# Remarks on the static potential in quantum chromodynamics

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We derive the Wilson loop formula for the static quark-antiquark potential using standard field-theory techniques. We also discuss the limitations of a static potential as a description of higher-order effects.

The success of nonrelativistic potential models in explaining the level structure of the charmonium states associated with the $J/\psi$ particle has prompted several investigations[1-5] into the nature of the static potential between a heavy quark and antiquark which interact by colored gluon exchange. The Wilson loop formula[6] has often been introduced on an intuitive basis as a useful tool for calculating this potential which, in conjunction with the Schrödinger equation, should approximately describe the observed level structure. In this article we derive the Wilson loop expression with standard field-theory methods and discuss the limits of validity in using it or related techniques to define a static potential. In particular, we show that nonlocal effects, which invalidate a static-potential model, arise in an inescapable way when one attempts to go beyond the level of accuracy of the standard Breit Hamiltonian.

This difficulty is well known in the analogous atomic physics calculations. The possibility that the same problems may arise in quarkonium models has been noted by authors cited above, especially in Refs. 1 and 5. Our derivation of the Wilson loop formula and its corrections from a standard formalism enables us to discuss the static-potential limit as well as its recoil and spin-dependent corrections and the breakdown of the static-potential picture within a unified framework. In addition, we point out that the virial theorem casts doubt on the validity of resummation schemes which have been proposed in hope of defining a static potential nonperturbatively.

To illustrate our points, it suffices to consider a single quark flavor described by a spin-$\frac{1}{2}$ Fermi field $\psi(x)$ in the defining representation of the SU(3) color group. The Lagrange function for this Fermi field, including its interaction with the colored gluon vector potential $A_{\mu a}(x)$, is given by

$$\mathcal{L}_\psi = -\bar{\psi}\left[\gamma^\mu\left(\frac{1}{i}\partial_\mu - gA_\mu\right) + M\right]\psi . \tag{1}$$

Here the gauge potential is written in a matrix form,

$$A_\mu = A_{\mu a}\lambda_a , \tag{2}$$

with $\lambda_a$ the defining matrices of the SU(3) color group which obey the group law

$$[\lambda_a, \lambda_b] = if_{abc}\lambda_c . \tag{3}$$

The full Lagrange function of this quantum-chromodynamic model is obtained by adjoining the non-Abelian gluon field contribution:

$$\mathcal{L} = -\tfrac{1}{4}F_a^{\mu\nu}F_{\mu\nu a} + \mathcal{L}_\psi , \tag{4}$$

where

$$F_{\mu\nu a} = \partial_\mu A_{\nu a} - \partial_\nu A_{\mu a} + gf_{abc}A_{\mu b}A_{\nu c} . \tag{5}$$

We are concerned with the character of states where the fermions behave essentially as loosely bound, nonrelativistic particles and pair-creation effects are negligible. We are therefore interested in an expansion involving inverse powers of the heavy-quark mass $M$. This expansion may be obtained by first performing a Foldy-Wouthuysen transformation[7] from the field $\psi$ to a new field $\psi'$ which yields

$$\mathcal{L}_\psi = \psi'^\dagger\left\{i\frac{\partial}{\partial t} - \gamma^0 M - \gamma^0\frac{\vec{\Pi}^2}{2M} - gA^0 + \gamma^0\frac{g}{2M}\vec{\sigma}\cdot\vec{B} + \gamma^0\frac{(\vec{\Pi}^2)^2}{8M^3} - \frac{ig}{8M^2}[\vec{\sigma}\cdot\vec{\Pi}, \vec{\sigma}\cdot\vec{E}] + \cdots\right\}\psi' , \tag{6}$$

where the dots stand for higher-order terms while

$$\vec{\Pi} = \frac{1}{i}\vec{\nabla} - g\vec{A} \tag{7}$$

and

$$E^k = F_a^{0k}\lambda_a , \tag{8a}$$

$$B^k = \tfrac{1}{2}\epsilon^{klm}F_a^{lm}\lambda_a . \tag{8b}$$

We work in the representation where

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{9}$$

so that we may identify quark and antiquark spinors by

 

$$\psi'(x)=\begin{pmatrix} Q(x) \\ \sigma_2\overline{Q}^\dagger(x)\end{pmatrix} \tag{10}$$

and write

$$\mathcal{L}_\psi=\mathcal{L}_Q+\mathcal{L}_{\overline{Q}}\,. \tag{11}$$

Referring to Eq. (6), we see that

$$\mathcal{L}_Q=Q^\dagger\left(i\frac{\partial}{\partial t}-M-h_0-h_1-h_2-\cdots\right)Q\,, \tag{12}$$

in which

$$h_0=\frac{\vec{\Pi}^2}{2M}+gA^0\,, \tag{13}$$

$$h_1=-\frac{g}{2M}\vec{\sigma}\cdot\vec{B}\,, \tag{14}$$

and

$$h_2=-\frac{(\Pi^2)^2}{8M^3}+\frac{ig}{8M^2}[\vec{\sigma}\cdot\vec{\Pi},\vec{\sigma}\cdot\vec{E}]\,. \tag{15}$$

The Lagrange function for the antiquark field has almost exactly the same structure. Placing $\overline{Q}^\dagger$ on the left introduces an overall minus sign from anticommuting the fields and transposes the matrices in the Lagrange function. This produces

$$\sigma_2\vec{\sigma}^T\sigma_2=-\vec{\sigma} \tag{16}$$

and

$$\overline{\lambda}_a=-\lambda_a^T\,, \tag{17}$$

where the superscript $T$ denotes matrix transposition. The $\overline{\lambda}_a$ represent the SU(3) color group in the charge-conjugated, antiquark representation. The derivatives in the Lagrange function may again be taken to operate to the right (with a sign change) by adding an irrelevant total derivative. Discarding the total derivative gives

$$\mathcal{L}_{\overline{Q}}=\mathcal{L}_Q[Q\to\overline{Q},\lambda_a\to\overline{\lambda}_a]\,. \tag{18}$$

Before employing this Foldy-Wouthuysen reduction, a few general remarks are appropriate. The use of the expansion implies that the rest mass $M$ is large compared to other energies and momenta in the problem. Since there is no extrinsic mass scale, this is a dynamical assumption. For bound states it implies that the binding energies are small compared to the rest mass; such levels are called threshold bound states. Because the virial theorem implies that the expectation value of the kinetic energy is of the same order of magnitude as the expectation value of the potential energy, the threshold bound states are nonrelativistic. The virial theorem further implies that it is not possible to treat the velocity $p/M$ and the coupling constant $g$ as independent expansion parameters in computing binding effects.[8]

The $1/M$ expansion assumes that $M$ is large in comparison with the momenta exchanged by the quanta of the $A_\mu$ fields. There are, however, radiative corrections which are cut off only by the fermion propagators at loop momenta on the order of $M$. The Foldy-Wouthuysen reduction introduces spurious ultraviolet divergences into such processes.[9] For example, $h_0$ contains a term $(g^2/2M)A^\mu A_\mu$. The Wick contraction of two such terms, one from each fermion line, gives a gluon bubble connecting the two fermions. In configuration space this graph gives a contribution to the potential proportional to $g^4/M^2R^3$, where $R$ is the spatial separation of the two fermions. The singularity at $R=0$ and the ultraviolet divergence of the corresponding scattering kernel in momentum space are spurious effects. Before the $1/M$ expansion, this term is a part of box graphs which are ultraviolet convergent.[5]

Let us turn now to define the static potential which corresponds to the formal $M\to\infty$ limit. In this limit the fermion Lagrange function becomes

$$\mathcal{L}_\psi\simeq Q^\dagger\left(i\frac{\partial}{\partial t}-M-gA^0\right)Q+\overline{Q}^\dagger\left(i\frac{\partial}{\partial t}-M-g\overline{A}^0\right)\overline{Q}\,, \tag{19}$$

where $\overline{A}^0$ is obtained from $A^0$ by replacing $\lambda_a$ with $\overline{\lambda}_a$. We should emphasize that even in this static limit the fermion fields are dynamical entities. They possess internal color degrees of freedom which are altered by the interaction with the gauge potential $A^0$. It is not possible to give a correct derivation of the static potential in non-Abelian gauge theories which treats the matter fields as fixed, external sources. To derive the Wilson loop formula, we express the static, four-point quark-antiquark time-ordered Green's function as a functional integral,

$$\langle 0|\,T(\overline{Q}(\underline{x})Q(x)Q^\dagger(x')\overline{Q}^\dagger(\underline{x}'))\,|0\rangle=\int[dA][dQ][dQ^\dagger][d\overline{Q}][d\overline{Q}^\dagger]\exp\left(iS_{\rm eff}(A)+i\int(dx)\mathcal{L}_\psi\right)\overline{Q}(\underline{x})Q(x)Q^\dagger(x')\overline{Q}^\dagger(\underline{x}')\,. \tag{20}$$

Here $S_{\rm eff}(A)$ is the action of the gauge field including gauge-fixing terms and the Faddeev-Popov determinant. The fermion piece of the functional integral is trivially performed, for it produces simply a product of Green's functions for the motion of the quark and antiquark in the external field $A_{\mu a}$. The prescription that the mass $M$ be considered as the limit $M-i\epsilon$, $\epsilon\to 0^+$, requires that these Green's functions are retarded. Hence the determinants of the Green's functions, which give the associated vacuum-vacuum

transformation functions, are unity. (There is no pair production in the $M \to \infty$ limit.) The Green's function for the quark obeys

$$\left(i\frac{\partial}{\partial t} - M - gA^0\right) G(\vec{x},t;\vec{x}',t') = i\delta(\vec{x}-\vec{x}')\delta(t-t'), \tag{21}$$

and is therefore given by

$$G(\vec{x},t;\vec{x}',t') = \theta(t-t')e^{-iM(t-t')}\delta(\vec{x}-\vec{x}')\mathcal{T}\exp\left(-ig\int_{t'}^{t} dx^0 A^0(\vec{x},x^0)\right). \tag{22}$$

The time-ordering symbol $\mathcal{T}$ refers to the order of the $\lambda_a$ matrices attached to $A_a^0(\vec{x},x^0)$; the fields $A_a^0(\vec{x},x^0)$ themselves are, of course, numerical quantities in this formula. The Green's function for the antiquark is obtained by simply replacing $A^0$ with $\bar{A}^0$. Accordingly, we now secure

$$\langle 0|\, T(\bar{Q}_\alpha(\underline{x})Q_\beta(x)Q^\dagger_\gamma(x')\bar{Q}^\dagger_\delta(\underline{x}'))\,|0\rangle = \delta(\vec{x}-\vec{x}')\delta(\underline{\vec{x}}-\underline{\vec{x}}')\theta(\underline{t}-\underline{t}')e^{-iM(t-t')}\theta(\underline{t}-\underline{t}')e^{-iM(\underline{t}-\underline{t}')}$$
$$\times \int [dA]e^{iS_{\text{eff}}(A)}\left[\mathcal{T}\exp\left(-ig\int_{t'}^{t} dx^0 A^0(\vec{x},x)\right)\right]_{\beta\gamma}\left[\mathcal{T}\exp\left(-ig\int_{\underline{t}'}^{\underline{t}} dx^0\bar{A}^0(\vec{x},x^0)\right)\right]_{\alpha\delta}. \tag{23}$$

For clarity the color labels $\alpha,\beta,\gamma,\delta$ have been displayed explicitly.

The static potential is identified if the quark and antiquark are created and destroyed at common times.[10] Introducing a complete set of intermediate energy eigenstates and reverting back to a matrix notation for the color indices, we have

$$G_2(\vec{x},\underline{\vec{x}},\tfrac{1}{2}T;\vec{x}',\underline{\vec{x}}',-\tfrac{1}{2}T) = \langle 0|\bar{Q}(\underline{\vec{x}},\tfrac{1}{2}T)Q(\vec{x},\tfrac{1}{2}T)Q^\dagger(\vec{x}',-\tfrac{1}{2}T)\bar{Q}^\dagger(\underline{\vec{x}}',-\tfrac{1}{2}T)|\,0\rangle$$
$$= \sum_n \langle 0|\bar{Q}(\underline{\vec{x}},0)Q(\vec{x}',0)|n\rangle\, e^{-iE_nT}\langle n|Q^\dagger(\vec{x}',0)\bar{Q}^\dagger(\underline{\vec{x}}',0)|\,0\rangle. \tag{24}$$

To identify the two-particle potential we must take the large-$T$ limit which isolates the contribution of the lowest-energy intermediate state. Comparing Eq. (24) with the functional integral representation (23) in this limit, we infer that

$$C^\dagger e^{-iV(\underline{\vec{x}}-\vec{x})T}C = \int [dA]e^{iS_{\text{eff}}(A)}\mathcal{T}\left[\exp\left(-ig\int_{-T/2}^{T/2} dx^0 A^0(\vec{x},x^0)\right)\exp\left(-ig\int_{-T/2}^{T/2} d\underline{x}^0\bar{A}^0(\underline{\vec{x}},\underline{x}^0)\right)\right], \tag{25}$$

where $C$ is a time-independent matrix wave function and it is implicitly assumed that the phase accumulation associated with self-energy effects in isolated, free-particle propagation is to be deleted. The quark and antiquark can be in either the singlet ($S$) or adjoint ($A$) representation of the SU(3) color group. We show in the Appendix that the projection matrices into these states are given by

$$P^{(S)}_{\alpha\beta,\gamma\delta} = \tfrac{1}{9}(\delta_{\alpha\delta}\delta_{\beta\gamma} - 6\bar{\lambda}_{\alpha\delta}\cdot\lambda_{\beta\gamma}) \tag{26a}$$

and

$$P^{(A)}_{\alpha\beta,\gamma\delta} = \tfrac{1}{9}(8\delta_{\alpha\delta}\delta_{\beta\gamma} + 6\bar{\lambda}_{\alpha\delta}\cdot\lambda_{\beta\gamma}). \tag{26b}$$

Using these projection matrices in conjunction with Eq. (25) isolates the potentials for the singlet and adjoint states, $V_S$ and $V_A$.

The singlet potential can be obtained by setting $\beta=\alpha$, $\gamma=\delta$ and summing over each pair of common indices, for

$$P^{(S)}_{\alpha\alpha,\delta\delta} = 3, \tag{27a}$$

while

$$P^{(A)}_{\alpha\alpha,\delta\delta} = 0. \tag{27b}$$

Recalling that $\bar{\lambda}_a = -\lambda_a^T$, we see that

$$\left[\mathcal{T}\exp\left(-ig\int_{-T/2}^{T/2} d\underline{x}^0\bar{A}^0(\underline{x},x^0)\right)\right]_{\alpha\delta}$$
$$= \left[\bar{\mathcal{T}}\exp\left(-ig\int_{T/2}^{-T/2} d\underline{x}^0 A^0(\underline{x},x^0)\right)\right]_{\delta\alpha}, \tag{28}$$

where $\bar{\mathcal{T}}$ calls for antitime ordering. Therefore, neglecting constant factors, we secure

$$e^{-iV_S(\underline{\vec{x}}-\vec{x})T}$$
$$= \int [dA]e^{iS_{\text{eff}}(A)}\text{tr}\mathcal{P}\exp\left(ig\oint dx'^\mu A_\mu(x')\right). \tag{29}$$

This is the Wilson loop formula.[6] We avoid the $A^0=0$ gauge so that all components $A^\mu(x,t)$ vanish for large times. Hence the loop integral in Eq. (29) may be taken to run around a closed rectangular contour with vertices at $(\vec{x},-\frac{1}{2}T)$, $(\vec{x},+\frac{1}{2}T)$, $(\underline{\vec{x}},+\frac{1}{2}T)$, $(\underline{\vec{x}},-\frac{1}{2}T)$. The symbol $\mathcal{P}$ indicates that the $\lambda_a$ matrices in the potential $A_\mu(x')$ are to be ordered about this path and tr denotes the trace in the color space. Since the functional integral defines the vacuum expectation value of time-ordered operators in the interacting non-Abelian field

system, we can write the loop formula as

$$e^{-iV_S(\vec{x}-\vec{\underline{x}})T}=\left\langle T\,\mathrm{tr}\mathcal{P}\exp\left(ig\oint dx'^{\mu}A_{\mu}(x')\right)\right\rangle. \quad (30)$$

Note that $T$ orders the quantized operators $A_{\mu a}(x')$ in time, while $\mathcal{P}$ places the matrices $\lambda_a$ in a sequence around the closed loop which differs from the time-ordering of their attached $A_{\mu a}(x')$ operators.[11]

Before proceeding, we wish to discuss a bit more the meaning of computing the potential from Eq. (30). So far we have neglected kinetic-energy contributions which, however, we have noted will not be negligible compared to the effect of the potential energy. The relevant argument must be based, therefore, on an approximation of the Born-Oppenheimer type. One is assuming that the gluon fields exchanged between the quarks adjust themselves to the positions of the quarks on a time scale which is short compared to the orbital period of the quarks. We shall now work in the radiation gauge. Such an assumption is obviously valid for the instantaneous potential associated with the time component of the vector mesons, but it is suspect when considering the exchange of transverse virtual gluons which can live for long times. In fact, we shall see that the static approximation breaks down for these transverse gluons in a manner familiar from hydrogen or positronium calculations.[12] Therefore, we begin by calculating the Coulomb-like potential resulting from the instantaneous exchange and then develop a perturbation method to include the effects of transverse exchange and the spin and recoil corrections.

The Coulomb exchange approximation evaluates the functional integral of some functional $F[A]$ as

$$\int[dA]e^{iS_{\mathrm{eff}}(A)}F[A]=\exp\left(\frac{i}{2}\int(dy)(dz)\frac{\delta}{\delta A_a^0(y)}D_{ab}^{00}(y-z)\frac{\delta}{\delta A_b^0(z)}\right)F[A]\Big|_{A=0}. \quad (31)$$

Since the Coulomb propagator

$$D_{ab}^{00}(y-z)=\frac{\delta_{ab}\delta(y^0-z^0)}{4\pi|\vec{y}-\vec{z}|} \quad (32)$$

is instantaneous, using Eq. (31) to compute the potential from formula (25) produces matrix combinations $\lambda\cdot\bar{\lambda}$ with both $\lambda_a$ and $\bar{\lambda}_a$ at the same position in the time ordering. Hence the result involves only $\lambda\cdot\bar{\lambda}$ which, of course, commutes with itself. There is no problem with matrix ordering and we derive easily

$$V(\vec{R})=\frac{g^2\lambda\cdot\bar{\lambda}}{4\pi R}. \quad (33)$$

Comparing this with the projection matrices (26) gives the potential in the singlet state,

$$V_S(\vec{R})=-\frac{4}{3}\frac{g^2}{4\pi R}, \quad (34a)$$

and the potential in the adjoint state,

$$V_A(\vec{R})=+\frac{1}{6}\frac{g^2}{4\pi R}. \quad (34b)$$

The kinetic energy modifies the Green's function defined in Eq. (21) by replacing $gA^0$ with $p^2/2M+gA^0$. Otherwise the analysis of the instantaneous interaction goes through as described above. The effect of this modification is to produce a two-particle Green's function $G_2(\vec{\underline{x}},\vec{x},\frac{1}{2}T;\vec{x}',\vec{\underline{x}}',-\frac{1}{2}T)$ [Eq. (24)] which corresponds to the Hamiltonian

$$H_0=\frac{\vec{p}^2}{2M}+\frac{\vec{\underline{p}}^2}{2M}+V(\vec{x}-\vec{\underline{x}}). \quad (35)$$

Both the corrections to the spin-independent potential from transverse gluon effects and the spin-dependent interaction terms of higher order in $M^{-1}$ displayed in Eqs. (14) and (15) can be treated as perturbations, changing $G_2$ to $G_2+\delta G_2$. To illustrate the character of these corrections which are nonlocal in time, we consider the contribution arising from $h_1$ of Eq. (14). We work to all orders in the instantaneous Coulomb exchange but to lowest order in the transverse exchange. Therefore, the functional integral over the transverse gluons reduces to a simple Wick contraction of the fields, replacing $iA_a^k(y)A_b^l(z)$ by the transverse propagator $D_{ab}^{kl}(y-z)$. The correction we are now considering is illustrated by the Feynman graph of Fig. 1. Before the contraction of the transverse gluons is performed, the Coulomb exchange perturbed by $h_1$ on the quark and $\bar{h}_1$ on the antiquark produces

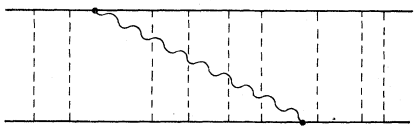

FIG. 1. Feynman diagram for the spin-spin interaction. Solid lines indicate quark or antiquark propagation; dashed lines correspond to instantaneous, "Coulomb" exchange; the wavy line corresponds to the transverse gluon which mediates the basic magnetic interaction.

$$\delta G_2^{\mathrm{Coul}}(T/2;-T/2)=\int_{-T/2}^{T/2}dt\int_{-T/2}^{t}d\underline{t}\,G_2(T/2;t)(-ih_1)G_2(t;\underline{t})(-i\bar{h}_1)G_2(\underline{t};-T/2)+(h_1\rightarrow\bar{h}_1)\,. \tag{36}$$

We are now using the usual operator notation of two-particle quantum mechanics. The integration limits appear as shown since the Green's functions are retarded. The contraction for the transverse gluon exchange now yields

$$\delta G_2(T/2;-T/2)=-i\int\frac{(d^4k)}{(2\pi)^4}\,\frac{\delta^{lm}\vec{k}^2-k^lk^m}{k^2-i\epsilon}\,\delta_{ab}\int_{-T/2}^{T/2}dt\int_{-T/2}^{t}d\underline{t}\,e^{-iH_0(T/2-t)}\frac{ig}{2M}\sigma^l\lambda_a e^{i(\vec{k}\cdot\vec{x}-k^0t)}$$
$$\times\, e^{iH_0(t-\underline{t})}\frac{ig}{2M}\bar{\sigma}^m\bar{\lambda}_b e^{-i(\vec{k}\cdot\vec{x}-k^0\underline{t})}e^{-iH_0(\underline{t}+T/2)}\,, \tag{37}$$

where we have dropped the second term involving the interchange of quark and antiquark variables since the structure in Eq. (37) suffices to illustrate the points which we wish to make.

Since we are interested in the energy shift which follows from this variation of the Green's function, we take its diagonal matrix element in some state $|\phi\rangle$ which is an energy eigenstate of the unperturbed Hamiltonian,

$$H_0|\phi\rangle=E|\phi\rangle\,. \tag{38}$$

Since $t-\underline{t}$ is positive, we can perform the $k^0$ integral by closing the integration contour in the lower half plane to obtain

$$\langle\phi|\delta G_2(T/2;-T/2)|\phi\rangle=-\frac{g^2}{4M^2}e^{-iET}\int\frac{(d^3k)}{(2\pi)^3}\,\frac{1}{2|\vec{k}|}(\delta^{lm}\vec{k}^2-k^lk^m)$$
$$\times\int_{-T/2}^{T/2}dt\int_{-T/2}^{t}d\underline{t}\langle\phi|\sigma^l\lambda_a e^{i\vec{k}\cdot\vec{x}}e^{-i(t-\underline{t})(H_0-E+|\vec{k}|)}\bar{\sigma}^m\bar{\lambda}_a e^{-i\vec{k}\cdot\vec{x}}|\phi\rangle\,. \tag{39}$$

The time integrals yield

$$\int_{-T/2}^{T/2}dt\int_{-T/2}^{t}d\underline{t}\,e^{-i(t-\underline{t})(H_0+|\vec{k}|-E)}=\frac{-iT}{H_0+|\vec{k}|-E}+\frac{1}{(H_0+|\vec{k}|-E)^2}\left[1-e^{-i(H_0+|\vec{k}|-E)T}\right]. \tag{40}$$

The oscillatory term effectively vanishes in the limit $T\rightarrow\infty$, while the time-independent term is a wave-function renormalization effect. The term linear in $T$ gives an energy according to

$$e^{-i(E+\Delta E)T}\simeq e^{-iET}(1-i\Delta ET)\,, \tag{41}$$

and so we obtain

$$\Delta E=-\frac{g^2}{4M^2}\int\frac{(d^3k)}{(2\pi)^3}\,\frac{1}{2|\vec{k}|}(\delta^{lm}\vec{k}^2-k^lk^m)\left\langle\phi\left|\sigma^l\lambda_a e^{i\vec{k}\cdot\vec{x}}\frac{1}{H_0+|\vec{k}|-E}\bar{\sigma}^m\bar{\lambda}_a e^{-i\vec{k}\cdot\vec{x}}\right|\phi\right\rangle. \tag{42}$$

The energy shift is given by a nonlocal operator rather than by a correction to the potential. Let us now take $|\phi\rangle$ to be a color-singlet state. Since the operation of $\bar{\lambda}_a$ changes $|\phi\rangle$ to the adjoint representation, only the potential $V_A$ appears in $H_0$. If we had ignored the kinetic energy in our calculation, the energy denominator in Eq. (42) would be a numerical quantity and the energy shift would be given by a local spin-dependent potential,

$$\Delta V_s(\vec{R})=\frac{g^2}{4M^2}\,\frac{4}{3}\int\frac{(d^3k)}{(2\pi)^3}e^{i\vec{k}\cdot\vec{R}}\,\frac{\vec{k}^2\vec{\sigma}\cdot\vec{\underline{\sigma}}-\vec{\sigma}\cdot\vec{k}\vec{\underline{\sigma}}\cdot\vec{k}}{2|\vec{k}|\left[|\vec{k}|+V_A(\vec{R})-V_s(\vec{R})\right]}\,. \tag{43}$$

The graph has been considered in this approximation by Dine[5] who notes also that the neglect of quark motion in such a process is not justified in perturbation theory.

For Coulombic states in the weak-coupling limit, the typical wave numbers which contribute to the integral in Eq. (42) are given by $k\sim a_0^{-1}$, where $a_0=(\alpha M)^{-1}$ is the Bohr radius with $\alpha=g^2/4\pi$. Since the unperturbed binding energies are of order $\alpha^2M$, the $H_0-E$ contribution to the denominator is a higher-order effect. If we neglect it, we can carry out the integral to obtain the spin-spin part of the Breit Hamiltonian. The remaining pieces of the Breit Hamiltonian are obtained from the other $M^{-2}$ effects which arise from perturbing with $h_1$ and $h_2$ while neglecting binding corrections. All of these terms give corrections to Coulombic energies of order $\alpha^4M$. The effects of the $H_0-E$ term which invalidates the instantaneous potential model are of higher order by an additional factor of $\alpha$. These nonlocal terms contribute to the energy at the same order as the Lamb shift which is

also nonlocal and requires the inclusion of binding effects.

In the ground state, the effect of the energy denominators is to decrease the spin-spin interaction strength from that predicted by the Breit Hamiltonian. In an intermediate coupling regime such as that of the $J/\psi$ family, it is tempting to think that such an effect may be relevant in understanding the $\psi-\eta_c$ splitting. However, we cannot say anything definite about this since we have not considered effects such as that of anomalous color magnetic moments which might be important in this region.[13,14]

Therefore, it does not seem meaningful to us to derive a potential between static quarks to arbitrary orders of perturbation theory and assume the result is relevant for computing bound states in a nonrelativistic Schrödinger equation for heavy quarks. We have illustrated this point by considering the binding-energy corrections to the magnetic interaction produced by the exchange of a single transverse gluon. (See Fig. 1.) Similar phenomena arise in any process which involves the exchange of transverse virtual gluons since these gluons can propagate over long time intervals. For example, similar energy-denominator effects arise in the process displayed in Fig. 2 which involve corrections to the spin-independent potential from the exchange of a transverse gluon between two Coulomb gluons.

Kinetic-energy effects should also be included in attempts to calculate the Van der Waal's forces between Coulombic singlet bound states which have been considered only in the static quark limit by Bhanot, Fischler, and Rudaz.[15]

The Coulombic states we have used should describe very heavy quarks whose effective coupling constant is small. However, the argument that potential and kinetic energies are of the same magnitude, which implies the breakdown of the local potential picture, follows from the virial theorem and should also apply to an intermediate coupling region relevant to the $J/\psi$ and $\Upsilon$ families of states. For example, the physical relevance of the potential between static sources calculated by Giles and McLerran[16] using a nonperturbative semiclassical method is suspect on this basis.

We have not addressed ourselves directly to attempts to compute long-range potentials or spin-dependent effects which result from contributions of nonperturbative gauge field configurations (instantons and merons) to the Wilson loop formula.[17,18] From the general arguments presented here, it appears that a local potential can be associated with such effects only if their duration is short compared to the orbital period of the fermions. The same observation has been made previously by Dine.[5] Such a limit seems to be present in the *ad hoc* instanton scale cutoff used by Callan *et al.*[17,18] Whether or not the effects of small-scale instantons do indeed dominate over those of long duration appears to be an open question.[19]

We cannot rule out the possibility that some summation of diagrams or other nonperturbative effect may provide *a posteriori* justification for a static potential. The motivation in searching for such a result is the apparent success of potential models in charmonium spectroscopy. The quantitative evidence, however, does not rule out the level-dependent, energy-denominator effects we have discussed here. The most striking feature of the quarkonium states is the success of describing them as levels of a two-body nonrelativistic system, which can survive in either case.

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## APPENDIX

We consider an SU($N$) color group. The $N^2-1$ generators of the defining representation $\lambda_a$ are taken to have an isospin normalization,

$$\mathrm{tr}\lambda_a\lambda_b=\tfrac{1}{2}\delta_{ab}\,. \tag{A1}$$

Hence

$$\lambda\cdot\lambda=\sum_a\lambda_a\lambda_a=\frac{N^2-1}{2N}\times 1\,. \tag{A2}$$

The generators of the charge-conjugate representation are given by $\bar{\lambda}_a=-\lambda_a^T$. The direct product of the two representations has the generating matrices

$$I_a=\lambda_a+\bar{\lambda}_a\,, \tag{A3}$$

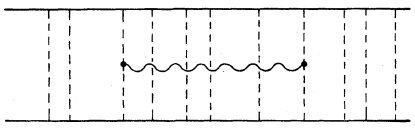

FIG. 2. Feynman diagram for the lowest-order non-Abelian gluon interaction.

which give the invariant

$$I\cdot I=\frac{N^2-1}{N}\times 1+2\lambda\cdot\bar{\lambda}\,. \tag{A4}$$

The direct product of the two $N$-dimensional representations reduces into a singlet with one component and an adjoint with $N^2-1$ components. In the singlet subspace $I_a=0$ and hence

$$(I\cdot I)_S=0\,. \tag{A5a}$$

To compute the value of the invariant in the adjoint subspace we take the trace of Eq. (A4):

$$\mathrm{tr}I\cdot I=(N^2-1)(I\cdot I)_A=\frac{N^2-1}{N}N^2\,, \tag{A6}$$

whence

$$(I\cdot I)_A=N\,. \tag{A5b}$$

Projection matrices into the two subspaces, $P^{(S)}$ and $P^{(A)}$, can be immediately constructed from the corresponding values

$$\lambda\cdot\bar{\lambda}=\begin{cases}-\dfrac{N^2-1}{2N}\,, & \text{singlet}\\[2ex] \dfrac{1}{2N}\,, & \text{adjoint}\end{cases} \tag{A7}$$

and the completeness property

$$P^{(S)}+P^{(A)}=1\,. \tag{A8}$$

Thus

$$P^{(S)}=\frac{1-2N\lambda\cdot\bar{\lambda}}{N^2} \tag{A9a}$$

and

$$P^{(A)}=\frac{N^2-1+2N\lambda\cdot\bar{\lambda}}{N^2}\,. \tag{A9b}$$

With these operators and standard algebraic techniques one can project the singlet and adjoint pieces which are generated from any terms in the expansion of the Green's function.

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[7]This is the familiar transformation of quantum electrodynamics which, for example, is explained in some detail in J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964), Chap. 4, except that care must be exercised in the order of the $\lambda_a$ matrices. As far as we are aware, the simple form of the order $M^{-2}$ contribution exhibited in Eq. (6) has not appeared in the literature. Since we consider the fields as numerical integration variables in a functional integral, there is no explicit problem with operator ordering. However, the resulting functional Jacobian determinant does give rise to closed-loop terms, but we shall neglect these pair creation effects.

[8]It is usually assumed that the heavy quark masses in the asymptotically free chromodynamic theory are large in comparison with the renormalization mass scale for which the coupling constant is of order unity. That is, heavy quarks imply weak coupling.

[9]In principle, these divergences should be cancelled by the functional Jacobian determinant mentioned in Ref. 7.

[10]In some circumstances it is convenient to keep all the times different so as to avoid spurious time dependences associated with the sudden creation and annihilation of the quark and antiquark. The interaction energy is then identified from the phase accumulated during the time $T$ between the latest creation and earliest annihilation.

[11]The derivations of the loop formula in Refs. 1 and 3 appear to be in error both in the use of static sources and in the lack of distinction between the two different sorts of ordering. However, their perturbative calculations conform to our Eq. (30). Our notation agrees with that of G. Tiktopoulos, Nucl. Phys. B145, 510 (1978).

[12]The limitations of a static potential model for the two-body problem in quantum electrodynamics are recognized in Refs. 1 and 5. However, those authors express the hope that analogous problems can be avoided in quantum chromodynamics by treating the kinetic and potential energies as independent expansion parameters. As discussed before, such an assumption is in apparent contradiction with the virial theorem.

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