opposed to link variables. But a map[14] has been established to the formulation[2] of Wilson and Kogut and Susskind showing the equivalence of the two versions of lattice theory.

We find this approach appealing in that it provides a systematic method for constructing a lattice theory from the continuum theory. For example, using the $B_{\mu\nu}$ form of $F_{\mu\nu}$ just by drawing pictures, we can easily derive that (no sum on μ)

$$F_{\mu\nu}B_\mu^\dagger(x)=U(x,C_1)-U(x,C_2)\,, \tag{3.3a}$$

$$B_\mu F_{\mu\nu}(x)=U^\dagger(x,C_2)-U^\dagger(x,C_1)\,, \tag{3.3b}$$

where the ordered path integrals $U(x,C_i)$ are taken over the curves C_i shown in Fig. 4. (They are just products of $B_{\mu\nu}$.) Taking the product of Eqs. (3.3a) and (3.3b) we find (no sum on μ, ν)

$$[F_{\mu\nu}F_{\mu\nu}(x)]_a^b=[-2+U(x,C_{12})+U^\dagger(x,C_{12})]_a^b\,, \tag{3.4}$$

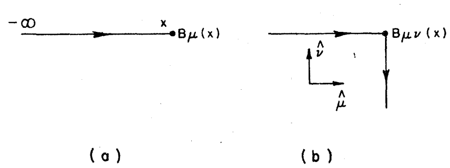


FIG. 3. Paths used for the line integrals in Eq. (3.1) for B_μ in (a) and for $B_{\mu\nu}$ in (b).

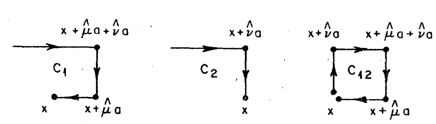


FIG. 4. Line integrals used to construct $F_{\mu\nu}$ in Eqs. (3.3) and (3.4).

where the closed curve C_{12} is given in Fig. 4. The unitary matrix $U(x, C_{12})$ may now be written as a product of either the corner variables $B_{\mu\nu}$ located at the corners of the "plaquette" in Fig. 4 or the link variables which represent line integrals over the sides of the "plaquette."[14] It is seen that in our approach the action $\mathrm{Tr}\,F_{\mu\nu}F_{\mu\nu}$ as well as other quantities such as $\mathrm{Tr}\,\boldsymbol{F}_{\mu\nu}F_{\mu\nu}$ can be constructed on the lattice in a straightforward and systematic fashion. In this paper we will use the Wilson version of the lattice theory since it is more familiar generally but the analysis could have been carried out entirely in terms of the corner variables $B_{\mu\nu}$. The Wilson loop now takes the well-known form[2] (using $g^2(a)$ instead of g^2a^{4-d})

$$\left\langle \frac{\mathrm{Tr}W}{N}\right\rangle = \frac{\int [dU]\exp[(1/g^2)\sum \mathrm{Tr}(UUUU+\mathrm{H.c.})]\frac{1}{N}\mathrm{Tr}(W)}{\int [dU]\exp[(1/g^2)\sum \mathrm{Tr}(UUUU+\mathrm{H.c.})]}\,. \tag{3.5}$$

The sum in the exponent is taken over all plaquettes in all space-time. The (dU)'s correspond to the invariant group measures for the link variables U, and W is a product of link variables over the latticized quark trajectories.

B. The approximation

Concentrating on the integration over a single link U we find we must evaluate an integral of the form

$$I(A) = \int dU \exp[(1/g^2)\,\mathrm{Tr}(AU + U^\dagger A^\dagger)]\,, \tag{3.6}$$

where A is the sum of unitary matrices in all plaquettes that share U as a link. For example, in two dimensions, as shown in Fig. 5, only two plaquettes share U_7, and they can be rewritten as

$$\mathrm{Tr}(U_1U_2U_3U_7 + U_7^\dagger U_3^\dagger U_2^\dagger U_1^\dagger) + \mathrm{Tr}(U_7^\dagger U_4U_5U_6 + U_7U_6^\dagger U_5^\dagger U_4^\dagger) = \mathrm{Tr}(AU_7 + U_7^\dagger A^\dagger)\,, \tag{3.7}$$

where

$$A = U_1U_2U_3 + U_6^\dagger U_5^\dagger U_4^\dagger\,. \tag{3.8}$$

In d space-time dimensions A has $2(d-1)$ terms in the sum

$$A = \sum_{i=1}^{2(d-1)} A_i\,,\quad A_i^\dagger A_i = 1 \quad \text{(no sum)}\,. \tag{3.9}$$

Later we will need also $AA^\dagger$ which is given by

$$AA^\dagger = \sum_{i,j} A_iA_j^\dagger = 2(d-1) + \sum_{i<j}(A_iA_j^\dagger + A_jA_i^\dagger)\,. \tag{3.10}$$

We wish to find an approximation for $I(A)$.

We proceed to compute $I(A)$ by expanding the exponent and evaluating the integral for each term in the series by using invariance arguments. We have attempted to find more direct ways of computing $I(A)$ by parametrizing U and writing out the measure explicitly, but these did not prove to be of any practical use (see the Appendix). Thus, we write

$$I(A) = \sum_{m,n}(1/g^2)^{m+n}(1/m!n!) \times \int dU(\mathrm{Tr}AU)^m(\mathrm{Tr}U^\dagger A^\dagger)^n\,. \tag{3.11}$$

Let us consider the general properties of these integrals for both U(N) and SU(N). It is clear from the invariance properties of the measure that $I(A)$ can be a function only of $\mathrm{Tr}(AA^\dagger)^l$ for any integer l, $\det A$, and $\det A^\dagger$. For U(N) the determinants must occur only in the combination $\det(AA^\dagger)$ which can be rewritten as a combination of $\mathrm{Tr}(AA^\dagger)^l$. Thus, for simplicity we specialize to the U(N) case since now only $m=n$ contributes. We do not expect that U(N) instead of SU(N) will drastically change the properties of the Wilson loop. Since we are interested in the large-N limit and since $\det AA^\dagger$ does not appear in the sum until $m=n=N$, we can evaluate the $m,n<N$ terms without dealing with the determinant. In the appendix we show that the general structure of the integral for $m=n<N$ is given by

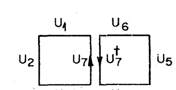


FIG. 5. Two neighboring plaquettes.

$$\left(\frac{1}{g^2}\right)^{2n}\frac{1}{(n!)^2}\int dU(\mathrm{Tr}AU)^n(\mathrm{Tr}A^\dagger U^\dagger)^n=\frac{(1/g^2)^{2n}}{N^2(N^2-1)\cdots[N^2-(n-1)^2]}\sum_{\Sigma lk_l=n}C_n^N(k_1,\ldots,k_n)\times\prod_{l=1}^{n}\frac{1}{k_l!}\left[\mathrm{Tr}\frac{(AA^\dagger)^l}{l}\right]^{k_l},\tag{3.12}$$

where the sum over the integers $k_1, k_2, \ldots, k_n$ extends over all possible terms with the restriction that $n=k_1+2k_2+\cdots+nk_n$. This simply means that the number of factors of $(AA^\dagger)$ is exactly equal to n. The coefficients $C_n^N(k_1,\ldots,k_n)$ are polynominals in N of order $k=k_1+k_2\cdots+k_n\leq n$. In the Appendix we also have calculated all the relevant coefficients C_n^N up to $n=4$ from which we have learned the following facts by extrapolation to $4<n<N$:

(i) When $AA^\dagger$ is proportional to the unit matrix $AA^\dagger=a^2\times 1$ the coefficients C_n^N conspire to give the simple exact answer

$$\left(\frac{1}{g^2}\right)^{2n}\frac{1}{(n!)^2}\int dU(\mathrm{Tr}AU)^n(\mathrm{Tr}A^\dagger U^\dagger)^n=\left(\frac{a}{g^2}\right)^{2n}\frac{1}{n!}\,.\tag{3.13}$$

(ii) When

$$AA^\dagger=\begin{bmatrix} a^2 & & & & & \\ & 0 & & & & \\ & & 0 & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & 0\end{bmatrix}$$

similarly we obtain

$$\left(\frac{1}{g^2}\right)^{2n}\frac{1}{(n!)^2}\int dU(\mathrm{Tr}AU)^n(\mathrm{Tr}A^\dagger U^\dagger)^n=\left(\frac{a}{g^2}\right)^{2n}\frac{(N-1)!}{(N+n-1)!\,n!}\,.\tag{3.14}$$

These can be checked explicitly up to $n=4$ by using the coefficients given in the Appendix. Note that the poles in Eq. (3.12) have canceled.[23]

Since the expression in Eq. (3.12) is difficult to deal with we will try to obtain an approximation for the integral $I(A)$ for a general A in the $N\to\infty$ limit. We first note that for fixed n the leading term is given when $k_1=n$, $k_i=0$, $i\neq 1$, because the $C_n^N(k_1\cdots k_n)$ are polynominals in N of order $k_1+k_2+\cdots+k_n$ with the k_i subject to the constraint $k_1+2k_2+\cdots+nk_n=n$. Furthermore, this leading coefficient has the form

$$C_n^N(n,000\cdots)=N^n+O(N^{n-2})\,,\tag{3.15}$$

as shown in the Appendix explicitly up to $n=4$. Hence we approximate the series in the $N\to\infty$ limit by keeping only the leading term in $C_n^N(n,00\cdots)$. In addition, we substitute for the denominator the leading form N^{2n} to obtain

$$\left(\frac{1}{g^2}\right)^{2n}\frac{1}{(n!)^2}\int dU(\mathrm{Tr}AU)^n(\mathrm{Tr}A^\dagger U^\dagger)^n \underset{N\to\infty}{\approx}\left(\frac{1}{g^2}\right)^{2n}\frac{1}{N^n}\left[\frac{(\mathrm{Tr}AA^\dagger)^n}{n!}+O\left(\frac{1}{N}\right)\right]\,.\tag{3.16}$$

Substituting this form into Eq. (3.11) and summing the series we arrive at our approximation

$$I(A)\approx\exp\left(\frac{1}{g^4}\,\frac{\mathrm{Tr}AA^\dagger}{N}\right)\,.\tag{3.17}$$

In our final evaluation of the Wilson loop we shall be interested in keeping g^2N fixed as $N\to\infty$, and we will use the notation $\beta=1/g^2N$. Substituting Eq. (3.9) in Eq. (3.17) and using Eq. (3.10) we arrive at the strikingly simple form

$$I(A)=\int dU\exp[\beta N\,\mathrm{Tr}(AU+A^\dagger U^\dagger)]\cong\exp[\beta^2N^2 2(d-1)]\times\exp\left[\beta^2N\,\mathrm{Tr}\sum_{i<j}(A_iA_j^\dagger+A_jA_i^\dagger)\right].\tag{3.18}$$

Note now that $A_iA_j^\dagger$ is the unitary matrix associated with the closed loop formed by combining two neighboring plaquettes. Hence *further integrations on the lattice will be of the same form as I(A) and the same approximation can be used repeatedly to completely integrate the lattice theory.* This will be described in detail in the coming sections.

We emphasize now a number of questions that arise with regard to our procedure of taking the $N\to\infty$ limit. In the standard $1/N$ expansion, in principle, the answer should have been obtained by first computing $I(A)$ exactly, fixing β, and *then* taking the $N\to\infty$ limit. Instead, we interchanged the sum and limit and kept only the leading term for each term in the series, regardless of the magnitude of g^2. Thus, our approximation could differ from the standard large-N limit. However, we have been able to check our approximation against the standard $N\to\infty$ approach for the special forms of $AA^\dagger$ given in (i) and (ii) above. For these cases, no approximation was used in arriving at Eqs. (3.13) and (3.14). Hence the integral can be evaluated by summing the series and *then* taking the limit $N\to\infty$. In so doing we extrapolate the result of $n<N$ to all n. Thus, we obtain the results (I_{N-1} is the modified Bessel function)

$$\text{(i)}\quad I(A)=\exp[(a/g^2)^2]\,, \tag{3.19}$$

$$\text{(ii)}\quad I(A)=(N-1)!\,(a/g^2)^{1-N}I_{N-1}(2a/g^2)\,. \tag{3.20}$$

Note that (ii) is exact (see the Appendix). Furthermore, our extrapolation of Eq. (3.13) to all n which ignores $\det AA^\dagger=a^{2N}$ is permissible for small a since we intend to take $N\to\infty$. After replacing $1/g^2=\beta N$, fixing β, and letting $N\to\infty$, these reduce to

$$\text{(i)}\quad I(A)=\exp(\beta^2N^2a^2)\,, \tag{3.21}$$

$$\text{(ii)}\quad I(A)=\frac{1}{\sqrt{\lambda}}\exp\left[N\left(\lambda-1+\ln\frac{2}{1+\lambda}\right)\right]\,, \tag{3.22}$$

where

$$\lambda\approx(1+4a^2\beta^2)^{1/2}+O\left(\frac{1}{N}\right).$$

Although the result (i) agrees with our approximation Eq. (3.17), result (ii) agrees only as $\beta\to 0$ for which it simplifies to the expected form

$$I(A)=\exp(N a^2\beta^2)\,. \tag{3.23}$$

Thus, henceforth we understand our approximation to be valid only for small β or large g^2N. This shows some relationship to the strong-coupling limit.

The fact that β must be fixed to a small value as $N\to\infty$ leads to an approximation scheme which is a special case of the standard large-N limit which is valid for any β. Because our approximations has to be applied to each link integration separately it means that it has to be understood as a local approximation as opposed to a global expansion in powers of β over the entire lattice. This will lead to subleties that will help clarify the local nature of our approximation, as will be discussed below. The main virtue of our approach is that it allows a complete integration of the theory and reveals an extremely interesting structure with no further approximations.

Now that we have established the approximation and have shown that the integral over each link leads to further integrations of the same type, all that remains is to keep track of the integrations systematically until the entire theory has been integrated. It is clear that this can be carried out in any dimension. In order to illustrate this fact, we treat two-dimensional and three-dimensional QCD for which we can draw pictures of the lattice to help keep track of the integrations. Four-dimensional QCD will be discussed by extension of the results in lower dimensions. Although the two-dimensional theory has already been discussed in the literature,[24,25] we shall nevertheless treat it in detail since it serves to illustrate as well as to support our method in a readily tractable case.

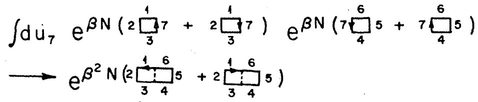
C. Two-dimensional QCD

A representative link integration in two dimensions has already been discussed in Eqs. (3.7) and (3.18) and Fig. 5. Using our approximation the result is

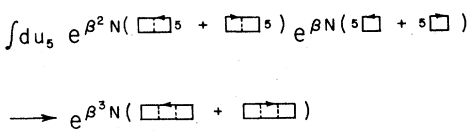
$$\int dU_7\exp[\beta N\,\mathrm{Tr}(U_1U_2U_3U_7+U_7^\dagger U_3^\dagger U_2^\dagger U_1^\dagger)]\exp[\beta N\,\mathrm{Tr}(U_7^\dagger U_4U_5U_6+U_7U_6^\dagger U_5^\dagger U_4^\dagger)]$$

$$\cong\exp[2\beta^2N^2]\exp[\beta^2N\,\mathrm{Tr}(U_1U_2U_3U_4U_5U_6+U_6^\dagger U_5^\dagger U_4^\dagger U_3^\dagger U_2^\dagger U_1^\dagger)]\,. \tag{3.24}$$

Thus, there is a factor $\exp(2\beta^2N^2)$ and a term involving larger loops over whose links integrals remain to be performed. Since factors of the type $\exp(2\beta^2N^2)$ always cancel between numerator and denominator in Eq. (3.5) we may ignore them. Therefore, we represent Eq. (3.24) graphically by


(3.25)

where the dotted line indicates the integrated link. Let us refer to β as the "weight" associated with the elementary plaquette. Equation (3.25) shows that the weight associated with the loop generated by joining two plaquettes is β^2. This has the consequence that subsequent integrations will lead to the joining of loops with different weights. For example, once again using Eqs. (3.9) and (3.18) one obtains



In general, for two arbitrary loops with n and m plaquettes having a common link, the link integration yields a larger loop whose weight is the product of the two weights, yielding $\beta^n \beta^m = \beta^{n+m}$.

A typical Wilson loop on the lattice is shown in Fig. 6. To evaluate the expression in Eq. (3.5), let us perform first the integrals over the links in the order shown on the figure. After the fourth integration we have a loop of the form shown in Fig. 7(a). Using the unitarity of the U's on the link L where the loop doubles on itself, this is equivalent to the loop in Fig. 7(b). The last dotted line is inserted since the integral over the link L is done trivially $\int dU_L = 1$ as the integrand at this stage is independent from U_L. We encounter similar situations after the integrals over 5 and 7 are performed. After all the integrations over the links inside the Wilson loop, we obtain the result

$$\exp[\beta^8 N \,\mathrm{Tr}(W + W^\dagger)], \tag{3.27}$$

where W is the product of the links comprising the Wilson loop and β^8 is the weight of W in this particular example. For the general Wilson loop containing n plaquettes the resulting weight will be β^n, after performing the integrations over the links inside it.

A similar procedure can now be followed to perform the integrals outside of the Wilson loop all the way to the boundary of the two-dimensional world. The result takes the form shown in Fig. 8. The weights of the inside and outside loops are β^n and β^m, respectively. The integrations over the links comprising these loops can be reduced to just three integrations over W, U, and V as shown in the figure, by using the invariance properties of the measures. Thus, we arrive at a numerator

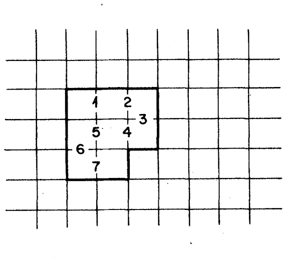


FIG. 6. Wilson loop in two dimensions, integration to be performed in the order indicated.

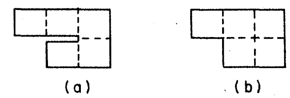


FIG. 7. Loops that emerge by combining plaquettes as a result of integration over the dotted links.

$$\int dU\, dV\, dW \exp[\beta^m N \,\mathrm{Tr}(VUV^\dagger W^\dagger + WVU^\dagger V^\dagger)] \times \exp[\beta^n N \,\mathrm{Tr}(W + W^\dagger)] \left(\frac{1}{N}\,\mathrm{Tr}\, W\right) \tag{3.28}$$

and a similar expression for the denominator where the last parenthesis is replaced by 1. We use the invariance properties of the measures once again to perform transformations on U from left and right, thus substituting $VUV^\dagger W^\dagger = U'$ everywhere. The integrations over V and U' are now decoupled from W and thus cancel among numerator and denominator. The final result is

$$\frac{1}{N}\langle \mathrm{Tr} W\rangle = \frac{\int dW \exp[\beta^n N \,\mathrm{Tr}(W + W^\dagger)]\, \frac{1}{N}\,\mathrm{Tr}\, W}{\int dW \exp[\beta^n N \,\mathrm{Tr}(W + W^\dagger)]}$$

$$= \frac{1}{2N} \frac{\partial}{\partial(\beta^n N)} \ln\left\{\int dW \exp[\beta^n N \,\mathrm{Tr}(W + W^\dagger)]\right\}. \tag{3.29}$$

This last integral is given in Eq. (3.17) except for β replaced by β^n, which leads to

$$\frac{1}{N}\langle \mathrm{Tr}\, W\rangle = \beta^n . \tag{3.30}$$

Since $n = A/a^2$, where A is the total area bounded by the Wilson loop and a^2 is the area of a plaquette, we can write

$$\frac{1}{N}\langle \mathrm{Tr}\, W\rangle = \beta^{A/a^2} = e^{-A/2\pi\alpha'} \tag{3.31}$$

with

$$\frac{1}{2\pi\alpha'} = -\frac{1}{a^2}\ln\beta = \frac{1}{a^2}\ln[Ng^2(a)]. \tag{3.32}$$

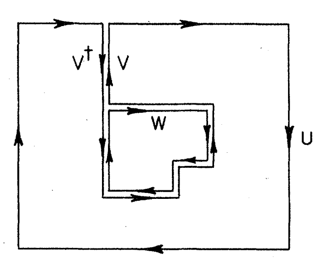


FIG. 8. Two closed loops associated with the unitary matrices W and the product $VUV^\dagger W$.

This agrees with the result arrived at previously by Migdal via other methods if his results are taken in the limit of $N\to\infty$ with fixed small β.[24] This confirms the validity of our approximation. Although in two dimensions the area law of Eq. (3.31) could have been arrived at from the lattice theory via Migdal's method[24, 25] without taking the limits $N\to\infty$ and β = small, our method, unlike Migdal's, has the advantage of generalizing readily to higher dimensions.

D. Three-dimensional cube

To illustrate the new features that appear in higher dimensions we begin with a simplified world of a cube in three dimensions, as shown in Fig. 9(a). We take the Wilson loop at the bottom of the cube and do the integrations in the order indicated in the figure. The integrations 1, 2, 3 combine the side plaquettes into the larger loop shown in Fig. 9(b). Using the invariance properties of the measures, the leftover integrations take the following form for the numerator:

$$\int dU\,dV\,dW \exp[\beta N\,\mathrm{Tr}(U+U^\dagger)]$$
$$\times \exp[\beta^4 N\,\mathrm{Tr}(WVUV^\dagger + VU^\dagger V^\dagger W^\dagger)]$$
$$\times \exp[\beta N\,\mathrm{Tr}(W+W^\dagger)]\left(\frac{1}{N}\,\mathrm{Tr}\,W\right), \quad (3.33)$$

and a similar form for the denominator with the last parenthesis replaced by 1. The first and last factors in this expression represent the plaquettes for the top and bottom faces of the cube which were part of the original action. The middle factor represents the loop in Fig. 9(b) which resulted from the first three integrations. Note that the exponent with weight β^4 is a measure of the four plaquettes that were combined as a result of the three integrations. We next perform the U integral which combines the first two factors in Eq. (3.33) to yield $\exp[\beta^5 N\,\mathrm{Tr}(W+W^\dagger)]$. The V integral is now performed trivially: $\int dV=1$. The weight β^5 is associated with the surface formed by the five plaquettes on the top and the sides. Thus, the Wilson loop takes the form

$$\frac{1}{N}\langle \mathrm{Tr}\,W\rangle = \frac{\int dW \exp[(\beta+\beta^5)N\,\mathrm{Tr}(W+W^\dagger)]\,\frac{1}{N}\,\mathrm{Tr}\,W}{\int dW \exp[(\beta+\beta^5)N\,\mathrm{Tr}(W+W^\dagger)]}$$
$$=\beta+\beta^5, \quad (3.34)$$

where we have used the same method as in Eqs. (3.29) and (3.30). The two weights β and β^5 correspond to the two surfaces bounded by the Wilson loop in this simplified world. Thus, we can write

$$\frac{1}{N}\,\mathrm{Tr}\langle W\rangle = \sum_i \exp(-A_i/2\pi\alpha'),$$
$$\frac{1}{2\pi\alpha'} = \frac{1}{a^2}\ln[Ng^2(a)] \quad (3.35)$$

where the sum $\sum_i$ is over all possible areas (here only two). It is easy to show that Eq. (3.35) holds for any other Wilson loop, such as the ones shown in Fig. 10. We note that *each area contributes* only once, i.e., the coefficients in the sum multiplying the exponential $\exp(-A_i/2\pi\alpha')$ are independent of the area and equal to 1.

E. Infinite lattice

The examples of two dimensions and the single cube in three dimensions have taught us that the best procedure for keeping track of integrations over the entire lattice is the following: Starting in the neighborhood of the Wilson loop we integrate over each link which is *not* part of the Wilson loop W. The W integration is done only after the rest of the lattice has been integrated. After doing those integrations we will generate factors of the form $\exp[\beta^n N\,\mathrm{Tr}(W+W^\dagger)]$, where β^n is the weight corresponding to some surface bounded by W. Thus, in the end we will obtain an expression of the form

$$\frac{1}{N}\langle \mathrm{Tr}\,W\rangle = \frac{\int dW \exp[(\sum\beta^n)N\,\mathrm{Tr}(W+W^\dagger)]\,\frac{1}{N}\,\mathrm{Tr}\,W}{\int dW \exp[(\sum\beta^n)N\,\mathrm{Tr}(W+W^\dagger)]}, \quad (3.36)$$

where $\sum\beta^n$ denotes a sum over weights corresponding to distinct surfaces. All W-independent factors cancel between numerator and denomina-

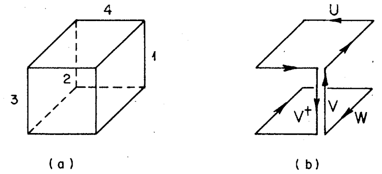


FIG. 9. (a) Cube in three dimensions. (b) Loop resulting after the integrations 1, 2, 3.

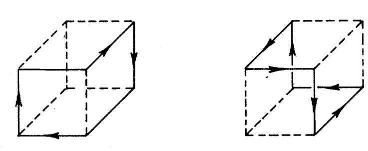


FIG. 10. Possible Wilson loops as boundaries of surfaces.

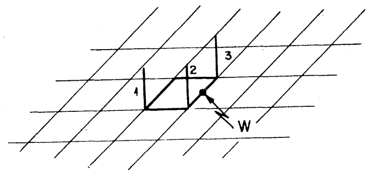


FIG. 11. Elementary Wilson loop.

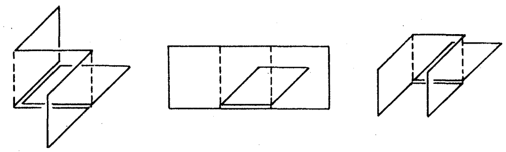


FIG. 13. Some loops of weight β^3 generated by two integrations.

tor. The last integral over W is easy to evaluate as before, and we obtain

$$\frac{1}{N}\,\mathrm{Tr}\langle W\rangle=\sum\beta^n\ . \qquad (3.37)$$

Thus, our problem is reduced essentially to finding the exact form and interpretation of the sum over weights.

To see how this works, consider a Wilson loop consisting of a single plaquette, as shown in Fig. 11. For clarity the figure displays plaquettes only in a horizontal plane, in addition to the first three vertical links (1, 2, and 3) over which we integrate. The remaining links and plaquettes are not shown in the figure, but they are there implicitly. The integrations can be done graphically just as explained for the previous examples as it follows from our approximate integral in Eq. (3.18). Thus, we combine all possible pairs of loops meeting at the integrated link into larger loops. For example, in Fig. 12 the new larger loops generated after the first integration are shown. After the first integration, the mathematical expression for these diagrams is the factor

$$\exp\{\beta^2 N\,\mathrm{Tr}[(U_a+U_a^\dagger)+(U_b+U_b^\dagger)+(U_c+U_c^\dagger)+(U_d+U_d^\dagger)+(U_e+U_e^\dagger)+(U_f+U_f^\dagger)]\} \qquad (3.38)$$

as generated by Eq. (3.18). Here U_a, etc., stand for the closed line integrals corresponding to the loops obtained by erasing the dotted link in the corresponding figure. The unitary matrices U_a, etc., can be written as a product of the links forming the loop. Subsequent integrations in Fig. 11 generate more complicated loops, *some* of which are shown in Fig. 13 after two integrations, and in Fig. 14 after three. There are, of course, more loops at the level of two and three integrations which are not shown in the figure. They are obtained in the same way by combining plaquettes with the new loops as well as with other plaquettes after each integration. The next integration over the link labeled by l in Fig. 14 generates a factor involving just the Wilson loop with weight β^5 by combining the loop of Fig. 14 with the plaquette on top of the cube, just as in the example of the single cube. This same integration will generate more complicated loops by combining the loop of Fig. 14 with other adjoining plaquettes.

The integrations continue in this graphical way. From Eq. (3.18) it is seen that with each resulting loop l we must associate the *multiplicative* factor $\exp[\beta^n N\,\mathrm{Tr}(U+U^\dagger)]$, where U is the product of links comprising the loop and β^n is its weight. The generated loops always correspond to the boundary of the *surface* formed by the combined plaquettes. The number of such plaquettes (n) determines the weight β^n. It will happen that a given loop is the boundary of many surfaces. Then the weight of the loop reduces to the sum of the weights corresponding to all the surfaces for which it serves as boundary. By proceeding in this manner, we will clearly obtain all possible weights associated with all possible surfaces bounded by the Wilson loop W, just as in the single cube case. *Each distinct surface occurs only once*, since there is only one way of forming each of these surfaces by attaching plaquettes.

In the previous discussion we started out from

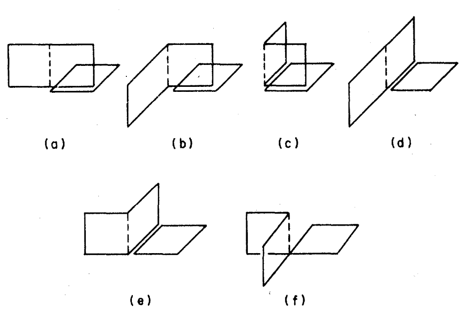


FIG. 12. Loops of weight β^2 generated by integration over the dotted link. The horizontal plaquette represents W drawn in the figures for reference.

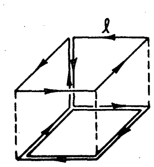


FIG. 14. A loop of weight β^4 generated by three integrations.

a Wilson loop the size of a plaquette. The same arguments can be applied to any size Wilson loop, the only difference being that a number of integrations are already needed just to form the smallest area, as in the two-dimensional case. Thus, for any Wilson loop, integrating over the entire lattice, we find the remarkable result

$$\frac{1}{N}\langle \operatorname{Tr} W\rangle = \sum_i \exp(-A_i/2\pi\alpha') \tag{3.39}$$

just as in the single-cube case, where the sum runs over all possible *distinct areas* bounded by the Wilson loop. Note that at any stage of our calculation it was only surface structures which appear and thus it should be no surprise that Eq. (3.39) is a sum over surface areas.

In Sec. IV we will show that only simply connected surfaces contribute to Eq. (3.39), due to a suppression factor of $1/N^2$ for surfaces with one handle and higher powers of $1/N$ for more complicated topologies. The simply connected surfaces fall into two general categories:

(1) Normal surfaces, formed by joining distinct plaquettes. These include both nonintersecting and self-intersecting surfaces [see Figs. 15(a) and 15 (b)].

(2) Singular surfaces, formed when overlapping plaquettes are joined [see Fig. 15(c) and Fig. 16].

In Fig. 15 we represent the integrated links by dotted lines and the boundary of the resulting surface by a heavy line which coincides with the Wilson loop. The surface of Fig. 15(a) is formed by the 12 outer faces of the cubes. The surface of Fig. 15(b) is formed by the six faces of the cube plus the two faces on the Wilson loop, combined together into a single surface self-intersecting along the indicated link. In Fig. 15(c) the dotted plaquette corresponds to a folded surface forming a pocket stitched on the outside whose open edges extend by combining with the two plaquettes bounded by the Wilson loop. This surface arises as graphically described in Fig. 16. It is easy to see that all normal surface areas appear in the

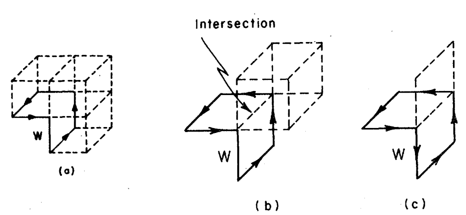


FIG. 15. Surfaces bounded by the Wilson loop W. (a) Normal nonintersecting surface of weight β^8. (b) Normal self-intersecting surface of weight β^8. (c) Singular surface of weight β^4.

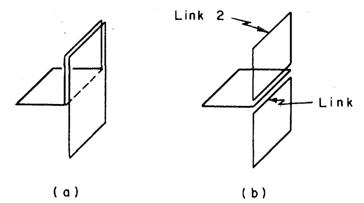


FIG. 16. (a) Loops of (β^2 each) which generate Fig. 15 (c) by integrating the common links. (b) Plaquettes involved in Figs. 15(c) and 16(a).

sum of Eq. (3.39). On the other hand, we find that a different set of singular surface areas appear when the integrations are performed in different order. Indeed, from Fig. 16(b) we see that if link 2 is integrated before link 1, the configuration of Fig. 16(a) never appears and the singular surface of Fig. 15(c) does not get generated.

The apparent ambiguity with regard to singular surfaces is resolved with a reanalysis of our approximation as described below. The net result is that for consistency within our approximation we must drop completely all singular surfaces in the sum of Eq. (3.39). This conclusion is arrived at as follows: We remember that our approximation for small β was based on a single link integral rather than on a global expansion over the whole lattice. To understand the approximation further let us first introduce a different weight β_p for each distinct plaquette in the action. At the end of any calculation we will take the limit $\beta_p \to \beta$ for all plaquettes p. Now return to our integral in Eq. (3.6) and the approximation as given in Eq. (3.18). We see that the β^2 in the exponent should now be replaced by $\beta_i\beta_j$ which is bilinear in the weights of each pair of plaquettes (i,j) that are combined. The terms that are dropped in the *exponent* in our approximation, by fixing β_p to a small value, are higher order in each β_p as in Eqs. (3.22) and (3.23). Thus, for consistency, when we calculate the weights of larger loops, any higher order terms in any single β_p that is generated in the course of the calculation must be dropped. It can easily be seen that the weights of all normal surfaces are multilinear in the distinct β_p that contribute to it, while singular surfaces have weights which are at least quadratic or higher order in some β_p. Since higher orders in each β_p were dropped to begin with, singular surfaces must also be dropped for consistency. This analysis results in an algorithm if we insist in working with a single β over the entire lattice: After *n neighboring* integrations the weight of any resulting loop is a polynominal. Terms in this polynomial of order β^{n+2} and higher should be dropped. It turns out that singular surfaces are associated with

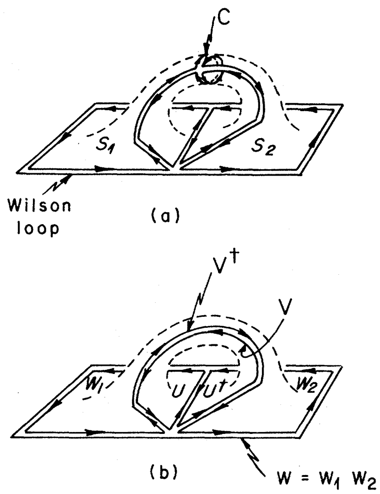


FIG. 17. Formation of a handle.

powers of β which are not allowed by the algorithm at any stage of the integration. On the contrary *all* normal nonintersecting and self-intersecting surfaces are included.

We therefore emphasize the following points:

(1) Only simply connected surfaces appear in Eq. (3.39).

(2) All normal surfaces arise.

(3) Each surface appears only once.

F. Suppression of handles and higher topologies

We must first envision how our integrations can lead to a surface with a handle. The two loops of Fig. 17(a) corresponding to surfaces S_1 and S_2 can be combined by integrating along the line C. This yields the loop of Fig. 17(b) which clearly is associated with the doubly connected surface of a handle. The easiest way in which to see the $1/N^2$ suppression of such a surface is to imagine a world in which this is the *only* surface we can form. Then, referring to the labeling of the various branches of the loop in Fig. 17(b), the Wilson loop integral takes the form

$$\frac{1}{N}\operatorname{Tr}\langle W\rangle = \frac{\int dW_1 dW_2\, dU\, dV \exp[\beta^m N(\operatorname{Tr} W_1 V^\dagger W_2 U^\dagger V U + \text{H.c.}]\frac{1}{N}\operatorname{Tr} W_1 W_2}{\int dW_1 dW_2\, dU\, dV \exp[\beta^m N \operatorname{Tr}(W_1 V^\dagger W_2 U^\dagger V U + \text{H.c.})]}, \tag{3.40}$$

where β^m is the combined weight of surfaces S_1 and S_2. This integral can be easily done by transforming W_1 and W_2:

$$\begin{aligned} W_2' &= W_2 U^\dagger , \\ W_1' &= U W_1 , \end{aligned} \tag{3.41}$$

which leaves the measure and $\operatorname{Tr} W_1 W_2$ invariant. After performing the now trivial U integration, we once again transform to

$$\begin{aligned} W_2'' &= W_2' V , \\ W_1'' &= W_1' V^\dagger . \end{aligned} \tag{3.42}$$

Then we obtain, for the numerator of Eq. (3.40),

$$\int dW_1 dW_2\, dV \exp[\beta^m N \operatorname{Tr}(W_1 W_2 + \text{H.c.})]\frac{1}{N}\operatorname{Tr} W_1 V W_2 V^\dagger = \int dW_1 dW_2 \exp[\beta^m N \operatorname{Tr}(W_1 W_2 + \text{H.c.})]\frac{1}{N}\operatorname{Tr} W_1 \frac{1}{N}\operatorname{Tr} W_2 . \tag{3.43}$$

If we transform one last time to $W_1' = W_1 W_2$ and evaluate the W_2 integral we obtain

$$\frac{1}{N^2}\int dW_1 \exp[\beta^m N(\operatorname{Tr} W_1 + \text{H.c.})]\left(\frac{1}{N}\operatorname{Tr} W_1\right) . \tag{3.44}$$

Thus, Eq. (3.40) becomes

$$\begin{aligned} \frac{1}{N}\operatorname{Tr}\langle W\rangle &= \frac{1}{2N^4}\frac{\partial}{\partial(\beta^m)}\ln\int dW_1 \exp[\beta^m N \operatorname{Tr}(W_1 + \text{H.c.})] \\ &= \frac{1}{N^2}\beta^m . \end{aligned} \tag{3.45}$$

Had this been a simply connected surface, no $1/N^2$ would have been present. One can also show with some work in the general case, with more surfaces being generated, that any loop of the type in Fig. 17(b) leads to a factor of $1/N^2$ multiplying the weight of the corresponding surface. In addition, the same argument applies to any number of handles, such that the suppression factor is $(1/N^2)^k$ for k handles. Thus, in the large-N limit these surfaces are excluded from Eq. (3.39).

G. Four dimensions

It is clear from the nature of our procedure that only surfaces will be built, regardless of the number of dimensions. The number of dimensions only affects the number of plaquettes meeting at a given link $[2(d-1)$ for d dimensions$]$, and according to (3.18) these are always combined *in pairs*, regardless of the number of dimensions. Hence the foregoing results including the discussion of the $1/N^2$ suppression also holds in four space-time dimensions.

In summary, our final result for a Wilson loop of arbitrary shape embedded in any dimension $d \geq 2$ has the form

$$\frac{1}{N}\langle \mathrm{Tr} W\rangle = \sum_i \exp\left\{-A_i\left[\frac{1}{a^2}\ \ln N g^2(a)\right]\right\}, \quad (3.46)$$

where the sum is over *all* distinct "normal" simply connected surfaces bounded by the Wilson loop. Since we were able to evaluate the integrals only for small β (i.e., large g^2N), we would not be justified in taking the continuum limit $a \to 0$. However, it is significant to note that the result is closely related to the functional integral of a *quantum* string theory. The functional integral for a string theory (ignoring end points) is of the form

$$\int (dx_{\mathrm{string}})\exp\left\{-\frac{1}{2\pi\alpha'}\int d\tau\, d\sigma[-g(x)]^{1/2}\right\}, \quad (3.47)$$

where $\int d\tau\, d\sigma[-g(x(\tau,\sigma))]^{1/2}$ is the surface bounded by W. If the surface spanned by the string is latticized in space-time then Eq. (3.47) reduces just to our result Eq. (3.46) for the Wilson loop. We note that for an agreement with the string theory each normal surface must occur *only once* in the sum Eq. (3.46). Indeed, we have shown this to be true for QCD in our limit.

Some discussion of the continuum limit will follow in Sec. IV. For now, if we naively substitute Eq. (3.47) for $(1/N)\langle \mathrm{Tr} W\rangle$ and combine this result with that of Sec. II, we obtain an effective theory for a free meson:

$$S_{\mathrm{eff}} = S_1(\mathrm{quark}) + S_2(\mathrm{quark}) + S(\mathrm{string}), \quad (3.48)$$

where S_1 and S_2 are given in Sec. II. This describes a quantum theory of a quark and an antiquark interacting via a string.[8,9] It certainly shows confinement in addition to many other string properties.

IV. DISCUSSION AND CONCLUSIONS

In the first part of this paper we have presented a complete model which is closely related to QCD. It is based on the Yang-Mills action for U(N) probed by Dirac particles carrying flavor, color, spin, and mass and moving on world lines. The model was motivated by the results established in two dimensions on QCD and strings, including interactions. Based on the discussion given, our model is expected to resemble QCD in four dimensions only if the comparison is made in the $N \to \infty$ limit in the infinite momentum frame. This is because our model does not allow pair creation-annihilation and quarks are forced to move on smooth world lines. We have shown that these features of the model are naturally related to a previously suggested string model[8,9] with definite modification at the end points. If granted that in the $1/N$ expansion the zeroth-order approximation of QCD corresponds to some theory of free strings, then the interactions to order $1/N$ could only correspond to string-string end-point interaction. As emphasized in the text, mainly because of the interactions at the end points the resulting string theory is expected to deviate from the standard dual models and their problems (e.g., $d=26$). This already occurs in the two-dimensional theory which is known to be fully consistent.

In the second part of the paper we presented an approximation for $N \to \infty$ with g^2N fixed to a large value for U(N) QCD on the lattice. The approximation involved an integration on a single link. The result was such that further integrations were of the same form, so that by repeated application of the approximation the entire Feynman path integral on the lattice could be performed. It is important to keep in mind that this was not a global expansion in $1/g^2N$, but rather an approximation applied to each link separately. The local nature of this approximation was further clarified in Sec. III E, and any independent method which attempts a comparison with our results should follow a similiar procedure.

Our method of approximation should be applicable to many problems, such as hadron-hadron scattering, hadron decays, hadronic form factors, baryons, etc. In this paper we concentrated on the meson propagator and were led to a calculation of the Wilson loop. Our result established a firm connection between strings and QCD in this approximation. This connection was established by noting that only simply connected normal areas, both intersecting and nonintersecting, occurred in the result. Handles and other topologies were shown to be suppressed by factors of at least $1/N^2$. The fact that each normal distinct area occurred only once in the sum is a crucial factor in establishing the correspondence with the areas described by Nambu's string action. This differs markedly from Wilson's strong-coupling expansion, in which the lowest-order contribution to the Wilson loop involves only the smallest area bounded by the loop, with higher-order corrections including more complicated topologies. Thus, statements that the strong-coupling expansion is related to a string theory are not entirely correct. The strong-coupling limit was shown by Wilson to lead to confinement via a linear potential, but this falls short of establishing a connection with strings. Our result here implies that QCD, by being firmly related to Nambu strings, is capable of explaining much more than just linear confinement. This includes such experimental phenomena as linearly rising trajectories, Regge behavior, and the richness of hadronic states implied by the Hagedorn temperature as well as duality.

Since our calculation was performed on a lattice, the connection with strings was established only for a latticized version of the string action. We are not really justified in taking the continuum limit since our answer is valid only for large g^2N rather than for all values of g^2N. A proper treatment for all values of g^2N is expected to reveal asymptotic freedom at short distances according to folklore. Of course, at this point we would expect a deviation from the string action. It is interesting to follow the behavior of the coupling constant as we change the size of the lattice. Since we have identified the string tension

$$1/2\pi\alpha' = \frac{1}{a^2}\ln[Ng^2(a)],$$

which is a physical quantity and therefore independent of the lattice cutoff a, we can solve for a rapidly varying coupling constant

$$Ng^2(a) = \exp[a^2/2\pi\alpha'] ,$$

which is valid *only for large values* of $Ng^2(a)$. Hence we see that the behavior of $Ng^2(a)$ is consistent with what one expects from a confining theory. This behavior of the coupling constant shows consistency with a renormalization-group analysis using our approximation.

We also point out that our method does not exhibit any phase transitions as a function of dimensions. Since our result is valid for large g^2N, this does not contradict mean field approaches.[26]

Our approximation method should be applicable among other things to the analysis of dynamical chiral-symmetry breaking. However, for this purpose it is essential that we use fully interacting Dirac fields as opposed to our world-line approach. This is because, as discussed in the text, the behavior of the quark propagator is the essential factor in determining the symmetry breaking. An analysis of this problem is underway.

ACKNOWLEDGMENTS

Parts of this work were performed while the authors visited several research centers including Yale University. We thank many colleagues for discussions and hospitality in the following institutions: Aspen Center for Physics, Harvard University, and The Institute for Advanced Study. This research was sponsored in part by the Department of Energy under Grant No. EY-76-S-02-2220.

APPENDIX

We want to evaluate the U(N) integral

$$F_n(A) = \int dU (\mathrm{Tr} AU)^n (\mathrm{Tr} A^\dagger U^\dagger)^n . \qquad \text{(A1)}$$

Similar group integrals for SU(N), not U(N), were discussed in Ref. 27. In order to evaluate the integral we consider the expression

$$J_n(A_i) = \int dU [\mathrm{Tr} A_1 U \, \mathrm{Tr} A_2 U \cdots \mathrm{Tr} A_n U] \times [\mathrm{Tr} A_1^\dagger U^\dagger \, \mathrm{Tr} A_2^\dagger U^\dagger \cdots \mathrm{Tr} A_n^\dagger U^\dagger] . \qquad \text{(A2)}$$

We note that if we take $(A_n)_p^q = \delta_p^i \delta_k^q$, $(A_n^\dagger)_r^s = \delta_r^k \delta_j^s$ and sum over k, leaving the remaining A_i unchanged, the last factors in each square bracket become

$$\mathrm{Tr} A_n U \, \mathrm{Tr} A_n^\dagger U^\dagger \rightarrow U_k^i U_j^{\dagger k} = \delta_j^i , \qquad \text{(A3)}$$

thus reducing the integral to a simpler one involving $(n-1)$ factors. That is,

$$J_n(A_1, \ldots, A_{n-1}, A_n) \rightarrow \delta_j^i J_{n-1}(A_1, \ldots, A_{n-1}) . \qquad \text{(A4)}$$

We use this fact in order to evaluate J_n as follows: We first note that because the measure is both left and right invariant, J_n must be U(N) × U(N) symmetric. Using this invariance property we write the answer for $J_n(A_i)$ in terms of all possible U(N) × U(N) invariants that are multilinear in $A_1, \ldots A_n, A_1^\dagger, \ldots A_n^\dagger$. Next symmetrize the answer in both sets $(A_1 \cdots A_n)$ and $(A_1^\dagger \cdots A_n^\dagger)$ since J_n has such a permutation symmetry. This expression has a number of unknown coefficients. In the answer if we substitute the special forms of $A_n, A_n^\dagger$ above we obtain a recursion relation through Eq. (A4) which determines the unknown parameters. Therefore, by starting from $n = 1$ we can build the result for arbitrary n. Here is how our procedure works. For $n = 1$ we write

$$\int dU\, \mathrm{Tr} A_1 U\, \mathrm{Tr} A_1^\dagger U^\dagger = C\, \mathrm{Tr} A_1 A_1^\dagger . \qquad (A5)$$

Applying Eq. (A4), the left-hand side reduces to $\delta_j^i \int dU = \delta_j^i$, while the right-hand side gives $CN\delta_j^i$. Therefore, we must have $C = 1/N$. Now we take $n=2$:

$$\int dU [\mathrm{Tr} A_1 U\, \mathrm{Tr} A_2 U][\mathrm{Tr} A_1^\dagger U^\dagger\, \mathrm{Tr} A_2^\dagger U^\dagger]$$
$$= C_1[\mathrm{Tr} A_1 A_1^\dagger\, \mathrm{Tr} A_2 A_2^\dagger + \mathrm{Tr} A_1 A_2^\dagger\, \mathrm{Tr} A_2 A_1^\dagger] + C_2[\mathrm{Tr} A_1 A_1^\dagger A_2 A_2^\dagger + \mathrm{Tr} A_1 A_2^\dagger A_2 A_1^\dagger] . \qquad (A6)$$

Using again Eq. (A4) the left-hand side reduces to Eq. (A5) for which we now know the answer $\delta_j^i(1/N)\, \mathrm{Tr} A_1 A_1^\dagger$. The right-hand side gives

$$C_1[\mathrm{Tr} A_1 A_1^\dagger N \delta_j^i + (A_1 A_1^\dagger)_j^i] + C_2[N(A_1A_1^\dagger)_j^i + \delta_j^i\, \mathrm{Tr} A_1 A_1^\dagger] . \qquad (A7)$$

By comparing the two sides we learn that

$$NC_1 + C_2 = 1/N , \qquad C_1 + NC_2 = 0 . \qquad (A8)$$

This gives $C_1 = 1/(N^2-1)$ and $C_2 = -1/N(N^2-1)$. The process continues in this way. At the end, we set $A_i = A$ and $A_i^\dagger = A^\dagger$ for all i to obtain our original integral. Here we list our explicit answer up to $n=4$:

$$\begin{aligned}
F_1(A) &= \frac{1}{N^2}[N\, \mathrm{Tr} A A^\dagger] ,\\
F_2(A) &= \frac{(2!)^2}{N^2(N^2-1)}\left[N^2 \frac{(\mathrm{Tr} A A^\dagger)^2}{2!} - N\frac{\mathrm{Tr}(AA^\dagger)^2}{2}\right] ,\\
F_3(A) &= \frac{(3!)^2}{N^2(N^2-1)(N^2-4)}\left[N(N^2-2)\frac{(\mathrm{Tr} AA^\dagger)^3}{3!} - N^2\, \mathrm{Tr} AA^\dagger \frac{\mathrm{Tr}(AA^\dagger)^2}{2} + 2N\frac{\mathrm{Tr}(AA^\dagger)^3}{3}\right] ,\\
F_4(A) &= \frac{(4!)^2}{N^2(N^2-1)(N^2-4)(N^2-9)}\left[(N^4-8N^2+6)\frac{(\mathrm{Tr} AA^\dagger)^4}{4!} - N(N^2-4)\frac{(\mathrm{Tr} AA^\dagger)^2}{2!}\frac{\mathrm{Tr}(AA^\dagger)^2}{2}\right.\\
&\quad \left. + (2N^2-3)\, \mathrm{Tr} AA^\dagger \frac{\mathrm{Tr}(AA^\dagger)^3}{3} + (N^2+6)\frac{1}{2!}\left(\frac{\mathrm{Tr}(AA^\dagger)^2}{2}\right)^2 - 5N\frac{\mathrm{Tr}(AA^\dagger)^4}{4}\right] .
\end{aligned} \qquad (A9)$$

The higher values of n can similarly be obtained. From above we see the general trend which allows us to write the general form of Eq. (3.12) in the text. The explicit coefficients displayed above allow us to further extract more specific information as discussed in the text.

We have tried other approaches to evaluate the integral

$$f(A) = \int dU \exp[\mathrm{Tr}(AU + U^\dagger A^\dagger)] \qquad (A10)$$

which have not proven to be as useful as the series method above. For future reference we explain them here. Both of the methods below attempt to parametrize the unitary matrix U. The first is a coset decomposition that we developed, the second which is applicable only when A is proportional to 1 is a parametrization due to Weyl.[28]

(i) *Coset decomposition.* For the U_{M+N} group, let us identify the subgroups U_M and U_N and decompose the $(M+N)\times(M\times N)$ unitary matrix U_{M+N} as follows:

$$U_{M+N} = U(Z)\begin{pmatrix} U_M & 0 \\ 0 & U_N \end{pmatrix} , \qquad (A11)$$

where $U(Z)$ is a unitary matrix parametrized by the MN complex variables Z_{ij},

$$Z = \begin{bmatrix} Z_{11} & Z_{12} \cdots Z_{1N} \\ Z_{21} & Z_{22} \cdots Z_{2N} \\ \vdots & \vdots \qquad \vdots \\ Z_{M1} & Z_{M2} \cdots Z_{MN} \end{bmatrix} , \qquad (A12)$$

$$U(Z) = \begin{bmatrix} \dfrac{1}{(1+ZZ^\dagger)^{1/2}} & Z\dfrac{1}{(1+Z^\dagger Z)^{1/2}} \\ -\dfrac{1}{(1+Z^\dagger Z)^{1/2}}Z^\dagger & \dfrac{1}{(1+Z^\dagger Z)^{1/2}} \end{bmatrix} . \qquad (A13)$$

Then the measure takes the form

$$[dU_{M+N}] = [dU_M][dU_N][dU(Z)] , \qquad (A14)$$

where

$$dU(Z) = \frac{dZ_{11}dZ_{11}^* \cdots dZ_{12}dZ_{12}^* \cdots dZ_{MN}dZ_{MN}^*}{[\det(1+ZZ^\dagger)]^N[\det(1+Z^\dagger Z)]^M} C_{MN} \qquad (A15)$$

and C_{MN} is a normalization constant. We now decompose the $(M+N)\times(M+N)$ matrix A_{M+N} as follows:

$$A_{M+N} = \begin{bmatrix} A_M & B \\ -C^\dagger & A_N \end{bmatrix} . \qquad (A16)$$

It is now possible to write

$$f_{N+M}\begin{bmatrix} A_M & B \\ -C^\dagger & A_N \end{bmatrix} = \int dU(Z)F_M\left((A-BZ^\dagger)\frac{1}{(1+ZZ^\dagger)^{1/2}}\right) \times F_N\left((A_n - C^\dagger Z)\frac{1}{(1+Z^\dagger Z)^{1/2}}\right). \tag{A17}$$

The idea is to use this form in order to generate a recursion relation. We were successful in simplifying this form further for the special case of $M=1$, $N=$ arbitrary, and both A_N and A_M proportional to unity, while $B=C^\dagger=0$. Then Z is just a row matrix

$$Z=(Z_1, Z_2, \ldots, Z_N) \tag{A18}$$

and we obtain

$$f_{N+1}\begin{bmatrix} a & 0 & \cdots & 0 \\ 0 & b & & \\ \vdots & & \ddots & \\ 0 & & & b \end{bmatrix} = C_N\int \frac{d^{2N}Z}{(1+ZZ^\dagger)^{N+1}} f_1\left(\frac{a}{(1+ZZ^\dagger)^{1/2}}\right) \times f_N\left(\frac{b}{(1+Z^\dagger Z)^{1/2}}\right). \tag{A19}$$

Note that $1/(1+ZZ^\dagger)^{1/2}$ is a scalar while $1/(1+ZZ^\dagger)^{1/2}$ is a matrix. Applying a U(N) rotation, Z can be put into the form $Z \Rightarrow (|Z|, 0, 0 \cdots 0)$ so that

$$\frac{b}{(1+Z^\dagger Z)^{1/2}} = \begin{bmatrix} \frac{b}{(1+ZZ^\dagger)^{1/2}} & 0 & \cdots & 0 \\ 0 & b & & \\ \vdots & & \ddots & \\ 0 & & & b \end{bmatrix}. \tag{A20}$$

Now both f_1 and f_N depend only on the magnitude

$$Z_1Z_1^* + Z_2Z_2^* + \cdots + Z_NZ_N^* \equiv \frac{1}{x} - 1. \tag{A21}$$

The angular integrations can be performed, giving the form

$$f_{N+1}\begin{bmatrix} a & 0 & \cdots & 0 \\ 0 & b & & \\ \vdots & & \ddots & \\ 0 & & & b \end{bmatrix} = N\int dx_1(1-x_1)^{N-1}F_1(a\sqrt{x}) \times f_N\begin{bmatrix} b\sqrt{x} & 0 & \cdots & 0 \\ 0 & b & & \\ \vdots & & \ddots & \\ 0 & & & b \end{bmatrix}. \tag{A22}$$

Now we note that f_1 is just the modified Bessel function

$$f_1(a\sqrt{x}) = \int_0^{2\pi}\frac{d\phi}{2\pi}\exp[a\sqrt{x}(e^{i\phi}+e^{-i\phi})] = I_0(2a\sqrt{x}). \tag{A23}$$

Therefore, we have arrived at a recursion relation from which we can attempt to calculate f_{N+1} by applying the same formula repeatedly in order to write the integral only in terms of the Bessel function I_0. The resulting integral is still cumbersome. The simplest case is when $b=0$, since then $f_N(0)=1$, so that

$$f_{N+1}\begin{bmatrix} a & \\ & 0 \end{bmatrix} = N\int_0^1 dx(1-x)^{N-1}I_0(2a\sqrt{x}) = N!\,(a)^{-N}I_N(2a). \tag{A24}$$

The normalization is chosen so that $f_{N+1}(0)=1$. This exact result agrees with our series method as described in the text, Eq. (3.20).

(ii) *Weyl's parametrization when* $A=ax1$. In this case we can diagonalize U:

$$U=T\begin{bmatrix} e^{i\phi_1} & & & \\ & e^{i\phi_2} & & \\ & & \ddots & \\ & & & e^{i\phi_N} \end{bmatrix}T^\dagger. \tag{A25}$$

The integrand is independent from T. Writing the measure as $dU=dT\,d\mu(\phi)$, we can evaluate the integral over T. The measure $d\mu(\phi)$ was given by Weyl as

$$d\mu(\phi)=\frac{d\phi_1}{2\pi}\cdots\frac{d\phi_N}{2\pi}\Delta(\phi)\bar{\Delta}(\phi), \tag{A26}$$

where $\Delta(\phi)$ is a determinant. The most convenient form for our purpose is

$$\Delta(\phi)=\frac{1}{\sqrt{N!}}\epsilon_{i_1i_2\cdots i_N}e^{i\phi_1(N-i_1)} \times e^{i\phi_2(N-i_2)}\cdots e^{i\phi_N(N-i_N)}. \tag{A27}$$

Note that for $A=a$ we have

$$e^{\beta N\,\mathrm{Tr}(AU+AU^\dagger)}=e^{2a\beta N(\cos\phi_1+\cos\phi_2+\cdots+\cos\phi_N)}. \tag{A28}$$

The integrals can be evaluated immediately:

$$f_N(a\beta N)=\frac{1}{N!}\epsilon_{i_1i_2\cdots i_N}\epsilon_{j_1j_2\cdots j_N} \times (I_{i_1-j_1}I_{i_2-j_2}\cdots I_{i_N-j_N}), \tag{A29}$$

where we have used the integral representation for the modified Bessel functions

$$I_n(2a\beta N)=\int_0^{2\pi}\frac{d\phi}{2\pi}e^{2a\beta N\cos\phi\pm in\phi}. \tag{A30}$$

Thus, we have the exact answer

$$f_N(a\beta N)=\begin{vmatrix} I_0 & I_1\cdots & I_{N-1}\\ I_1 & I_0\cdots & I_{N-2}\\ \vdots & \vdots & \vdots\\ I_{N-1} & I_{N-2}\cdots & I_0\end{vmatrix}, \qquad (A31)$$

where the Bessel functions have argument $2a\beta N$. In the text, using the series method, we arrived at the asymptotic form of this integral as $N\to\infty$:

$$f_N(a\beta N)\xrightarrow[N\to\infty]{} e^{a^2\beta^2N^2},$$

for $a\beta$ = small. It is difficult to take the limit $N\to\infty$ directly of this determinant, both because the determinant grows and because the arguments as well as the indices of the Bessel functions go to infinity at different rates.

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