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Integration of source-charge constraints in quantum chromodynamics with fixed quark and antiquark sources

Stephen L. Adler
The Institute for Advanced Study, Princeton, New Jersey 08540
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I study the problem of satisfying the source-charge constraints in operator-symmetrized quantum chromodynamics (QCD) with static sources. I show that the color-charge algebras generated by the QCD outer product $P^A(u,v)=(i/2)f^{ABC}(u^Bv^C+v^Cu^B)$ can always be put in the form $P^A(w_a,w_b)=iC_{abc}w_c$, with structure constants C_{abc} which are totally antisymmetric, but which do not in general satisfy the Jacobi identity. However, total antisymmetry of the C's is enough for the corresponding overlying classical field equations to be derivable from a Lagrangian and to possess a conserved stress-energy tensor, involving a finite number of undetermined constants of integration. I postulate conditions for determining the integration constants when the sources are in a color-singlet state, and use them to fix the overlying classical theory in the $q\bar{q}$ and the qqq ($\bar{q}\bar{q}\bar{q}$) cases.

I return in this note to the problem of integrating the source-charge constraint equation in quantum chromodynamics (QCD) with static sources, using a modified form of the algebraic approach proposed in earlier papers.[1,2] As the starting point for the analysis, I take the following operator-symmetrized form of the QCD field equations,

$$f^A_{\mu\nu}=\frac{\partial}{\partial x^\nu}b^A_\mu-\frac{\partial}{\partial x^\mu}b^A_\nu-igP^A(b_\mu,b_\nu),$$

$$D_\nu f^{A\mu\nu}=\frac{\partial}{\partial x^\nu}f^{A\mu\nu}+igP^A(b_\nu,f^{\mu\nu})=gJ^{A\mu}, \quad (1)$$

$$P^A(u,v)=\frac{i}{2}f^{ABC}(u^Bv^C+v^Cu^B);$$

$$J^{A\mu}=\bar{q}\gamma^\mu\tfrac{1}{2}\lambda^Aq, \quad (2)$$

with λ^A, $A=1,\ldots,n^2-1$ the usual SU(n) matrices. The gluon operator symmetrization used in Eq. (1) is suggested by the first-order formalism for the quantization of QCD,[3] in which the radiation (or Coulomb) gauge can be imposed without the introduction of ghost fields. (Specifying a gluon operator-ordering prescription is essential, since as part of the canonical quantization procedure one uses the constraint equation to express b^A_0 as a nonlocal functional of canonical coordinates and momenta, and thus b^A_0 does not commute with b^A_j and f^A_{0j} for spacelike separations.[4]) In effect, the operator-ordering prescription of Eq. (1) is a postulate on which the detailed form of the subsequent analysis depends.[5]

From the quark canonical commutation relations, one finds that the quark source density $J^A_0=q^\dagger\frac{1}{2}\lambda^Aq$ satisfies the equal-time color algebra

$$[J^A_0(\vec{x},t),J^B_0(\vec{y},t)]=i\delta^3(\vec{x}-\vec{y})f^{ABC}J^C_0(\vec{x},t). \quad (3)$$

The static-quark limit of QCD is obtained by letting the quark mass become infinite, so that the quark and antiquark sources are at rest in the observer's frame and can be arbitrarily well localized. Only the quark color degrees of freedom, and the gluon fields, remain as dynamical variables in this limit. In the sector containing N_q quark and $N_{\bar{q}}$ antiquark sources, the static limit is described by making the replacements

$$\begin{aligned}&J^A_j(x)\to 0,\\ &J^A_0(x)\to\sum_{l=1}^{N}Q^A_l\delta^3(\vec{x}-\vec{x}_l),\quad N=N_q+N_{\bar{q}},\\ &Q^A_l=\tfrac{1}{2}\lambda^A_l \text{ for a quark source at } \vec{x}_l,\\ &Q^A_l=-\tfrac{1}{2}\lambda^{*A}_l \text{ for an antiquark source at } \vec{x}_l,\\ &[Q^A_l,Q^B_m]=0,\quad l\neq m.\end{aligned} \quad (4)$$

The right-hand side of Eq. (4) is an operator acting on the finite-dimensional direct product of the color Hilbert spaces for the N source particles, and clearly gives a representation of the equal-time color algebra of Eq. (3). Equations (1) and (4) constitute the static-source model which I analyze below.

To study the structure of the gluon field equations in the static-source limit, it is necessary to first determine the properties of the quark color-charge algebra $\mathfrak{A}_{N_q,N_{\bar{q}}}$, defined as the minimal set of (n^2-1)-plet operators w^A containing N_q quark charges, $N_{\bar{q}}$ antiquark charges, and closed under composition with the outer product $w^A=P^A(u,v)$ defined in Eq. (1). It is easily shown[6] that $\mathfrak{A}_{N_q,N_{\bar{q}}}$ is a finite-dimensional algebra. Let us choose a basis w^A_a, $a=1,\ldots,\dim(\mathfrak{A}_{N_q,N_{\bar{q}}})$ spanning $\mathfrak{A}_{N_q,N_{\bar{q}}}$, with the properties

$$w_a^A = w_a^{A\dagger},$$
$$\mathrm{Tr}(w_a^A w_b^A) \equiv \prod_{l=1}^{N} \mathrm{tr}_l(w_a^A w_b^A) = N(n, N_q, N_{\bar{q}})\,\delta_{ab}, \tag{5}$$

with $N>0$ a convenient overall normalization constant. That such a choice is possible follows from the fact that for an arbitrary Hermitian basis v_a^A spanning $\mathfrak{A}_{N_q, N_{\bar{q}}}$, one has

$$\mathrm{Tr}(v_a^A v_b^A) = M_{ab}, \tag{6}$$

with M a real, symmetric matrix. Letting O_{ab} be the real, orthogonal matrix which diagonalizes M, we can define a suitable basis w_a by the transformations

$$u_a^A = O_{ab} v_b^A, \tag{7}$$
$$w_a^A = N^{1/2} \frac{u_a^A}{[\mathrm{Tr}(u_a^A u_a^{\dagger A})]^{1/2}}. \tag{8}$$

[The index a in the denominator of Eq. (8) is not summed.] Let us now define structure constants C_{abc} for the color-charge algebra by writing

$$P^A(w_a, w_b) = i C_{abc} w_c^A; \tag{9}$$

since P is antisymmetric in its arguments, we evidently have

$$C_{abc} = -C_{bac}. \tag{10}$$

The tensor C_{abc} is also invariant under cyclic permutation of its indices, a fact which follows immediately on using Eqs. (1), (5), and (9) to write

$$\begin{aligned} C_{abc} &= N^{-1}\mathrm{Tr}[w_c^A \tfrac{1}{2} f^{ABC}(w_a^B w_b^C + w_b^C w_a^B)] \\ &= N^{-1}\mathrm{Tr}[w_b^C \tfrac{1}{2} f^{CAB}(w_c^A w_a^B + w_a^B w_c^A)] = C_{cab}. \end{aligned} \tag{11}$$

Taken together, Eqs. (10) and (11) imply that C_{abc} is totally antisymmetric in its indices. We will see later on that C_{abc} does not in general satisfy the Jacobi identity, and so it is useful to introduce the further tensor

$$D_{abcd} = C_{abe}C_{ecd} + C_{ade}C_{ebc} + C_{ace}C_{edb}. \tag{12}$$

The total antisymmetry of C_{abc} implies that D_{abcd} is totally antisymmetric in all four indices.[7]

We can now explicitly display the action of the various operators in our problem on the quark color Hilbert space by decomposing them on the basis w_a^A,

$$\begin{aligned} &b_\nu^A = \sum_a b_\nu^a w_a^A + \cdots, \quad f_{\mu\nu}^A = \sum_a f_{\mu\nu}^a w_a^A + \cdots, \\ &Q_l^A = \sum_a Q_{l\,\mathrm{eff}}^a w_a^A, \quad J_j^A = \sum_a J_j^a w_j^A = 0, \\ &J_0^A = \sum_a J_0^a w_a^A, \quad J_0^a = \sum_{l=1}^{N} Q_{l\,\mathrm{eff}}^a \delta^3(\vec{x} - \vec{x}_l). \end{aligned} \tag{13}$$

The effective charges $Q_{l\,\mathrm{eff}}^a$ and the amplitudes b_ν^a and $f_{\mu\nu}^a$ are c numbers, while all terms lying outside the quark color-charge algebra (including all nontrivial operator structure in the gluon Hilbert space) have been denoted by $\cdots$. To proceed further, I make the tree approximation of dropping the terms indicated by $\cdots$ in Eq. (13). The operator equations of Eqs. (1) and (4) can then be rewritten directly as a system of overlying classical equations

$$\begin{aligned} &f_{\mu\nu}^a = \frac{\partial}{\partial x^\nu} b_\mu^a - \frac{\partial}{\partial x^\mu} b_\nu^a + g C_{abc} b_\mu^b b_\nu^c, \\ &D_\nu f^{a\mu\nu} = \frac{\partial}{\partial x^\nu} f^{a\mu\nu} - g C_{abc} b_\nu^b f^{c\mu\nu} = g J^{a\mu}, \\ &J^{aj} = 0, \quad J^{a0} = \sum_{l=1}^{N} Q_{l\,\mathrm{eff}}^a \delta^3(\vec{x} - \vec{x}_l). \end{aligned} \tag{14}$$

These equations have the same structure as those of a classical Yang-Mills theory (with fixed point-singularity sources), except for the fact that the structure constants C_{abc}, while totally antisymmetric, do not satisfy the Jacobi identity. I call such a theory a classical semigauge theory.

Let us now study the properties of the semigauge system of Eq. (14). Setting $\mu = 0$ in the field equation gives the familiar charge constraint equation

$$D_i f^{a0i} = g J^{a0}. \tag{15}$$

The condition for Eq. (15) to be preserved in time is most easily obtained in the form

$$D_\mu D_\nu f^{a\mu\nu} = \tfrac{1}{2}(D_\mu D_\nu - D_\nu D_\mu) f^{a\mu\nu} = g D_\mu J^{a\mu}. \tag{16}$$

Because the C's do not satisfy the Jacobi identity, the left-hand side of Eq. (16) is not identically zero. By use of the identity

$$(D_\mu D_\nu - D_\nu D_\mu) h^a = g C_{abc} f_{\mu\nu}^b h^c + g^2 D_{abcd} b_\mu^b b_\nu^c h^d, \tag{17}$$

Eq. (16) may be rewritten as

$$\tfrac{1}{2} D_{abcd} b_\mu^b b_\nu^c f^{d\mu\nu} = -\sum_{l=1}^{N} C_{abc} b_0^b(x_l) Q_{l\,\mathrm{eff}}^c \delta^3(\vec{x} - \vec{x}_l). \tag{18}$$

Since the left-hand side of Eq. (18) is a continuous function of $\vec{x}$, the left- and right-hand sides must vanish separately, yielding the additional equations of constraint[8]

$$D_{abcd} b_\mu^b b_\nu^c f^{d\mu\nu} = 0, \tag{18a}$$
$$C_{abc} b_0^b(x_l) Q_{l\,\mathrm{eff}}^c = 0, \quad l = 1, \ldots, N. \tag{18b}$$

Equation (18) explicitly involves b_0^b, and since the

field equations do not determine $\partial_0 b_0^b$, no further constraint equations are obtained from the condition that Eq. (18) be preserved in time. From the antisymmetry of C_{abc}, we see that the identity

$$\frac{\partial}{\partial x^\nu}(f^a g^a) = (D_\nu f^a) g^a + f^a (D_\nu g^a) \tag{19}$$

remains valid in the semigauge case. Hence using

$$\delta f^a_{\mu\nu} = D_\nu \delta b^a_\mu - D_\mu \delta b^a_\nu \tag{20}$$

and defining a Lagrangian density $\mathcal{L}$ by

$$\mathcal{L} = K(-\tfrac{1}{4} f^a_{\mu\nu} f^{a\mu\nu} - g J^{a\mu} b^a_\mu), \tag{21}$$

we find that[9]

$$\begin{aligned} 0 &= \delta \int d^4x \mathcal{L} \\ &= K \int d^4x (-f^{a\mu\nu} D_\nu \delta b^a_\mu - g J^{a\mu} \delta b^a_\mu) \\ &= K \int d^4x (D_\nu f^{a\mu\nu} - g J^{a\mu}) \delta b^a_\mu , \end{aligned} \tag{22}$$

yielding the equation of motion given in Eq. (14). Thus a classical semigauge theory is derivable from a local Lagrangian density. Defining the symmetrized gluon stress-energy tensor by

$$T^{\mu\,\text{gluon}}_{\ \ \nu} = K(-f^{a\mu\lambda} f^a_{\nu\lambda} + \tfrac{1}{4} \delta^\mu_\nu f^a_{\lambda\sigma} f^{a\lambda\sigma}), \tag{23}$$

we find

$$\begin{aligned} \frac{\partial}{\partial x^\mu} T^{\mu\,\text{gluon}}_{\ \ \nu} = K(&-D_\mu f^{a\mu\lambda} f^a_{\nu\lambda} - f^{a\mu\lambda} D_\mu f^a_{\nu\lambda} \\ &- \tfrac{1}{2} D_\sigma f^a_{\nu\lambda} f^{a\lambda\sigma} - \tfrac{1}{2} D_\lambda f^a_{\sigma\nu} f^{a\lambda\sigma} \\ &+ \tfrac{1}{2} Q^a_{\nu\lambda\sigma} f^{a\lambda\sigma}), \end{aligned} \tag{24}$$

with

$$Q^a_{\nu\lambda\sigma} = D_\nu f^a_{\lambda\sigma} + D_\sigma f^a_{\nu\lambda} + D_\lambda f^a_{\sigma\nu} = -g^2 D_{abcd}\, b^b_\nu b^c_\lambda b^d_\sigma . \tag{25}$$

The first term of Eq. (24) can be evaluated by using Eq. (14), the sum of the second, third, and fourth terms vanishes, while the last term vanishes by virtue of the constraints of Eq. (18a). Hence the gluon stress-energy tensor satisfies a conservation law of the usual form,

$$\frac{\partial}{\partial x^\mu} T^{\mu\,\text{gluon}}_{\ \ \nu} = K g f^a_{\nu\lambda} J^{a\lambda} . \tag{26}$$

The constant K in Eq. (26) is effectively an undetermined constant of integration. If the color-charge algebra of Eq. (9) diagonalizes into R disjoint subalgebras, then the classical analysis of Eqs. (15)–(26) can be carried out independently for each subalgebra, leading to the introduction of R non-negative constants of integration $K_{(r)}$, $r = 1, \ldots, R$ in the classical Lagrangian and stress-energy tensor,

$$\begin{aligned} \mathcal{L} &= \sum_{r=1}^{R} \mathcal{L}^{(r)} \\ &= \sum_{r=1}^{R} K_{(r)} (-\tfrac{1}{4} f^{(r)a}_{\mu\nu} f^{(r)a\mu\nu} - g J^{(r)a\mu} b^{(r)a}_\mu), \end{aligned} \tag{27}$$

$$\begin{aligned} T^{\mu\,\text{gluon}}_{\ \ \nu} &= \sum_{r=1}^{R} T^{\mu\,\text{gluon}(r)}_{\ \ \ \ \nu} \\ &= \sum_{r=1}^{R} K_{(r)} (-f^{(r)a\mu\lambda} f^{(r)a}_{\nu\lambda} + \tfrac{1}{4} \delta^\mu_\nu f^{(r)a}_{\lambda\sigma} f^{(r)a\lambda\sigma}) . \end{aligned}$$

In order for the above formalism to give a uniquely defined classical stress-energy tensor, it is necessary to determine the constants of integration $K_{(r)}$ which appear in Eq. (27). These constants carry information, which up to this point has not figured in the analysis, about the color state of the system of N source particles. For quark and antiquark sources in a color-singlet state $|0\rangle$, defined by[10]

$$\sum_{i=1}^{N} Q^A_i |0\rangle = 0 , \tag{28}$$

I postulate the following rules for determining (or partially determining) the constants $K_{(r)}$:

(i) The constants $K_{(r)}$ should obey the sum rule

$$\begin{aligned} \frac{n^2 - 1}{2n} &= Q^A_k Q^A_k \\ &= \sum_{r=1}^{R} K_{(r)} Q^{(r)a}_{k\,\text{eff}} Q^{(r)a}_{k\,\text{eff}} , \quad k = 1, \ldots, N . \end{aligned} \tag{29}$$

This rule is motivated by the expectation that a classical scheme for calculating static quark forces should reproduce the leading Coulombic quark self-energy divergence, as calculated by the usual perturbation theory rules.

(ii) For any proper diagonal subalgebra $\mathfrak{A}^{(s)}_{N_q, N_{\bar{q}}}$ which annihilates the color-singlet state [i.e., $w_{(s)a}|0\rangle = 0$ for all a spanning the subalgebra], choose $K_{(s)} = 0$. This rule[11] is motivated by the expectation that such subalgebras should not be excited when the quarks are in a color-singlet state, and hence should make a vanishing contribution to the classical stress-energy tensor.

I close by applying the formalism developed above, and the rules for restricting the constants of integration, to the $q\bar{q}$ and $qqq(\bar{q}\bar{q}\bar{q})$ systems.

A. $q\bar{q}$

The color charge algebra $\mathfrak{A}_{1,1}$ has been computed by Giles and McLerran.[2] It is spanned by a basis w^A_a, $a = 1, \ldots, 4$, with

$$w_1^A = \frac{1}{n}(Q_q^A + Q_{\bar{q}}^A),$$

$$w_2^A = -\frac{2}{n} d^{ABC} Q_q^B Q_{\bar{q}}^C + \frac{2}{n^2}(Q_q^A - Q_{\bar{q}}^A),$$

$$w_3^A = -\frac{2}{n} f^{ABC} Q_q^B Q_{\bar{q}}^C, \tag{30}$$

$$w_4^A = \begin{cases} 0, & n=2 \\ \dfrac{(n^2-4)^{1/2}}{n^2}(Q_q^A - Q_{\bar{q}}^A) + \dfrac{4}{n}\dfrac{1}{(n^2-4)^{1/2}} d^{ABC} Q_q^B Q_{\bar{q}}^C, & n>2. \end{cases}$$

The basis of Eq. (30) has been normalized as in Eq. (5), with $N(n, 1, 1) = (n^2-1)/n$, that is,

$$\mathrm{Tr}(w_a^A w_b^A) \equiv \mathrm{tr}_q \mathrm{tr}_{\bar{q}}(w_a^A w_b^A) = \frac{n^2-1}{n}\delta_{ab}, \tag{31}$$

and satisfies the outer-product algebra

$$\begin{aligned} &P(w_a, w_b) = iC_{abc} w_c = i\tfrac{1}{2}\epsilon_{abc} w_c, \\ &P(w_a, w_4) = 0, \quad a, b, c = 1, 2, 3. \end{aligned} \tag{32}$$

Evidently $\mathfrak{A}_{1,1}^{(1)} \equiv \{w_{1,2,3}\}$ and $\mathfrak{A}_{1,1}^{(2)} \equiv \{w_4\}$ are proper diagonal subalgebras of the algebra $\mathfrak{A}_{1,1}$, so let us relabel the bases as

$$\begin{aligned} &w_{(1)a} = w_a, \quad a = 1, 2, 3, \\ &w_{(2)} = w_4. \end{aligned} \tag{33}$$

The fact that $\mathfrak{A}_{1,1}$ contains no proper diagonal subalgebra of dimension $\dim(\mathfrak{A}_{1,1}^{(r)}) \geq 4$ guarantees that it is in fact a Lie algebra, since the totally antisymmetric tensor D_{abcd} must vanish on subalgebras of dimension 1, 2, or 3. The classical Lagrangian and stress-energy tensor of Eq. (27) involve two undetermined constants of integration $K_{(1)}$ and $K_{(2)}$. We wish to determine these constants for a $q\bar{q}$ pair in the color-singlet state $|0\rangle$, defined by

$$(Q_q^A + Q_{\bar{q}}^A)|0\rangle = 0. \tag{34}$$

Using Eq. (34) to eliminate $Q_{\bar{q}}^A$ we get

$$\begin{aligned} w_{(2)}^A|0\rangle &= \left[\frac{2(n^2-4)^{1/2}}{n^2} Q_q^A \right. \\ &\quad \left. - \frac{4}{n}\frac{1}{(n^2-4)^{1/2}} d^{ABC} Q_q^B Q_q^C\right]|0\rangle \\ &= 0, \end{aligned} \tag{35}$$

and so by rule (ii) above we set $K_{(2)} = 0$. Although $w_{(1)1}|0\rangle = 0$, the other two elements of the (1) subalgebra do not annihilate $|0\rangle$, and so we expect $K_{(1)} \neq 0$. To fix $K_{(1)}$, we apply rule (i) above. The effective charges are

$$\begin{aligned} Q_{q\ \mathrm{eff}}^{(1)a} &= \left(\frac{n}{2}, 1, 0\right)^a, \\ Q_{\bar{q}\ \mathrm{eff}}^{(1)a} &= \left(\frac{n}{2}, -1, 0\right)^a, \end{aligned} \tag{36}$$

and thus by Eq. (29) we get

$$K_{(1)} = \frac{n^2-1}{2n}\left(\frac{n^2}{4}+1\right)^{-1} = \frac{2}{n}\frac{n^2-1}{n^2+4}, \tag{37}$$

completing the determination of the integration constants. This uniquely fixes the overlying classical theory, giving the SU(2) Yang-Mills form assumed in Ref. 1 as the basis for a nonperturbative discussion of the static $q\bar{q}$ potential.

B. qqq ($\bar{q}\bar{q}\bar{q}$)

The color algebra $\mathfrak{A}_{3,0}$ [which for the outer product of Eq. (1) is isomorphic to $\mathfrak{A}_{0,3}$] has been computed by Lee.[12] He finds an 18-dimensional algebra, consisting of an irreducible 16-dimensional proper diagonal subalgebra $\mathfrak{A}_{3,0}^{(1)}$ and two 1-dimensional proper diagonal subalgebras $\mathfrak{A}_{3,0}^{(2,3)}$. The subalgebra $\mathfrak{A}_{3,0}^{(1)}$ has a nonvanishing tensor D_{abcd}, and so is not a Lie algebra. For $n=3$ [that is, when the color group is SU(3)] both 1-dimensional subalgebras annihilate the color-singlet state. Hence application of rules (i) and (ii) above determines the constants $K_{(1),(2),(3)}$ as follows:

$$\begin{aligned} &K_{(1)} = \tfrac{4}{3}[Q_{q\ \mathrm{eff}}^{(1)a} Q_{q\ \mathrm{eff}}^{(1)a}]^{-1}, \\ &K_{(2)} = K_{(3)} = 0, \end{aligned} \tag{38}$$

again uniquely fixing the overlying classical theory. Further details of the structure of the algebra $\mathfrak{A}_{3,0}$ are given in Ref. 12.

Added notes.

1. In the discussion in the text, I have restricted the quark and antiquark source charge matrices to be constants in time. The formalism may be extended to include time development of the color sources (arising from dynamics of the quark field) as follows: At time $t=0$ assume the color sources to have the canonical form of Eq. (4), and let the time-independent basis w_a^A be the one constructed, as in the text, from the $t=0$ color sources. At all later times the field operators can be decomposed on the fixed basis w_a^A as in Eq. (13), but now the effective charges $Q_{l\ \mathrm{eff}}^a$ will be time dependent. Hence Eq. (18) is modified to read

$$\tfrac{1}{2} g D_{abcd} b_\mu^b b_\nu^c f^{d\mu\nu} = \sum_{l=1}^{N}\left[\frac{\partial}{\partial x^0} Q_{l\ \mathrm{eff}}^a - g C_{abc} b_0^b(x_l) Q_{l\ \mathrm{eff}}^c\right]\delta^3(\vec{\mathbf{x}} - \vec{\mathbf{x}}_l). \tag{A1}$$

By continuity this still implies Eq. (18a), but now Eq. (18b) becomes

$$\frac{\partial}{\partial x^0} Q_{l\ \mathrm{eff}}^a = g C_{abc} b_0^b(x_l) Q_{l\ \mathrm{eff}}^c, \tag{A2}$$

which describes the time development of the color

sources. At times $t>0$ the time-evolved source charge matrices will not, in general, obey the canonical conditions

$$[Q_l^A, Q_m^B]=0, \quad l\neq m,$$

$$Q_l^A Q_l^B - \frac{1}{n}\mathrm{tr}_l(Q_l^A Q_l^B) = \begin{pmatrix} q^{ABC} \\ -q^{BAC} \end{pmatrix} Q_l^C \quad \text{for} \begin{pmatrix} \text{quark} \\ \text{antiquark} \end{pmatrix}, \tag{A3}$$

$$q^{ABC}=\tfrac{1}{2}(d^{ABC}+if^{ABC}).$$

Specifically, calculations by Lee[6] show that in the $q\bar{q}$ case the initial charges $Q_q^A(0), Q_{\bar{q}}^A(0)$ are the only pair of operators in the color algebra $\mathfrak{A}_{1,1}$ which obey the canonical conditions of Eq. (A3). His results show, in the $q\bar{q}$ sector, that if one wishes to maintain a strict canonical charge structure at all times, the static source constraint of Eq. (18b) must be imposed. The same statement is probably true in other sectors as well. However, I believe it is likely that for a correct quantization procedure in radiation-gauge QCD, one need only have a strict canonical charge structure at *one* time (e.g., $t=0$) along the system world line.

2. An important aspect of the discussion of the text is that not only are b_0^A and f_{0i}^A algebra-valued, as in the usual radiation-gauge canonical treatment,[3] but the spatial part of the gauge potential b_i^A is algebra-valued as well. The following simple argument shows that this unconventional feature is necessary in the fixed-source model, when truncated down to the finite Hilbert space spanned by the quark color-charge algebra. Suppose that b_i^A is not algebra-valued; then the overlying components b_i^a vanish identically, and Eqs. (14) simplify to read

$$f^{a0i}=-\frac{\partial}{\partial x^i}b_0^a, \tag{A4a}$$

$$\frac{\partial}{\partial x^i}f^{a0i}=-\nabla^2 b_0^a = gJ_0^a = g\sum_{l=1}^{N} Q_{l\,\mathrm{eff}}^a\,\delta^3(\vec{x}-\vec{x}_l), \tag{A4b}$$

$$\frac{\partial}{\partial x^0}f^{a0i}=gC_{abc}\,b_0^b f^{c0i}. \tag{A4c}$$

Taking the divergence of Eq. (A4c) and using Eqs. (A4a), (A4b) gives, as expected, the time-evolution equation of Eq. (A2). Solving Eq. (A4b) for b_0^a (assuming b_0^a vanishes at infinity) gives

$$b_0^a=\sum_l \frac{Q_{l\,\mathrm{eff}}^a(x_0)}{4\pi|\vec{x}-\vec{x}_l|}, \tag{A5a}$$

$$f^{a0i}=\sum_l \frac{Q_{l\,\mathrm{eff}}^a(x_0)}{4\pi|\vec{x}-\vec{x}_l|^3}(x-x_l)^i. \tag{A5b}$$

Substituting Eq. (A5), together with Eq. (A2), into the left-hand side of Eq. (A4c) gives

$$\frac{\partial}{\partial x^0}f^{a0i}=\sum_l \frac{(\partial/\partial x^0)Q_{l\,\mathrm{eff}}^a(x_0)}{4\pi|\vec{x}-\vec{x}_l|^3}(x-x_l)^i = \sum_l \frac{(x-x_l)^i}{4\pi|\vec{x}-\vec{x}_l|^3}\sum_{n\neq l}\frac{gC_{abc}Q_{n\,\mathrm{eff}}^b(x_0)Q_{l\,\mathrm{eff}}^c(x_0)}{4\pi|\vec{x}_l-\vec{x}_n|}, \tag{A6}$$

while substituting Eq. (A5) into the right-hand side of Eq. (A4c) gives

$$gC_{abc}\,b_0^b f^{c0i}=\sum_l \frac{(x-x_l)^i}{4\pi|\vec{x}-\vec{x}_l|^3}\times\sum_{n\neq l}\frac{gC_{abc}Q_{n\,\mathrm{eff}}^b(x_0)Q_{l\,\mathrm{eff}}^c(x_0)}{4\pi|\vec{x}-\vec{x}_n|}. \tag{A7}$$

The expressions in Eq. (A6) and Eq. (A7) differ by a nonvanishing divergence-free vector unless

$$C_{abc}Q_{n\,\mathrm{eff}}^b(x_0)Q_{l\,\mathrm{eff}}^c(x_0)=0 \quad \text{for all } l,n, \tag{A8}$$

a condition which is not satisfied by the effective charges [see, for instance, Eq. (36)]. Hence the conventional canonical assumption that b_i^A is not algebra-valued is inconsistent with the equations of motion of the truncated fixed-source model. This conclusion is independent of assumptions about operator ordering, since when b_i^A is not algebra-valued, the ordering prescription of Eq. (1) used in the text agrees with the alternative ordering prescription described in Ref. 5.

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[4] This operator-ordering problem occurs in the axial gauge as well as in the Coulomb gauge. For a comprehensive discussion of canonical quantization procedures, see A. Hanson, T. Regge, and C. Teitelboim, *Constrained Hamiltonian Systems* (Accademia Nazionale dei Lincei, Rome, 1976).

[5] An alternative ordering prescription for the $D_\nu f^{\mu\nu}$ equation would be to completely symmetrize b^ν with

respect to the factors b^μ, b^ν appearing in $f^{\mu\nu}$. In this case, one would have to perform an analysis similar in spirit to that of the text, but based on the algebra of the color charges when acted on by the original binary product

$$P^A(u,v)=\tfrac{1}{2}if^{ABC}(u^Bv^C+v^Cu^B)$$

and by a second binary product

$$\begin{aligned}Q^A(u,v)=&-\tfrac{1}{4}f^{ABC}f^{CDE}(u^Bv^Du^E+u^Bu^Ev^D+v^Du^Bu^E\\&+u^Eu^Bv^D+v^Du^Eu^B+u^Ev^Du^B)\\=&P^A(u,P(u,v))-\tfrac{1}{4}f^{ABC}f^{CDE}(v^Du^Bu^E+u^Eu^Bv^D).\end{aligned}$$

[6]S.-C. Lee, Phys. Rev. D 20, 1951 (1979). See also P. Cvitanović, R. J. Gonsalves, and D. E. Neville, *ibid.* 18, 3881 (1978); V. Rittenberg and D. Wyler, *ibid.* 18, 4806 (1978).

[7]The antisymmetry of C_{abc} also implies that the constants C_{abc} and D_{abcd} obey the identity

$$\sum_{\substack{\text{permutations}\\ p\begin{pmatrix}abcde\\rstuv\end{pmatrix}}}\epsilon_p C_{rsf}D_{ftuv}=0,$$

$$\epsilon_p=\begin{cases}+1 \text{ even permutations}\\ -1 \text{ odd permutations}.\end{cases}$$

[8]Note that Eq. (18b) is equivalent to covariant source current conservation $D_\mu J^{a\mu}=0$, and so we see that even in the semigauge case (with point sources) the conservation equation is a consequence of the field equations, and does not have to be introduced as an independent postulate. I wish to thank S-C. Lee for discussions which helped to clarify the structure of the constraint equations.

[9]In carrying out the variation of Eq. (22), I have assumed $J^{a\mu}$ to be functionally independent of b^a_μ. This assumption is consistent with the canonical expression for $J^{A\mu}$ given in Eq. (2), in which only quark fields appear.

[10]In the tree approximation of dropping the terms $\cdots$ in Eq. (13), the quantum structure of the theory is reduced to a finite-dimensional quark color Hilbert space. The color-singlet state must be invariant under global color rotations in this space, and hence is annihilated by the quark contribution F^A_{quark} to the color spin,

$$F^A_{\text{quark}}\mid 0\rangle=0,\quad F^A_{\text{quark}}=\sum_{l=1}^{N}Q^A_l.$$

However, the total color spin operator F^A contains gluon as well as quark contributions,

$$F^A=g^{-1}\int_{\substack{\text{sphere}\\ \text{at}\infty}}d^2S^j f^{a0j}w^A_a=F^A_{\text{quark}}+F^A_{\text{gluon}},$$

$$F^A_{\text{gluon}}\equiv\int d^3x\,C_{abc}b^a_j f^{b0j}w^A_c,$$

and so an additional argument is required to show that $F^A\mid 0\rangle=0$ for the $q\bar{q}$ and qqq systems of physical interest. In the $q\bar{q}$ case, the analysis of the overlying classical equations given in Ref. 1 shows that

$$\int_{\substack{\text{sphere}\\ \text{at}\infty}}d^2S^j f^{a0j}=0\Rightarrow F^A_{\text{gluon}}=-F^A_{\text{quark}},$$

and so in the tree approximation the color spin operator F^A vanishes identically. In the qqq case, symmetry of the overlying algebra in the variables of the three quarks, and the fact that $\mathfrak{A}_{3,0}\mid 0\rangle$ is spanned by $(Q^A_{q_1},Q^A_{q_2},Q^A_{q_3})\mid 0\rangle$, implies that

$$F^A_{\text{gluon}}\mid 0\rangle\propto F^A_{\text{quark}}\mid 0\rangle=0.$$

Hence in this case the color spin operator F^A does not vanish identically, but nonetheless still annihilates the color-singlet state. The picture that emerges is that for static $q\bar{q}$ and qqq systems in the color-singlet state, both the quark color spin and the color spin of the surrounding gluon cloud are separately zero, and couple trivially to give a state of zero total color spin. This is consistent with the usual phenomenology of the nonrelativistic quark model.

[11]An alternative to rule (ii) would be to postulate that for stable states of physical interest, the constants $K_{(r)}$ are fixed by minimization of the free-energy functional $\mathfrak{F}$ of Ref. 1, subject to the sum rule of Eq. (29) and the positivity conditions

$$K_{(r)}\geq 0,\quad r=1,\ldots,R.$$

If, as postulated in Ref. 1, confinement is associated with the nonlinear sectors of the overlying algebra, the minimum of $\mathfrak{F}$ is attained by maximizing the contribution of these sectors, which requires setting $K_{(s)}=0$ for any Abelian subalgebra $\mathfrak{A}^{(s)}$. Since in the $q\bar{q}$ and qqq cases the Abelian subalgebras are the only ones which annihilate the color-singlet state, this alternative rule implies the same values for the constants $K_{(r)}$ as are obtained from rule (ii) of the text.

[12]S-C. Lee, Phys. Rev. D (to be published).