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Approximation scheme for quantum chromodynamics

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An approximation scheme is proposed for calculating masses of $Q\bar{Q}$ mesons in quantum chromodynamics. The vacuum instability is removed, not by giving the magnetic field an expectation value, but by enhancing the amplitude of the low-frequency components of the virtual magnetic field. A simple truncation of the Schwinger-Dyson equations incorporates such an effect. Rough numerical calculations indicate that the equations for the gluon propagator have a solution which increases logarithmically at large distances. Clustering does not hold for multigluon Green's functions. Instead, if generally believed properties are found to be true for the two-gluon function, they are assumed to be true for the higher functions. Equations for the quark propagator and the $Q\bar{Q}$ bound states are set up. The ladder approximation is totally inadequate in the confinement region, but we can make a reasonable approximation which leads to static forces at large distances. Corrections probably involve vibrational modes of the dual string. At high energy, linearly rising $Q\bar{Q}$ Regge trajectories with the expected quantum numbers are found. The scheme is free from dimensionless parameters if the quarks have zero bare mass. Our equations possess the possibility of chiral-symmetry breaking. Massless "pions" would then appear as $Q\bar{Q}$ bound states; massless η's would not appear if closed quark loops were included.

I. INTRODUCTION

Our aim in this paper is to propose an approximation scheme for low-energy calculations in quantum chromodynamics (QCD). We shall direct our approach towards the calculation of $Q\bar{Q}$ mesonic mass spectra. If our scheme, or any other scheme, turns out to be successful in this respect, it can probably be extended to other calculations.

Let us state at the outset that we cannot motivate our scheme as the expansion in a small parameter or, alternatively, that we know of no consistent limiting case in which our approximation scheme would be exact. To our knowledge the only possible small parameter which has been suggested is $1/N$ (N being the number of colors) and, indeed, if a procedure was discovered which made no approximation other than the smallness of $1/N$, it would represent a breakthrough in our knowledge of strong interactions. At the present time no such procedure is known. The situation in this regard is not very different from that in many branches of physics. In the absence of a small-parameter approximation scheme, what is wanted is an approximation which incorporates the main qualitative features of the physics. It is then not unreasonable to hope that quantitative calculations could reproduce the correct results fairly well. At present we are still far from determining whether higher approximations of our scheme give a convergent sequence.

We aim for an approximation that produces the confinement properties of the theroy. It has been argued that the confining forces might only become important at distances large compared to the radii of low-lying hadrons and that an understanding of confinement might not be necessary for calculations involving such hadrons. The near linearity of certain Regge trajectories suggests, however, that the forces responsible for their lower members should be similar to those responsible for the high angular momentum states, where confinement is certainly crucial.

Most if not all of the suggestions of a confinement mechanism involve the assumption that the low-frequency components of the virtual color magnetic field in the true vacuum have a larger amplitude than those of the bare vacuum. Such a hypothesis is implied in the statement that the gluon propagator diverges faster than $(p^2)^{-1}$ at small p, and is also true for the monopole vacuum proposed by Mandelstam[1] and 't Hooft.[2] In the present paper we shall not introduce monopoles explicitly, but shall construct Green's functions for a vacuum with enhanced low-frequency components. Such a vacuum should have similar properties to a vacuum which is explicitly constructed as a monopole plasma. The equations used for the Green's functions will be truncated Schwinger-Dyson equations. To motivate our equations and, in particular, the method of truncation, we start from the magnetic instability of the QCD vacuum, which has recently been extensively studied.

Saviddy[3] and Wilczek[4] have pointed out that one obtains a lowering of the vacuum energy by giving the magnetic field a nonzero vacuum expectation value. Their calculations were performed in the one-loop approximation, but their results appeared to be a general feature of any infrared unstable system. The instability is related to the "Landau ghost," i.e., to the pole which appears in the gluon propagator when bubble diagrams are summed. The energy density caused by a magnetic field $\mathcal{H}$ is $c g^2\mathcal{H}^2 \ln\mathcal{H}$, and for small $\mathcal{H}$, this term will outweigh the classical $\mathcal{H}^2$ term. The fact that there is a stable value of $\mathcal{H}$ which minimizes the total (classical and quantum) energy may lead us to hope that there is a similar stability in the gluon propagator, i.e., that a better treatment would cause the pole to move out of the spacelike region, probably to the point $p^2=0$.

If QCD is the theory describing strong interactions, the vacuum expectation value of the magnetic field cannot be nonzero, since Lorentz invariance would then be spontaneously broken. We should like to suggest that the vacuum instability implies not that the zero-frequency component of the magnetic field has a vacuum expectation value, but that the low-frequency modes are enhanced. We have already mentioned that confinement is probably implied thereby. A method of implementing this feature is suggested by a calculation of the vacuum energy by Nielsen and Olesen,[5] who introduced a fixed, unquantized magnetic field and summed the energies of the gluon modes. As expected, the low-frequency gluon modes were enhanced. (Nielsen and Olesen showed that, though the energy was lowered, there was still a remaining instability in their vacuum. As it is not our intention to give $\mathcal{H}$ a nonzero expectation value, we shall not pursue this point.)

We shall replace the unquantized color magnetic field by a virtual quantized magnetic field. In order to remain as close as possible to the calculation of Nielsen and Olesen, we shall not allow this color magnetic field to interact with itself. In other words, we have two gluon fields, one representing the actual gluons and one representing the virtual field. Gluons of the first type can emit or absorb those of the second, but otherwise there are no interactions. If, in addition, we neglect nonplanar diagrams and closed gluon loops, we obtain a soluble equation for the gluon propagator, represented diagramatically in Fig. 1.

The gluon-propagator equation, improved by the inclusion of Faddeev-Popov ghosts, will be examined in Sec. II. As we have mentioned, it is a truncated Schwinger-Dyson equation, and a comparison will be given with other possible methods of truncation. It is not our intention in this paper to present detailed numerical computations, but we shall attempt rough solutions with simple parametrizations in order to determine whether or not a solution exists. We shall find that a fairly simple parametrization gives a solution accurate to 5%. We believe that this is fairly strong evidence that a solution exists; if no solution existed it would probably be impossible to come close. Our solution behaves like $(p^2)^{-2}$ for small p, or, in other words, like $\ln|x|$ at large x. We shall also examine the ultraviolet behavior of our solutions and shall compare it with the behavior given by renormalization-group analysis.

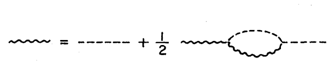

FIG. 1. Simplified integral equation for gluon propagator.

In Sec. III we shall treat the multigluon Green's functions. In any approximation scheme of any theory one has to make an assumption regarding such functions. One usually assumes that they are given by the sum of disconnected diagrams, but, in the present system, which does not possess clustering, such an assumption is not applicable. We shall base our assumptions on the Wilson-loop amplitude; if properties which are generally believed to be true (and, in particular, the shape independence of the area terms) do in fact hold for the two-gluon function, we shall assume that they will continue to hold for multigluon Green's functions. We shall then be able to obtain an ansatz for the loop integrals of multigluon Green's functions, which is all that we require. We shall find that the ladder approximation to the Bethe-Salpeter equation is totally inadequate in the confinement regime, but that it is a reasonable approximation to replace the $\ln|x|$ term by a linear static potential. Corrections to this approximation probably involve including vibrational modes of the "string."

In Sec. IV we shall study the quark propagator. As in the two-dimensional model,[6] the equations appear to be straightforward, with no divergences in the infrared or ultraviolet regions. The equations for the quark propagator, as well as those for the $Q\overline{Q}$ bound state, contain no arbitrary dimensionless parameter. We also examine the possibility of solutions with chiral-symmetry breaking. The general analysis of Maris, Herscovitz, and Jacob,[7] Pagels and Langacker,[8] Lane[9] and others is applicable here. In fact, the physical situation in this respect is not very different from that of the original model of Nambu and Jona-Lasinio; the direct four-fermion interaction of the latter model is here replaced by an interaction through gluon exchange. We shall show that a simplified form of our equations does lead to chiral-symmetry breaking. Treatment of our actual equations would involve a numerical computation, itself dependent on the computation of the gluon propagator, but the results just mentioned show that our equations contain the strong possibility of chiral-symmetry breaking.

The $Q\bar{Q}$ bound-state equation will be treated in Sec. V. Our two-variable integral equation simplifies greatly in the region of high energy and angular momentum—the "confinement" region. Linear Regge trajectories with the expected quantum numbers are found; not a surprising result in view of the fact that we have a linear static potential. If the chiral symmetry of the quark propagator is spontaneously broken, the bound-state equation will have solutions corresponding to massless pseudoscalar particles.

In most of our treatment we neglect closed quark loops, but we do examine the question whether the inclusion of such loops changes this last conclusion for the flavor-singlet sector. The original explanation of the η mass involved the Adler-Bell-Jackiw anomaly and instantons.[10] A gluon propagator which increases instead of decreasing at large distances implies that the vacuum possesses components with fields that decrease like $(r^2)^{-1}$. The topological distinction which characterizes instanton states then disappears. Witten has examined this question in the large-N limit of a two-dimensional model[11] ($N\to\infty$, g^2N fixed). He found that the η mass behaved like $N^{-2/3}$ rather than like e^{-N} and could be calculated without the explicit introduction of instantons. In our calculations we would expect a similar feature; the potentials or fields fall off sufficiently slowly at large distances so that the anomalous term in the axial-charge density diverges when integrated over space. There is no conserved axial charge and no reason to expect a massless η. These expectations are confirmed; if we include closed fermion loops we do not find a massless pseudoscalar particle in the flavor-singlet sector.

II. THE GLUON PROPAGATOR

A. Discussion of approximation

As we have already explained, we use the approximation shown in Fig. 1; wavy lines represent the full gluon propagator, dashed lines the bare propagator. For the moment we ignore the Faddeev-Popov ghosts.

We may regard our approximation as a set of truncated Schwinger-Dyson equations. The exact equations for the gluon propagator are shown in Fig. 2. (Tadpole diagrams have been neglected.) Ghost propagators have been represented by dotted lines. Comparing Figs. 1 and 2, we may be tempted to include further corrections to the former set of equations and, in particular, to use full gluon propagators for both internal lines in the bubble. It is well known, however, that such a procedure would be incorrect. The Ward identity shows that for low-frequency gluons there are cancellations between the corrections to the second internal propagator and those to the vertex function. In fact, if the propagator has a $(p^2)^{-2}$ behavior for small p, as is necessary for confinement, the vertex has an extra p^2 which compensates the factor p^{-2} in the second internal line. We must therefore use the bare propagator for the second internal line.

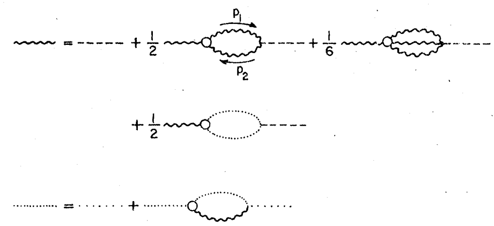

FIG. 2. Exact Schwinger-Dyson equations for gluon and ghost propagators.

Another attractive possibility might be to use the Ward identity directly to obtain information about the vertex function. The Ward identity provides information about the physical (i.e., transverse) vertex function if one momentum is much smaller than the other two, and an approximation which suggests itself is to regard the transverse Ward identity, valid when

$$p_1 \ll p_2\,,\quad p_1 \ll p_3 \tag{2.1}$$

as valid when

$$p_1 < p_2\,,\quad p_1 < p_3\,. \tag{2.2}$$

An approach similar in spirit has been followed by Baker, Ball, and Zachariasen.[12] However, we shall see in the sequel that a cancellation is necessary to obtain a $(p^2)^{-2}$ behavior. Such a cancellation does occur in our equations, but not in the equations derived from the Ward identity. This feature might at first appear disconcerting, especially since the canceling terms depend only on the application of the transverse Ward in the region (2.1), where it is valid. In fact, the absence of the cancellation is not surprising, since the use of the Ward identity, without inclusion of the three-gluon term in Fig. 2, treats the quartic couplings inconsistently. Some of the quartic couplings are implicitly taken into account *via* the Ward identity, while others should be included explicitly *via* the three-gluon term in Fig. 2.

If one wishes to use the transverse Ward identity to obtain information about the vertex function, one must therefore include the three-gluon term. Even then it is by no means clear how we can obtain the required cancellation, since it will depend on the behavior of the four-gluon vertex when the momentum of one gluon is small. The Ward identity relates the four-gluon vertex in this limit, not to the gluon propagator, but to the unknown three-gluon vertex. A more detailed study of the three-gluon term might indicate a possible method of procedure. Without the inclusion of this term we know of no consistent truncation of the Schwinger-Dyson equations which improves on that represented in Fig. 1.

We should remark that our comments above apply specifically to equations similar in structure to those of the present paper. The equations of Ref. 12 have a slightly different structure. It is our opinion that our above comments can probably be generalized to apply to equations of different structure, but we have not proved this. If it is shown that a solution with a $(p^2)^{-2}$ behavior can be obtained from direct use of the Ward identity, our comments will of course be negated.

It might be objected that the absence of a $(p^2)^{-2}$ behavior shows not that the equations are wrong, but that the propagator does not have such a behavior. In fact, our equations cannot have solutions with any other power behavior, and Baker, Ball, and Zachariasen show that the same is true with their equations. It thus appears that a consistent solution, with other than a $(p^2)^{-2}$ behavior, must involve $(p^2)^{-2}$ together with powers of logarithms. Such a behavior appears to be much less

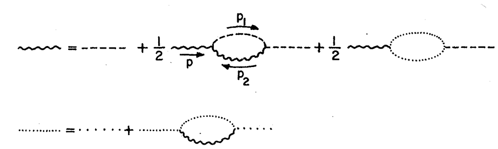

FIG. 3. Truncated Schwinger-Dyson equations for gluon and ghost propagators.

physically reasonable than a pure $(p^2)^{-2}$ solution; we shall comment further on this point in Sec. III. We therefore find it significant that a simple truncated Schwinger-Dyson equation does lead to a $(p^2)^{-2}$ solution.

The equations represented in Fig. 1 may be supplemented by contributions from the Faddeev-Popov ghosts. We then obtain the equations of Fig. 3, where the line with widely separated dots represents the bare ghost propagator. As there are no cancellations between the gluon-ghost vertex function and the propagators, we have used complete propagators for the internal lines of diagrams involving ghosts. In fact, the ghost contribution to the gluon propagator makes very little difference. We require knowledge of the ghost propagator, not mainly because of its contribution to the gluon propagator, but as an auxiliary function for identifying renormalization constants.

B. Setting up of equations

We now proceed to set up the equations for the diagrams in question. Let us start with the gluon bubble. Polarizations will be denoted by Greek indices λ, μ, σ, ρ. Color indices will be suppressed, as the bubble depends on them only through a Kronecker δ. We use the Landau gauge throughout. Integrals over components of momenta will be in Minkowski space with a timelike metric, but, when we integrate over invariants, we make a Wick rotation to Euclidean space. The bubble is then equal to

$$-\frac{g^2}{2(2\pi)^4}c_1\int dp_1dp_2\delta^4(p-p_1+p_2)[g^{\sigma\rho}(p_1+p_2)^\lambda-2g^{\lambda\rho}p_1^\sigma-2g^{\lambda\sigma}p_2^\rho]$$
$$\times F(p_2^2)\frac{1}{p_1^2p_2^2}\left(g_{\rho\rho'}-\frac{p_{1\rho}p_{1\rho'}}{p_1^2}\right)\left(g_{\sigma\sigma'}-\frac{p_{2\sigma}p_{2\sigma'}}{p_2^2}\right)[g^{\sigma'\rho'}(p_1+p_2)^\mu-2g^{\sigma'\mu}p_2^{\rho'}-2g^{\mu\rho'}p_1^{\rho'}]\,, \qquad (2.3)$$

where c_1 is the quadratic Casimir invariant for the adjoint representation of the group [N for SU(N)]. We have denoted the correction factor to the gluon propagator by $F(p^2)$. On doing the algebra we find

$$-\frac{g^2}{2(2\pi)^4}c_1\int dp_1dp_2\delta^4(p-p_1+p_2)F(p_2^2)\frac{1}{p_1^2p_2^2}$$
$$\times\Bigg\{(p_1-p_2)^\lambda(p_1-p_2)^\mu\left[-2-2p_1\cdot p_2\left(\frac{1}{p_1^2}+\frac{1}{p_2^2}\right)+\frac{(p_1\cdot p_2)^2}{p_1^2p_2^2}\right]+(p_1^\lambda p_2^\mu+p_2^\lambda p_1^\mu)\left[4+\frac{2(p_1\cdot p_2)^2}{p_1^2p_2^2}\right]$$
$$+[(p_1^\lambda+p_2^\lambda)(p_1^\mu-p_2^\mu)+(p_1^\mu+p_2^\mu)(p_1^\lambda-p_2^\lambda)](p_1\cdot p_2)\left(\frac{1}{p_2^2}-\frac{1}{p_1^2}\right)+g^{\mu\nu}\left[4(p_1^2+p_2^2)-4(p_1\cdot p_2)^2\left(\frac{1}{p_1^2}+\frac{1}{p_2^2}\right)\right]\Bigg\}\,. \qquad (2.4)$$

We first perform the angular integrations in (2.4). Let $p^2=-P$, $p_1^2=-P_1$, $p_2^2=-P_2$. We find that, after performing the angular integrations,

$$\int dp_1dp_2\delta^4(p-p_1+p_2)\to\frac{\pi}{2}\int\Delta P^{-1}dP_1dP_2\,\theta(P_1^{1/2}+P_2^{1/2}-P^{1/2})\theta(P_1^{1/2}+P^{1/2}-P_2^{1/2})\theta(P_2^{1/2}+P^{1/2}-P_1^{1/2})\,, \qquad (2.5a)$$

where

$$\Delta = [(P^{1/2}+P_2^{1/2})^2-P_1]^{1/2}[-(P^{1/2}-P_2^{1/2})^2+P_1]^{1/2}. \tag{2.5b}$$

The angular integrations over components of momenta give

$$\langle p_1^\lambda p_2^\mu\rangle \approx \langle p_2^\lambda p_1^\mu\rangle \approx -\langle p_1^\lambda p_1^\mu\rangle \approx -\langle p_2^\lambda p_2^\mu\rangle \approx \tfrac{1}{12}\Delta^2 P^{-1} g^{\lambda\mu}, \tag{2.6}$$

where the sign $\approx$ means up to terms involving p^λ or p^μ. Inserting these expressions in (2.4), we find the following expression for the bubble:

$$ic_1 g^{\lambda\mu}\frac{g^2}{48(2\pi)^3P^2}\int_0^\infty dP_2\int_{(P^{1/2}-P_2^{1/2})^2}^{(P^{1/2}+P_2^{1/2})^2} dP_1 P_1^{-2}P_2^{-2}F(P_2)[(P^{1/2}+P_2^{1/2})^2-P_1]^{3/2}[-(P^{1/2}-P_2^{1/2})^2+P_1]^{3/2}$$
$$\times[\tfrac{1}{2}P^2+5PP_2+\tfrac{1}{2}P_2^2+5P_1(P+P_2)+\tfrac{1}{2}P_1^2]+\text{longitudinal terms}. \tag{2.7}$$

The integration over P_1 may be performed explicitly, as it does not involve the unknown function $F(P_2)$. We find that the transverse part of the bubble is

$$\int K_1(P,P_2)F(P_2)dP_2, \tag{2.8a}$$

where

$$K_1=ic_1\frac{g^2}{192\pi^2}[18+(25P_2P^{-1}-\tfrac{7}{2}P_2^2P^{-2})\theta(P-P_2)$$
$$+(25P_2^{-1}P-\tfrac{7}{2}P_2^{-2}P^2)\theta(P_2-P)]. \tag{2.8b}$$

The other diagrams are somewhat simpler. For the ghost loop in the gluon propagator, we obtain

$$\frac{g^2}{2(2\pi)^4}c_1\int dp_1dp_2$$
$$\times\delta^4(p-p_1+p_2)p_1^\lambda p_2^\mu G(p_1^2)G(p_2^2)\frac{1}{p_1^2p_2^2}, \tag{2.9}$$

where G is the correction factor for the ghost propagator. Performing the angular integrations with the aid of (2.5) and (2.6), we find that the transverse part of the gluon loop is

$$\int K_2(P,P_1,P_2)G(P_1)G(P_2)dP_1dP_2, \tag{2.10}$$

where

$$K_2(P,P_1,P_2)=-ic_1\frac{g^2}{48(2\pi)^3}P^{-2}P_1^{-1}P_2^{-1}\Delta^3$$
$$\times\theta(P_1^{1/2}+P_2^{1/2}-P^{1/2})$$
$$\times\theta(P-(P_1^{1/2}-P_2^{1/2})^2). \tag{2.11}$$

The loop in the ghost propagator is given by the integral

$$\frac{g^2}{(2\pi)^4}c_1\int dp_1dp_2\delta^4(p-p_1+p_2)\left(p_1^2-\frac{(p_1\cdot p_2)^2}{p_2^2}\right)$$
$$\times G(p_1^2)F(p_2^2)\frac{1}{p_1^2p_2^2}, \tag{2.12}$$

and, after performing the angular integrations, we obtain

$$\int K_3(P,P_1,P_2)G(P_1)F(P_2)dP_1dP_2,$$

where

$$K_3(P,P_1,P_2)=-ic_1\frac{g^2}{8(2\pi)^3}P^{-1}P_1^{-1}P_2^{-2}\Delta^3$$
$$\times\theta(P_1^{1/2}+P_2^{1/2}-P^{1/2})$$
$$\times\theta(P-(P_1^{1/2}-P_2^{1/2})^2). \tag{2.13}$$

C. Solution of integral equations

The integral equations represented by Fig. 3 now take the form

$$F(P)=\Big[1-P^{-1}\int K_1(P,P_2)F(P_2)dP_2$$
$$-P^{-1}\int K_2(P,P_1,P_2)G(P_1)G(P_2)dP_1dP_2\Big]^{-1}, \tag{2.14a}$$

$$G(P)=\Big[1-P^{-1}\int K_3(P,P_1,P_2)G(P_1)F(P_2)dP_1dP_2\Big]^{-1}, \tag{2.14b}$$

the kernels K_1, K_2, and K_3 being given by (2.8), (2.11), and (2.13), respectively. To investigate the nature of their solutions, we shall first neglect the ghost term in (2.14a). We shall later return to consider it. Since the ghost term is much smaller than the gluon term (the relative contributions to the asymptotic behavior in perturbation theory are 1:25), we would not expect it to change the qualitative nature of the solution.

We thus have the equation

$$F(P)=\Big[1-P^{-1}\int K_1(P,P_2)F(P_2)dP_2\Big]^{-1}, \tag{2.15}$$

and we are interested in a solution where F behaves like P^{-1} as P approaches zero, corresponding to a $(p^2)^{-2}$ behavior of the propagator. First let us say precisely what we mean by such a behavior. On taking the Fourier transform, we find the following result:

$$\int [(p^2+i\epsilon)^{-2} + (8\pi)^{-1}\ln(k\epsilon r_0)\delta^4(p)]e^{ipr}dp = -2\pi^3\ln(r/r_0), \quad (2.16)$$

where the value of the constant k will not concern us. By a $(p^2)^{-2}$ behavior we shall always mean a logarithmic behavior in coordinate space. In other words, a factor $(p^2)^{-2}$ is to be replaced by the quantity in square brackets on the left of (2.16). We shall encounter integrals of the form $\int P_2{}^{-1}dP_2$, where the singularity at $P_2=0$ arises from the $(p^2)^{-2}$ singularity of the propagator. According to (2.16), we must replace the logarithmically divergent constant arising from the lower limit by an infrared finite constant whose value depends on r_0. The value of r_0 will not enter into any of our results; the only place it appears is in the gluon mass renormalization. Since we shall not be able to obtain a solution to our equations unless the reciprocal of the gluon propagator is zero at $p=0$, we choose the value of r_0 so as to give such a zero. In other words, we require the renormalized gluon mass to be zero. We then need not explicitly refer to the value of r_0. Notice that, in this way of looking at the problem, we do not introduce any physical information about the gluon mass. We simply choose the value of r_0 so as to solve our equations.

An alternative approach would be to recall that the reciprocal of the gluon propagator obtained from the exact, gauge-invariantly regularized, Schwinger-Dyson equations (including the tadpole terms) would vanish at $p=0$, as may be shown from the Ward identities. Hence, when truncating the Schwinger-Dyson equations, one might add a constant term to the reciprocal of the propagators so as to effect such vanishing. The constant r_0 would not then enter into any calculations. The actual method of procedure is independent of which approach we adopt.

We may now note that if we write

$$F(P_2) = A/P_2 + F_1(P_2), \quad (2.17)$$

and insert the term A/P_2 into the integral in the denominator of (2.15), the result is a mass-renormalization term of the form const $\times P^{-1}$ in the denominator. The term may therefore be neglected. This feature is due to the fact that the constant term in the brackets of (2.8b), i.e., the term 18, contained no θ function. If (2.8) had contained a term of the form $a\theta(P-P_2)+b\theta(P_2-P)$, a term $AP_2{}^{-1}$ in F would have given a contribution

$$[(b-a)\ln P + \text{const}]P^{-1}$$

to the denominator of P. Since there is no other term in the denominator which diverges like $P^{-1}\ln P$, the denominator could not possibly behave like P, and we could not obtain a solution of our equations which behaved like P^{-1} for small P. This is the cancellation referred to above, which occurs in our equations but not in certain other approaches to the problem.

Besides neglecting the first term on the right-hand side of (2.17) in the integral of (2.15), we can also neglect the term $18c_1g^2/(192\pi^2)$ in the kernel, as this term, too, gives a contribution proportional to P^{-1}. We may therefore rewrite Eq. (2.15) as

$$F(P) = AP^{-1} + F_1(P), \quad (2.18a)$$

$$F(P) = \left[1 - P^{-1}\int \tilde{K}_1(P,P_2)F_1(P_2)dP_2\right]^{-1}, \quad (2.18b)$$

$$\tilde{K}_1(P,P_2) = \frac{c_1g^2}{192\pi^2}[(25P_2P^{-1} - \tfrac{7}{2}P_2{}^2P^{-2})\theta(P-P_2) + (25P_2{}^{-1}P - \tfrac{7}{2}P_2{}^{-2}P^2)\theta(P_2-P)]. \quad (2.18c)$$

All mass-renormalization terms have been removed from (2.18). One still has to perform wave-function renormalization; if one is seeking a solution of the form $F(P)\to \text{const}\times P^{-1}$, $P\to 0$, one subtracts out the constant term (at small P) in the denominator of (2.18b).

We can now make an ansatz for the behavior of F at small P. First let us suppose that the terms $-\tfrac{7}{2}P_2{}^2P^{-2}\theta(P-P_2) - \tfrac{7}{2}P_2{}^{-2}P_2{}^2\theta(P_2-P)$ in (2.18c) were absent. If we try an input

$$F_1 = aP_2^\alpha, \quad P_2 < P_0 \quad (2.19a)$$

$$F_1 = \tilde{F}_1, \quad P_2 > P_0, \quad (2.19b)$$

where P_0 is an arbitrary point, we find

$$P^{-1}\int \tilde{K}_1(P,P_2)F_1(P_2)dP_2 \approx aP^\alpha + bP, \quad P\to 0 \quad (2.20)$$

where $\approx$ means equal up to constant terms. Hence, if we take $\alpha=1$, we obtain the required output

$$F = AP^{-1}. \quad (2.21)$$

By taking $F_1 = aP + bP^3 + \cdots$ for $P<P_0$, we obtain an output

$$F = AP^{-1} + \tilde{a}P + \tilde{b}P^3 + \cdots, \quad P\to 0, \quad (2.22)$$

which is consistent in form with the input.

When we take into account the complete kernel (2.18c) the situation is slightly more complicated, since an input $F_1(P_2)\propto aP_2$ produces a term $P\ln P$ in the integral (2.19). However, if we again start from the ansatz (2.19), we find

$$P^{-1}\int \tilde{K}_1(P,P_2)F_1(P_2)dP_2 \approx aP^\alpha + bP\ ,\quad P\to 0 \tag{2.23a}$$

where

$$a=-\frac{25}{\alpha+1}+\frac{7}{2(\alpha+3)}+\frac{25}{\alpha}-\frac{7}{2(\alpha-1)}\,. \tag{2.23b}$$

If

$$\alpha=-1+(186)^{1/2}/6 \doteq 1.273\,, \tag{2.24}$$

the constant a vanishes. An input to (2.18b) of the form (2.19) thus produces an output $F(P)\propto P^{-1}$ (P small). We can then improve the input; for instance, if we replace (2.19a) by

$$F_1=aP_2^{\alpha}+bP_2^{\alpha+2}+cP_2^{\alpha+4}\ ,\quad P_2<P_0 \tag{2.19c}$$

we obtain an output

$$F=AP^{-1}+\tilde{a}P^\alpha+\tilde{b}P^{\alpha+2}+O(P^{2\alpha+1})\ ,\quad P_2 \text{ small}\,. \tag{2.25}$$

The output is consistent in form with the input up to the term in $P^{\alpha+2}$. By adding further terms we can obtain consistency up to any order, though it is unlikely that the series converges.

The above reasoning does not prove that a solution with a P^{-1} behavior exists, but merely suggests the possibility. To explore the matter further, we attempted to find an input which would be numerically reproduced, to reasonable accuracy, by the output. After a few false starts, we used the following ansatz:

$$\begin{aligned}F=A(P/P_0)^{-1}+B[(1-k-l)(P/P_0)^\alpha\\ +k(P/P_0)^{\alpha+2}\\ +l(P/P_0)^{\alpha+4}]\ ,\quad P<P_0\ ,\end{aligned} \tag{2.26a}$$

$$\begin{aligned}F=A(P/P_0)^{-1}+B\\ +(\alpha+2k+4l)[B-B(P/P_0)]\ ,\quad P>P_0\,.\end{aligned} \tag{2.26b}$$

With this parametrization, F and its first derivative are continuous at $P=P_0$. Leaving A/B arbitrary, we fixed k and l by the requirements that (i) at $P=0$, the coefficients of the terms AP^{-1} and aP^α in the output should agree with those of the input, and (ii) at $P=P_0$, the output should agree with the input. We then fixed the value of A/B so as to optimize the agreement of the output with the input for $P>P_0$.

The parameters thus found were

$$A/B=25\ ,\quad k=-0.573\ ,\quad l=0.297\,. \tag{2.27}$$

The normalization of input and output depends on the renormalization convention; with arbitrary normalization, we found

input:

$$\begin{aligned}F&=25(P/P_0)^{-1}+1.276(P/P_0)^{1.273}-0.573(P/P_0)^{3.273}\\ &\quad+0.297(P/P_0)^{5.273}\ ,\quad P<P_0 &(2.28a)\\ &=23.68(P/P_0)^{-1}+2.315\ ,\quad P>P_0\,. &(2.28b)\end{aligned}$$

output:

$$\begin{aligned}F&=[43.04(P/P_0)-2.195(P/P_0)^{3.273}\\ &\quad+0.539(P/P_0)^{5.273}]^{-1}\ ,\quad P<P_0\ , &(2.29a)\\ &=[115.7(P/P_0)\ln(P/P_0)-57.8(P/P_0)\\ &\quad+122.2-25.17(P/P_0)^{-1}+0.9(P/P_0)^{-2}]^{-1}\ ,\quad P>P_0\,. &(2.29b)\end{aligned}$$

We then found values for $F(P_0)/F(P)$ as listed in Table I.

Except at the last point, we find agreement between input and output to within 5%. For larger values of P/P_0 we should have to adopt a different ansatz, since the input (2.28) does not represent the asymptotic behavior. It is not difficult to see that the solution to (2.18) has the asymptotic behavior const$\times(\ln P)^{-1/2}$. We should therefore restrict (2.28b) to a range $P_0<P<P_0'$, and take

$$F=\text{const}\times[\ln(P/P_0)]^{-1/2}\ ,\quad P>P_0'\,.$$

Choosing the joining point P_0' so that F and its first derivative are continuous across it, we find

$$P_0'/P_0=78\ , \tag{2.28c}$$

input:

$$F(P)=0.211F(P_0)[\ln(P/P_0)]^{-1/2}\ ,\quad P>P_0'\,. \tag{2.28d}$$

We have not checked the output corresponding to this new input, but, in view of the agreement between the old input and output, and the fact that the new input and output agree asymptotically up to a

TABLE I. Comparison of input and output for solution of gluon-propagator integral equation.

P/P_0	$F(P)/F(P_0)$ (input)	$F(P)/F(P_0)$ (output)
$\frac{1}{2}$	0.515	0.531
$\frac{3}{4}$	0.763	0.787
$1\frac{1}{2}$	1.436	1.489
2	1.838	1.927
3	2.547	2.674
5	3.69	3.78
10	5.55	5.50
20	7.43	7.35
50	9.32	9.91

constant factor, we do not expect any large discrepancy. We believe that the 5% agreement between input and output indicates pretty convincingly that our equations for F have a solution which behaves like P^{-1} at small P. Furthermore, our analysis of the small-P limit indicates that no other behavior is consistent. In particular, the ansatz $F(P) = \text{const}$ would lead to a logarithm in the denominator of (2.18b).

Let us now briefly examine the ghost terms in our equation. We first consider (2.14b), where, for F, we take the solution of (2.18) already obtained. The ansatz $G \sim P^{-1}$, $P \to 0$, is not consistent, since one would then obtain a term proportional to P^{-2} in the denominator of (2.14b). In fact, the situation is simpler than in the equation for F; we can obtain a consistent small-P behavior by iteration, starting with $F(P) = \text{const}$. The only divergent term is a gluon mass-renormalization term which we set equal to zero as before. We may also remark that the iterated expressions are free of Landau ghosts. It thus appears likely that the equation for G will have a consistent solution which approaches a constant at $P = 0$, though we have not carried through a numerical analysis. The overall constant again depends on our renormalization prescription.

Finally, inclusion of the second term in the denominator of (2.14a) makes little qualitative difference; the only change is that the constant term in the small-P expansion for F, which was previously set equal to zero, must now be chosen so that the $\ln P$ contributions from the two terms in the denominator of (2.14a) cancel. In view of the smallness of the ghost contribution to the gluon propagator, it is unlikely that the nature of the solution would be altered. It is necessary to say something about renormalization in order to determine the relative magnitude of the two terms in the denominator of (2.14a). We now turn to this question.

D. Ultraviolet behavior and renormalization

It is not difficult to see that the solution of (2.14) have the following high-energy behavior:

$$F(P) \to \left(\frac{75g^2c_1}{272\pi^2} \ln \frac{P}{P_0}\right)^{-1/2}, \quad P \to \infty \tag{2.30a}$$

$$G(P) \to \left(\frac{51g^2c_1}{400\pi^2} \ln \frac{P}{P_0}\right)^{-1/4}, \quad P \to \infty . \tag{2.30b}$$

We may compare (2.30) with the exact renormalization-group results:

$$F(P) \to \left(\frac{11g^2c_1}{48\pi^2} \ln \frac{P}{P_0}\right)^{-13/22}, \quad P \to \infty \tag{2.31a}$$

$$G(P) \to \left(\frac{11g^2c_1}{48\pi^2} \ln \frac{P}{P_0}\right)^{-9/44}, \quad P \to \infty . \tag{2.31b}$$

While our equations do not automatically give the correct high-energy behavior, we notice that the error made is not very large.

Equations (2.30) enable us to determine the relative normalization of F and G, which is required in the solution of (2.14). We use the same normalization factor; we then observe that

$$F/G^2 \to 17/25 , \quad P \to \infty .$$

We may also remark that g^2FG^2 is a renormalization-group invariant at high energies; this is true whether we use our equations (2.30) or the exact equations (2.31). We find that

$$g^2FG^2 \to \left(\frac{3c_1}{16\pi^2} \ln \frac{P}{P_0}\right)^{-1}, \quad \text{our equations} \tag{2.32a}$$

$$g^2FG^2 \to \left(\frac{11c_1}{48} \ln \frac{P}{P_0}\right)^{-1}, \quad \text{exact results} . \tag{2.32b}$$

It is possible to build the exact asymptotic behavior into our equations, at the cost of having an arbitrary joining point between two different forms of the kernel. We should like to get an expression for the vertex in Fig. 2 which is exact at high energies. In particular, we should like an expression for the vertex which is accurate when all three components of the momenta are large, and when $p_1 \gg p$, $p_2 \gg p$, since this is the region which determines the logarithmic behavior of the propagator. The Ward identity relates the longitudinal part of the three-gluon vertex to the gluon-ghost vertex and the propagators. At high momenta, the Landau-gauge gluon-ghost vertex is equal to the bare vertex. It may then easily be seen that the three-gluon vertex is given by the simple formula

$$V = V_B F^{-1}(p_r^2) G(p_r^2) , \tag{2.33}$$

where V_B is the bare vertex and p_r is any one of the momenta p, p_1, p_2, if these are of the same order of magnitude. [The effect of changing p_r between the three p's is smaller than the main term by a power of $\ln(p^2)$, and (2.33) is in any case only accurate to leading logarithms.] If one of the three p's is much smaller than the other two, we must take p_r to be one of the larger p's. The leading logarithm in the transverse part of V is the same as that of the longitudinal part, and we may take (2.33) as valid for the entire vertex at high energy.

Equation (2.33) is a kind of renormalization correction to V. As with all renormalization corrections, we must apply it to *both* vertices in Fig. 2. We have not proved this last statement, but it is easy to check it for the simplest nontrivial diagram, Fig. 4. Let us first consider the left-hand

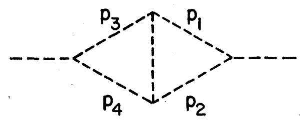

FIG. 4. Overlapping gluon-propagator diagram.

triangle as a single vertex. We are interested in the case where all p's are large and, if $p_1, p_2 \gg p$, we are interested in terms proportional to p^2/p_1^2 (among others), since these contribute to the high-energy behavior of the propagator after mass renormalization. The high-momentum part of the loop integration for the vertex in question, i.e., the region for which $p_3, p_4 \gg p_1, p_2$, then gives us the leading logarithm, which is equivalent to the second term in the expansion of (2.33). There will be an additional contribution of the form $(p^2/p_1^2)\ln p_1^2$ $(p \ll p_1)$ from the region where $p_3, p_4 \ll p_1, p_2$. To evaluate that contribution to the final result, we consider the right-hand triangle as a single vertex. We then obtain a second term equal to the first.

To obtain the asymptotic behavior, we must thus correct the gluon term in the gluon propagator by two factors $F^{-1}(p_i)G(P_i)$, where P_i is one of the internal momenta. We could take P_i either equal to P_1 or to P_2, since the asymptotic behavior is determined by the region $P_1 \approx P_2 \gg P_3$. For reasons of symmetry we take a factor

$$F^{-1}(P_1)F^{-1}(P_2)G(P_1)G(P_2)\,.$$

We have already included a factor $F^{-1}(P_1)$ by taking a bare gluon propagator for the p_1-internal line in (2.3). We must therefore make the replacement

$$F(P_2) \to G(P_1)G(P_2)\,, \quad P_1, P_2 > \tilde{P}_0$$

in (2.8a). Our final result will, of course, depend on the value of $\tilde{P}_0$ chosen. We should obviously take a value where the asymptotic formulas for the propagators are reasonably accurate. As long as we do so Eqs. (2.31) indicate that G^2 does not differ much from F, and the result should not depend sensitively on $\tilde{P}_0$. It is now not difficult to check that the asymptotic behavior of the solutions to our equations is given by (2.31).

As a check of the sensitivity of our equations to the choice of gauge, we might calculate the asymptotic behavior of the quantity g^2FG^2, which is gauge invariant. In Feynman gauge, we find

$$g^2FG^2 = \left(\frac{c_1}{4\pi^2}\ln\frac{P}{P_0}\right)^{-1}. \tag{2.34}$$

The Feynman-gauge and Landau-gauge results differ from one another by a factor $\frac{3}{4}$, with the correct result in between.

It may perhaps be worth mentioning that, in the gauge with bare propagator $g^{\mu\nu} - \frac{1}{3}p^\mu p^\nu (p^2)^{-1}$, the factor (2.33) is unity, and our calculations would give the correct asymptotic behavior.

III. MULTIGLUON GREEN'S FUNCTIONS

If the gluon Green's function behaves like $k \ln|x|$ at large distances, the loop integral

$$\frac{1}{N}\,\mathrm{Tr}\,\tau^\alpha\tau^\beta \int \langle\, T\,[\,dx\cdot A^\alpha(x)dx'\cdot A^\beta(x')]\rangle \tag{3.1}$$

behaves like $k(N^2-1)\pi\mathfrak{A}$, where $\mathfrak{A}$ is the area of the loop. We define the area of a loop which is not in a plane as follows:

$$\mathfrak{A} = \int |d\sigma| - \frac{1}{4\pi}\epsilon_{\kappa\lambda\rho\tau}\epsilon_{\mu\nu\sigma}{}^{\tau} \times \int d\sigma^{\kappa\lambda} d\sigma'^{\mu\nu}(x-x')^\rho(x-x')^\sigma[(x-x')^2]^{-2}\,, \tag{3.2}$$

where the integral is over any surface bounded by the loop; it will be independent of the surface chosen. For a plane loop, (3.2) is the area of the plane surface bounded by it.

Before we can perform calculations involving quarks we require information regarding multigluon amplitudes. Such information is necessary in any theory, and, if the two-particle propagator decreases at large distances, one usually assumes that the multiparticle Green's functions are given by the sum of disconnected diagrams. In the present theory, where the two-gluon Green's function increases at large distances, clustering does not occur, and we must replace the assumption of disconnectedness by another ansatz. Our proposal will be based on two assumptions.

Let us consider the second term obtained by expanding the exponential of the Wilson integral:

$$\frac{1}{N}\,\mathrm{Tr}\,\tau^\alpha\tau^\beta\tau^\gamma\tau^\delta \int \langle\, T[dx\cdot A^\alpha(x)dx'\cdot A^\beta(x')dx''\cdot A^\gamma(x'')dx'''\cdot A^\delta(x''')]\rangle\,. \tag{3.3}$$

This expression will contain a term proportional to $\mathfrak{A}^2$, obtained from the region of integration where all four x's are distant from one another, and terms smaller than $\mathfrak{A}^2$ (for large $\mathfrak{A}$), obtained from the region of integration where two or more x's are close to one another. Our first assumption is that only the most divergent terms in (3.3) are important; the terms smaller than $\mathfrak{A}^2$ will be assumed to be given by the sum of disconnected diagrams.

Our second assumption concerns the dependence of the most divergent terms on the shape of the area. The first term, given by (3.1), is independent of the shape. This fact was partly a result of our choice of integral equation—we chose the equation represented by Fig. 1 rather than that obtained by direct use of the Ward identity—but it is nevertheless true that a simple choice of integral equation led to this result. Most models of confinement possess the feature that the Wilson integral does not depend on the shape of the area. We shall assume that our result for the two-gluon propagator is not a coincidence, but is a property of the theory which is maintained for the higher Green's functions. We thus assume that the leading term of (3.3) is independent of the shape of the area.

We can treat the higher terms in the expansion of the Wilson integral in a similar way. The term involving the $2n$-gluon Green's function depends on $\mathcal{A}^n$, and will be assumed to be independent of the shape of the loop. The whole Wilson integral is thus independent of the shape of the loop. By considering a loop which itself consists of two well-separated loops, we conclude that the only possible form for the Wilson integral is $e^{-\alpha\mathcal{A}}$ (up to perimeter-dependent or shape-dependent, area-independent, factors). We can then identify the nth term in the expansion with the contribution of the $2n$-gluon Green's function. In particular, the factor α is just $k(N^2-1)\pi$, k being the constant of proportionality between the two-gluon propagator and $\ln|x|$.

As long as we are only interested in color-singlet states, we may in principle perform calculations by integrating over closed quark loops. For instance, the energies of bound states can be determined from the Green's function $\langle T[\overline{\Psi}(x)O_1\Psi(x)\overline{\Psi}(y)O_2\Psi(y)]\rangle$, where the operators O_1 and O_2 may involve γ matrices or derivatives. Such a Green's function can be determined by integrating over loops passing through x and y. If we neglect the effects of further virtual quark loops, each loop will be weighted by the vacuum-expectation value of the Wilson integral. The above arguments therefore give us all the information we require with respect to the large-distance behavior of the Green's functions.

Our expression for the Wilson integral is the same as that given by a model with Abelian, noninteracting gluons whose propagator behaves like $k\ln|x|$ at large x. Planar and nonplanar diagrams must be included; the nonplanar diagrams take into account the interactions between the actual, non-Abelian gluons. Notice that it would be completely inadequate to neglect the nonplanar diagrams. The higher terms in the Wilson expansion are dominated by such diagrams, and the planar diagrams alone would not give a behavior $e^{-\alpha\mathcal{A}}$. We have examined the Bethe-Salpeter equation in the ladder approximation, with a $(p^2)^{-2}$ gluon propagator, and we found no hint of a confinementlike spectrum.

The solution of the $Q\overline{Q}$ problem with the inclusion of crossed gluon lines would appear to be a formidable task, even without gluon interactions or virtual quark loops. However, an approximation which suggests itself is to replace the expression (3.2) for the area by the following, noncovariant expression

$$\mathcal{A}'=\int|x-x'|\delta^4(x_4-x_4')dx_4dx_4'\,. \qquad (3.4)$$

In other words, we divide the area into a number of infinitesimal strips by lines perpendicular to the x_4 axis, we then project each strip onto the plane formed by the x_4 axis and the long edge of the strip. To obtain an idea of the error made we may consider the case, relevant for the asymptotic behavior of Regge trajectories, where the edges of the area rotate with an angular velocity c around a fixed point. (For the moment we use real time.) The area is then out by a factor $\frac{1}{4}\pi$. The presence of the δ function in (3.6) gives us an instantaneous interaction, and the associated dyanmical problem is soluble without difficulty.

Our approximation involves the replacement, in the propagator, of

$$A(\delta_{\mu\nu}-\partial_\mu\partial_\nu\partial^{-2})\ln|x| \text{ by } A\pi|x|\delta_{\mu 0}\delta_{\nu 0} \qquad (3.5a)$$

or of

$$A[\delta_{\mu\nu}-p_\mu p_\nu(p^2)^{-1}](p^2)^{-2} \text{ by } A\delta_{\mu 0}\delta_{\nu 0}\,. \qquad (3.5b)$$

More precisely, we divide the propagator into a term proportional to $(p^2)^{-2}$ (or $\ln|x|$) and a remaining term. For the first term, we make the replacement (3.5); the remaining term is not changed. It is then reasonable to use approximations similar to those of the last section for treating systems involving quarks, whereas approximations of this type, without the replacement (3.7), would be totally inadequate for studying the confinement properties of the theory.

We have not examined the question of the improvement of the approximation (3.4). The formula $e^{-\alpha\mathcal{A}}$ for a Wilson loop suggests a dual-string Lagrangian, which (3.5) replaces by a static force between quarks. It therefore appears that one should improve on the approximation by taking into account the vibrational modes. Possibley one might treat those modes with wavelength greater than a certain value; for shorter distances one would use the covariant gluon propagator. Needless to say, the spirit of the approximation (3.5) is fundamentally different from the general spirit of our approximation scheme. We hope to return to examine possible methods of improvement and thereby to gain more insight into the approximation in question.

We have not attempted to obtain the multigluon Green's function themselves, but simply the integrals such as (3.3). We do not require additional information in order to calculate the meson spectrum. The Green's function will depend strongly on whether various combinations of points are collinear, coplanar, or (possibly) on the same three-dimensional surface, as may be seen by considering the expectation value of $F_{0i}(x_1)F_{0j}(x_2)$ in a state with a heavy well-separated $Q\bar{Q}$ pair. This quantity depends on the vacuum expectation values

$$\frac{1}{N}\,\mathrm{Tr}\,\tau^\alpha\tau^\beta\cdots\tau^\zeta\int\langle T[dy\cdot A^\alpha(y)dy'\cdot A^\beta(y')\cdots dy^{(n)}\cdot A^\zeta(y^{(n)})F^\eta_{0i}(x_1)F^\eta_{0j}(x_2)]\rangle\,, \tag{3.6}$$

the integrals to be taken over the quark world lines. As we would expect the expectation value to be large if x_1 and x_2 are collinear with the quarks, we conclude that the integrand must depend on the collinearity or coplanarity of the various points.

IV. THE QUARK PROPAGATOR

The results of the previous section cannot be used to calculate the actual quark propagator, as it was assumed that all amplitudes are to be functionally integrated over closed quark loops. The calculations in this section give us an "effective propagator" to be inserted into the $Q\bar{Q}$ bound state integral equation.

We shall use the equation for the quark propagator shown in Fig. 5. Thus

$$S^{-1}(p)=-i\gamma\cdot p+\frac{g^2}{2(2\pi)^4}\int dp_1dp_2\delta^4(p-p_1+p_2)\times\gamma^\mu S(p_2)\gamma^\nu D_{\mu\nu}(p_1)\,, \tag{4.1}$$

where S is the quark propagator and D the gluon porpagator. According to the results of Sec. III, the term $A[\delta_{\mu\nu}-p_\mu p_\nu(p^2)^{-1}](p^2)^{-2}$ in D is to be split off and replaced by the expression (3.7b).

The fact that D consists of a covariant and an instantaneous part may make (4.1) a bit awkward to handle practically. If the instantaneous term alone were present, Eq. (4.1) could be solved exactly as in the two-dimensional model. In fact, the first iteration for S^{-1} would be the exact solution. One may therefore hope to solve (4.1) by iteration, starting from the solution of the equation where the covariant term alone is present.

Next let us discuss the question of renormalization. By using an analysis similar to that used for the gluon propagator, we find that the integrand in (4.1) requires an extra factor $G^2(p_1)$ at high energy, where G is the correction factor for the ghost propagator. In deriving this result, we use the Taylor-Slavnov identity for the quark-gluon vertex,[13] together with the fact that both S and the auxiliary function H are ultraviolet finite in the Landau gauge. Thus

$$S^{-1}(p)=-i\gamma\cdot p+\frac{g^2}{2(2\pi)^4}\int dp_1dp_2\delta^4(p-p_1-p_2)\times\gamma^\mu S(p_2)\gamma^\nu D_{\mu\nu}(p_1)G^2(p_1)\,. \tag{4.2}$$

Since g^2DG^2 is a renormalization-group invariant at high energies, we can take Eq. (4.2) as it stands.

The function G^2 tends to a finite limit as $p_1\to 0$, and we can therefore use the integrand of (4.2) throughout the range of integration. In fact, the Landau-gauge Taylor-Slavnov identity

$$(Z_1Z_2^{-1})_{\text{fermion}}=(Z_2^{-1})_{\text{ghost}} \tag{4.3}$$

suggests that we should associate a factor G with g if we renormalize at some finite value of p_1. Equation (4.3) would then instruct us to normalize the quark propagator to be unity at an arbitrarily chosen value of the quark momentum and zero gluon momentum. The correction factor F to the gluon propagator would be normalized so that

$$g^2F=\left(\frac{75}{272\pi^2}\ln\frac{P}{P_0}\right)^{-1/2}\ \text{or}\ \left(\frac{11}{48\pi^2}\ln\frac{P}{P_0}\right)^{-13/22}$$

at high energies, depending on whether or not we modified the equations for F so as to obtain the right high-energy behavior. Finally, we should add a factor $G(0)$ at each vertex. The result would be independent of the quark momentum chosen. We cannot follow this complete procedure, since we are using the bare quark-gluon vertex, but we do confirm the presence of the factor $G(0)$ associated with each vertex function at zero gluon momentum.

We conclude this section with a treatment of possible chiral-symmetry breaking in the solutions of our equations. Chiral-symmetry breaking is independent of the confinement properties of the system and is not sensitively dependent on the low-p behavior of the gluon propagator. We shall therefore consider the covariant part of the gluon prop-

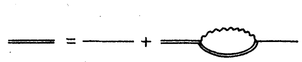

FIG. 5. Equation for the quark propagator.

agator alone. The angular integrations in (4.2) can then be performed. Let us define

$$S^{-1}(p)=-i\,[\gamma\cdot pH_1(P)+H_2(p)] \tag{4.4}$$

where, as usual, $P=p^2$. We then find the equations

$$H_1=1-\int K_4(P,P_1)H_1(P_1)\times\{P_1[H_1(P_1)]^2+[H_2(P_1)]^2\}^{-1}dP_1\,, \tag{4.5a}$$

$$H_2=\int K_5(P,P_1)H_2(P_1)\times\{P_1[H_1(P_1)]^2+[H_2(P_1)]^2\}^{-1}dP_1\,, \tag{4.5b}$$

where

$$K_4(P,P_1)=\frac{g^2}{16(2\pi)^3P^2}\times\int\frac{dP_2}{P_2}F(P_2)\times[2P_2-P-P_1-P_2^{-1}(P-P_1)^2]\Delta\,, \tag{4.6a}$$

$$K_5(P,P_1)=\frac{3g^2}{16(2\pi)^3P}\int\frac{dP_2}{P_2}F(P_2)\Delta\,. \tag{4.6b}$$

The range of integration is given by the θ functions in (2.11). We are interested in possible solutions of Eqs. (4.5) and (4.6) with $H_2\neq 0$.

The existence of solutions to the equations will, of course, depend on the precise form of F. Here we shall only investigate the general question of the likelihood of the existence of solutions to equations such as ours. We shall therefore make the following simplifications, which will enable us to use the result of Maris, Herscovtiz, and Jacob.[7]

(i) We shall take $H_1=1$ so as to decouple our equations. Since we are using the covariant part of F alone, corrections to H_1 will not change its form qualitatively.

(ii) We shall assume that F is approximately equal to 1; in other words, we use the bare gluon propagator. We do not wish to take F precisely equal to 1, since the equation would then be scale invariant and its qualitative features would be changed. We shall therefore begin with the case $F=1$, and shall subsequently investigate the change caused by a small change in F.

As in Ref. 7, we shall make the additional approximation of replacing the term $\{H_2\}^2$ in the denominator of (4.5b) by a constant a, say; we shall return to check the validity of this approximation. The solution to the equation has then been given by Maris, Herscovitz, and Jacob[7]:

$$H_2=a^{1/2}{}_2F_1\left(\tfrac{1}{2}+(\tfrac{1}{4}-\lambda)^{1/2},\tfrac{1}{2}-(\tfrac{1}{4}-\lambda)^{1/2},2,-Pa^{-1}\right)\,, \tag{4.7a}$$

where

$$\lambda=\frac{3g^2}{16(2\pi)^3}\,. \tag{4.7b}$$

The factor $a^{1/2}$ has been chosen so that $[H_2(0)]^2=a$.

We can now check the validity of the replacement of the term $\{H_2\}^2$ in the denominator of (4.5b) by a. We examined the case $\lambda=\frac{1}{2}$, where the hypergeometric function on the right-hand side of (4.11) is a sum of complete elliptic integrals. We found that our approximation made a change of less than 5% in the denominator of (4.5b) throughout the range of P. The expression (4.7a) is therefore a fairly accurate solution of our equations (with F, the gluon-propagator correction factor, equal to 1), and it is extremely likely that the equation does have a solution. Note that the constant a, which sets the scale, may be chosen arbitrarily.

Next let us investigate the effect of a slight change of the gluon-propagator correction factor F, which breaks the scale invariance. The change δH_2 of the solution of (4.5b) (with $H_1=1$) will be given by the equation

$$\delta H_2(p)=\int\delta K_5(P,P_1)H_2(P_1)\{P_1+[H_2(P_1)]\}^{-1}dP_1+\int K_5(P,P_1)\times\{P_1-[H_2(P_1)]^2\}\{P_1+[H_2(P_1)]^2\}^{-2}\times\delta H_2(P_1)dP_1\,, \tag{4.8}$$

where δK_5 is the change in K_5 caused by the change in F. Equation (4.8) is an inhomogeneous linear integral equation for δH_2. The corresponding homogeneous equation has a solution, since, by changing the scale of H_2, we can continue to satisfy (4.5b) without changing the kernel. If $\delta_s H_2$ is this homogeneous solution, Eq. (4.12) will be soluble provided that the inhomogeneous term I satisfies the equation

$$\int dP\,I(P)P\,\{P-[H_2(P)]^2\}\times\{P+[H_2(P)]^2\}^{-2}\delta_s H_2(P)=0\,. \tag{4.9}$$

The extra factor P in (4.9) arises from the fact that we must multiply $K_5(P,P_1)$ by P to obtain a symmetric kernel.

Suppose, now, that the change in F approaches zero when P approaches zero or infinity. If this change has a fixed sign, the inhomogeneous term I will have the same fixed sign. Furthermore, $\delta_s H_2$ has a fixed sign, at any rate if (4.7a) is a reasonably accurate solution of the scale-invariant equation; the change in H_2 has the same sign as the change in a. Since the factor $P - H_2^2$ in (4.9) changes sign at a certain value of P, it follows that we can change the sign of the left-hand side of (4.9) by scaling the change of F upwards or downwards. In particular, we can scale the change of F in such a way that (4.9) is satisfied; Eq. (4.8) is then soluble. Alternatively, for a given value of the change of F, we can scale the function H_2 in (4.8) in such a way that the equation becomes soluble.

The above arguments assume that the change of F is zero for small and large P but, since our equations are convergent at both ends of the range of integration, we may remove this restriction by a limiting process. Our result is thus that the family of solutions to Eq. (4.5b), with $H_1 = 1$ and $F=1$, becomes replaced by a single solution when the scale invariance is broken by a slight change in F.

We may mention that our last result is true whether or not the theory is asymptotically free. The feature of our results which would depend on asymptotic freedom would be the behavior of the solution if we were to imagine that the constant g^2 could be varied while the scale breaking of F were held fixed. We would then find that the pole in the quark propagator would move from zero or infinity according as F decreased or increased at large p.[14]

Finally, we must ask whether the solution with $H_2 \neq 0$, if it exists, is the one which is realized in nature. We shall investigate the question by "turning on" the coupling. Quantum chromodynamics (with massless bare quarks) contains no arbitrary dimensionless coupling constant, but we can introduce one by applying an ultraviolet cutoff—by putting the system on a lattice, for instance. Once we are on a lattice we can also give the gluons a bare mass to ensure weak coupling at low momenta, though the strong coupling at low momenta appears irrelevant to the present problem. With weak coupling and ultraviolet cutoff, our equations possess no solutions with $H_2 \neq 0$; we are in the regime with true chiral symmetry. We can now gradually increase the strength of the coupling by taking the ultraviolet cutoff to infinity. At a certain point, a solution with $H_2 \neq 0$ begins to appear. The system then has a massless pseudoscalar particle. Furthermore, since chiral symmetry is still realized as a true symmetry when the $H_2 \neq 0$ solution just begins to appear, the system also has a massless scalar particle. We are thus in a typical situation of the onset of chiral-symmetry breaking and, as the cutoff momentum is increased further, we would expect the system to choose the symmetry-broken mode.

V. $Q\bar{Q}$ BOUND-STATE EQUATION

A. Leading Regge trajectories

For $Q\bar{Q}$ bound states we use the Bethe-Salpeter equation:

$$\Psi(E,p) = \frac{ig^2}{2(2\pi)^4} S(p+\tfrac{1}{2}E) \times \int dp' \gamma^\mu \Psi(E,p') \gamma^\nu D_{\mu\nu}(p-p') S(p-\tfrac{1}{2}E), \tag{5.1}$$

where D is the gluon propagator modified as in Sec. III. By E we mean the vector $(E,0,0,0)$. We are using a matrix notation for Ψ; the antiquark spinor indices label the rows and the quark indices the columns. Notice that our equation for $Q\bar{Q}$ bound states is consistent with that for the propagator. If, in the inhomogeneous equation corresponding to (5.1), we insert the solution of (4.1) for S, take $E=0$, and join the quark lines at the ends, we recover the solution of (4.1).

Since the force between the quarks at large distances is a static force proportional to $|x|$, we should expect confinement. We shall verify this by showing that the Regge trajectories are linearly rising ($l \propto E^2$) for large l and E. In this regime we require S at *high* momenta, as may be demonstrated by rewriting the equation in coordinate space; the singularity, at $x=0$, of the Fourier transform of S, governs the high-l region. We may therefore take

$$S(p) = i(\gamma \cdot p + i\epsilon)^{-1}. \tag{5.2}$$

On the other hand, we require the large-distance or low-momentum part of $D_{\mu\nu}$, and we may take the expression (3.7b). Renormalizing as in Sec. IV, we replace the constant A by $A' = AG^2(0)$; the combination $g^2 A'$ will be independent of g^2.

We thus have the equation

$$\Psi(E,p) = -\frac{ig^2 A'}{2(2\pi)^4}[\gamma\cdot(p+\tfrac{1}{2}E)+i\epsilon]^{-1} \times \int dp' \gamma^0 \Psi(E,p')\gamma^0 \times \{(\vec{p}-\vec{p}\,')^2\}^{-1}[\gamma\cdot(p-\tfrac{1}{2}E)+i\epsilon]^{-1}. \tag{5.3}$$

As the kernel $[(\vec{p}-\vec{p}\,')^2]^{-1}$ is independent of p_0, we may eliminate the zeroth component by integrating over it. We thus define

$$\chi(E,\vec{p}) = \int_{-\infty}^{\infty} dp_0 \Psi(E,p). \tag{5.4}$$

Then

$$\chi(E,\vec{p})=\frac{g^2A'}{8(2\pi)^3}\int d\vec{p}'(\gamma^0-\vec{\gamma}\cdot\hat{\vec{p}})\gamma^0\chi(E,\vec{p}')\gamma^0(\gamma^0+\vec{\gamma}\cdot\vec{p})[(\vec{p}-\hat{\vec{p}}')^2]^{-1}(|p|-\tfrac{1}{2}E)^{-1}$$
$$-\frac{g^2A'}{8(2\pi)^3}\int d\vec{p}'(\gamma^0+\vec{\gamma}\cdot\hat{\vec{p}})\gamma^0\chi(E,\vec{p}')\gamma^0(\gamma^0+\vec{\gamma}\cdot\vec{p})[(\vec{p}-\hat{\vec{p}}')^2]^{-1}(|p|+\tfrac{1}{2}E)^{-1}. \tag{5.5}$$

We are interested in solutions where $|p|$ and E are large and nearly equal, and we may therefore neglect the second term on the right-hand side of (5.5) in comparison with the first. By taking the factor $(|p|-\frac{1}{2}E)^{-1}$ into the numerator of the left-hand side, we can write our equation as a differential equation in coordinate space. Separating variables in the usual way, we may write

$$|p|=\left[\frac{l(l+1)}{r^2}+p_r^{\,2}\right]^{1/2}.$$

In the regime of interest, the term $l(l+1)/r^2$ is much greater than $p_r^{\,2}$ and

$$|r-R|<R\ ,$$

where R is some average value; these facts may be verified from the solution of our equation. Thus

$$|p|\approx\frac{l}{r}+\frac{R}{2l}p_r^{\,2}. \tag{5.6}$$

The projection operators $(\gamma^0\pm\vec{\gamma}\cdot\hat{\vec{p}})\gamma^0$ will commute with $(1/64\pi)|x|$, the Fourier transform of the kernel

$$\frac{1}{8(2\pi)^3}[(\vec{p}-\vec{p}')^2]^{-2},$$

since the momentum $\vec{p}$ is mainly angular. We can therefore eliminate these projection operators by taking a wave function in the projected space, i.e., by writing

$$\chi=(\gamma^0-\vec{\gamma}\cdot\vec{p})\gamma^0\phi\gamma^0(\gamma^0+\vec{\gamma}\cdot\vec{p}). \tag{5.7}$$

Furthermore, if we write

$$\phi=\phi_1\gamma^5+\phi_2\vec{\gamma}\cdot\vec{p}+\phi_3+\phi_4\gamma^5\vec{\gamma}\cdot\vec{p}, \tag{5.8}$$

we may also commute the operators $\vec{\gamma}\cdot\vec{p}$ through the potential and write the differential equation as an equation for the ϕ_r's, $r=1,\ldots,4$. Note that the terms in (5.8), when sandwiched between the projection operators, span the space generated by these operators.

With these definitions and approximations, our equations become

$$\left(\frac{l}{r}+\frac{R}{2l}p_r^{\,2}-\tfrac{1}{2}E+\frac{g^2A'|x|}{64\pi}\right)\phi_l(r)=0, \tag{5.9}$$

where $\tilde{\phi}_{rl}$ is the radial wave function of the lth partial wave of $\tilde{\phi}_r$. Under the conditions of interest, the important values of r will be near the minimum R of the sum of the two r-dependent terms. Thus

$$R^2=64\pi l/g^2A'.$$

If $r=R+r'$, we may expand the term l/r in powers of r' and retain only the first two. Thus

$$\left(\frac{l}{R}+\frac{g^2A'R}{64\pi}+\frac{2lr'^2}{R^3}+\frac{R}{2l}p_r^{\,2}-\tfrac{1}{2}E\right)\phi_l(r)=0. \tag{5.10}$$

Equation (5.10) is a simple-harmonic-oscillator equation. We may in fact neglect the zero-point energy of the oscillator, which will be of order l^{-1}. Thus, using (5.9), we find

$$E^2=\frac{g^2A'}{4\pi}l, \tag{5.11}$$

$$\phi_l=\exp\left[-\frac{g^2A'}{64\pi}(r-R)\right]. \tag{5.12}$$

From (5.11), we confirm that we obtain linearly rising trajectories at large E or l.

The form (5.8) of our solutions shows that there are three types of trajectories, all with equal slope. As in the nonrelativistic quark model, the trajectories correspond to the π, ρ or σ, and A_1; the ρ trajectory is doubled.

B. Chiral-symmetry breaking

If our propagator S possesses chiral-symmetry breaking, we should expect our Bethe-Salpeter equation to have a zero-mass pseudoscalar solution. This has been shown directly by Lane[9] for the exact Bethe-Salpeter equation and, as long as our equations for the propagator and the two-quark bound state are consistent, a similar result should (and does) hold in our approximation.

We start again from our equation (4.1), but we now write it as an equation for S instead of S^{-1}. Let

$$S(p)=i[\gamma\cdot pE_1(P)+E_2(P)]. \tag{5.13}$$

If E_2 is nonzero, it will satisfy the equation

$$E_2=\frac{ig^2}{2(2\pi)^4}(E_2^{\,2}-PE_1^{\,2})\times\int dp_1dp_2\delta^4(p-p_1-p_2)\gamma^\mu E_2\gamma^\nu D_{\mu\nu}(p_1). \tag{5.14}$$

Hence

$$\gamma_5E_2=\frac{ig^2}{2(2\pi)^4}(\gamma\cdot pE_1+E_2)\times\int dp_1dp_2\delta^4(p-p_1-p_2)\gamma^\mu\times(\gamma_5E_2)\gamma^\nu(\gamma\cdot pE_1+E_2)D_{\mu\nu}(p_1),$$

i.e.,

$$\gamma_5 E_2 = \frac{ig^2}{2(2\pi)^4} S(p) \int dp_1 dp_2 \delta^4(p-p_1-p_2)\gamma^\mu \times(\gamma_5 E_2)\gamma^\nu S(p) D_{\mu\nu}(p_1). \quad (5.15)$$

Thus $\gamma_5 E_2$ does satisfy the Bethe-Salpeter equation (5.1) for $E=0$.

We have been neglecting closed quark loops throughout our treatment. At this point, however, it is interesting to determine whether they would affect this last conclusion, since we might expect the zero-mass pseudoscalar particle in the flavor-singlet sector to acquire a mass when closed loops are included. A logarithmic behavior of the gluon propagator would mean that the contribution of the triangle anomaly to the axial-vector current would diverge at large distances, and there would be no meaningful conserved axial-vector charge.

Once we include closed loops, we have coupled integral equations for the $Q\overline{Q}$ and two-gluon wave functions. If the equations were covariant, as they would be without confinement, there would be no possible two-gluon pseudoscalar wave function, and we would not have coupled equations for this particular state. With confinement we have seen that the equations are not covariant, and we can have a pseudoscalar two-gluon wave function of the form $\epsilon_{0\lambda\mu\nu}p_1^\lambda$, where μ and ν are the gluon polarization indices. It is not possible to derive the Bethe-Salpeter equation for the coupled wave functions at $E=0$ from that for the propagator, and the argument for the existence of a massless pseudoscalar particle does not go through.

It might be objected that the absence of a massless flavor-singlet pseudoscalar particle, depending as it does on the noncovariance of our approximation, could be a consequence of the approximation itself, and that the massless particle would reappear if the equations were improved. There is, however, nothing in the structure of the equations which would indicate the presence of a massless pseudoscalar particle. If the matrix element of the axial-vector current is calculated in the approximation to which we are working (including closed quark loops), triangle diagrams will occur. One therefore obtains an anomalous contribution

$$\epsilon_{0\lambda\mu\nu}\left(A_\lambda^\alpha \frac{\partial A_\nu^\alpha}{\partial x_\mu} + \tfrac{2}{3} g \epsilon^{\alpha\beta\gamma} A_\lambda^\alpha A_\mu^\beta A_\nu^\gamma\right) \quad (5.16)$$

to the axial-vector charge density. In the matrix element

$$\langle 0 | j_0^5(x) | \eta \rangle ,$$

the contribution from the first term of (4.2) will fall off like $|x|^{-1}$ as x approaches infinity, and the total charge will not be finite. It could happen that there was a cancellation between the two terms of (4.2); to the extent that such a cancellation is possible we have not disproved that a more correct version of our equations will yield a massless η. At present we see no reason for a cancellation and therefore no reason for a massless η.

VI. CONCLUDING REMARKS

The approximation scheme proposed here appears to predict many of the qualitative features of the hadron spectra, and it may be hoped that actual calculations will give reasonably accurate results. In principle one should be able to extend the scheme to higher approximations though, at present, we have no idea of the convergence of the series.

There are two immediate questions which we hope to investigate. The first concerns the ansatz regarding the multigluon Green's functions. One should be able to test the consistency of the ansatz by examining the equations for the four-gluon Green's function. If it turns out to be consistent, it is unlikely that it will be inconsistent for the higher Green's functions.

We should also like to examine the improvement of the approximation (3.4) for the area of a loop. As we have already mentioned, we believe that higher approximations would involve consideration of the vibrational modes of a string; the static force already introduced corresponds to the longitudinal mode. As we have been using the covariant form for the short-distance part of the gluon propagator, the string would be expected to interact with gluons. Of course, the string itself is introduced to take certain gluon effects into account, and one will have to be careful not to double count.

Note added in proof. It has been pointed out to the author by M. Peskin that the definition (3.2) for the area of a nonplanar loop is unreasonable, since a loop consisting of two widely separated loops should have an area equal to the sum of the areas of the individual loops, together with possible terms which decrease exponentially as the distance between the loops increases. The definition (3.2) does not satisfy this requirement for nonplanar loops. It is still true that the expression $e^{-\alpha\mathfrak{A}}$ can formally be written as the sum of contributions from n-point Green's functions; the sum will probably converge for nearly planar loops. Our calculation for the two-point function correctly gives us its contribution to $e^{-\alpha\mathfrak{A}}$ (for large loops). and our plausability arguments for identifying the contribution of the higher Green's functions remain as before. A consistency check for the four-point function is obviously needed.

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