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# Generalized two-dimensional QED and functional determinants

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We obtain a closed expression for the functional determinant of the Dirac operator in two-dimensional QED by confronting results on cluster decomposition with direct fermion integration in the presence of Atiyah-Singer zero modes. The result is shown to be identical with the one obtained from the ζ-function definition and the use of the modified anomaly equation.

## I. INTRODUCTION

In the study of the dynamical aspects of four-dimensional quantum chromodynamics ($QCD_4$), interest has mainly focused in the past on the properties (instantons, boundary effects, etc.) of the pure gauge theory, whereas the effects due to quarks have been regarded as a kind of perturbation. Whether this is justified or not is not clear. In two-dimensional gauge theories, where the pure gauge part has no dynamical degree of freedom, it is essential to include the matter part if one wants to have a nontrivial theory, and it has been demonstrated, in fact, that the quark mass and the vacuum structure play an important role in the understanding of the subtle differences existing between color screening and confinement.[1]

One way of incorporating the effects due to quarks into a semiclassical treatment would consist of performing the functional integration over the fermionic degrees of freedom. The resulting effective action $S_{\text{eff}}[A]$ would now involve, in addition to the original pure gauge field Lagrangian, the contribution due to the fermion determinant $\ln(\det i\not{D})$, where $\not{D}$ is the covariant Dirac operator. In the context of two-dimensional QED and its Abelian generalizations,[2] this procedure allows one to understand in a more conventional way the results derived previously[1] on the basis of the boson representation of fermions.

The calculation of fermionic determinants has recently been found useful also in the semiclassical treatment of theories involving no fermions in the Lagrangian, such as the complex projective, $CP^n$, models.[3,4] Fermion determinants in two-dimensional QED for nontrivial winding have also been previously considered.[5,6] However, its functional representation in compact form has not been given for generic field configurations with arbitrary winding. It is the object of this paper to fill this gap. We shall see that for external field configurations with nontrivial winding, $\ln(\det i\not{D})$ will involve the contribution of an additional nonpolynomial term in $A_\mu$, which, however, cancels in the computation of $U(1)_G$-invariant, but chirality-breaking correlation functions. The result agrees with the formulas previously obtained from cluster arguments.[5]

The plan of the paper is as follows: In Sec. II we discuss the ζ-function approach to the calculation of the determinants, with special emphasis on the simplifications resulting in two dimensions. In Secs. III and IV we then calculate $\ln(\det i\not{D})$ for generic field configurations using different methods: The method of Sec. III is based on explicit functional integration taking into account the Atiyah-Singer zero modes, and comparing them with previously derived results[5]; the calculation of Sec. IV is entirely based on the modified anomaly relation for the induced axial-vector current. The arbitrary integration constant arising in the latter treatment is determined in Sec. V by computing $\ln(\det i\not{D})$ for particular field configurations using the ζ-function method. We conclude in Sec. VI with some remarks.

## II. THE ζ-FUNCTION METHOD

The ζ-function method[7] has been used as a powerful mathematical tool[8] for the study of the Atiyah-Singer index of elliptic operators, as well as for the computation of their determinants.[9] In particular, for massless two-dimensional QED the operator of interest is the Euclidean gauge-covariant Dirac operator

$$i\not{D} = i\not{\partial} + e\not{A}\,, \tag{2.1}$$

where $A_\mu$ is assumed to tend to a pure gauge at asymptotically Euclidean points:

$$A_\mu(x) \xrightarrow[x^2\to\infty]{} g(x)\partial_\mu g^{-1}(x)\,. \tag{2.2}$$

Our conventions will be

$$\gamma_1 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The operator (2.1) appears in the action of the Euclidean Feynman path integral in the form of the bilinear $\bar{\psi} i \not{D} \psi$; in order to perform the functional integration over the fermion fields $\psi$ one expands $\psi$ in terms of a complete set or orthonormal functions [see Eq. (3.5)]. These are conveniently chosen to be eigenfunctions of the associated Dirac operator $\hat{\not{D}}$,

$$\hat{\not{D}} u_i = \lambda_i u_i \,, \tag{2.3a}$$

$$\hat{\not{D}} = \frac{R^2 + x^2}{2R} \not{D} \,, \tag{2.3b}$$

with the normalization

$$\int d^2x \frac{2R}{R^2 + x^2} u_i^\dagger(x) u_j(x) = \delta_{ij} \,, \tag{2.4}$$

where $R$ is an arbitrary parameter with dimensions of length, and the $\lambda_i$'s are dimensionless. $\hat{\not{D}}$ is self-adjoint with respect to the integration measure implied by (2.4). The completeness and discreteness of the spectrum follows from the observation[6,10] that for $A_\mu(x)$ satisfying (2.2) there exists a one-to-one mapping from the set $\{u_i\}$ to the eigenfunctions of the Dirac operator $D_{S_R{}^2}$ defined on the two-dimensional sphere of radius $R$ obtained by stereographic projection from $R_2$.

At this point, a comment on the physical relevance of the compactified Dirac equation (2.3) is in order. All classical quantities (representable by tree diagrams), such as the Euclidean Green's function (without zero modes),

$$i\not{D} G(x, y; A) = -\delta^2(x - y)$$

have a spectral representation in terms of the eigenfunctions of the associated Dirac equation:

$$G(x, y; A) = -\sum_i \frac{u_i(x) u_i^\dagger(y)}{\lambda_i} \,.$$

The basic reason for this lies in the conformal covariance of the massless Dirac equation. Such spectral representations are particularly useful for computing nonclassical quantities as the fermion determinant and induced current, where loop graphs are involved. The generalized $\zeta$-function formalism based on Eq. (2.3) provides here a natural and nonperturbative regularization. Unlike the case of classical (tree-graph) quantities, the correspondingly regularized nonclassical quantities will in general depend on the choice of $R$ in (2.3). For two-dimensional QED, a superrenormalizable theory, this dependence on $R$ of the determinant will only occur via zero modes and, in fact, cancels in the correlation functions, which lose complete memory of this arbitrary parameter. Those features peculiar to two-dimensional QED will emerge in the course of the discussion in this section.

Because of the $\gamma_5$ invariance of $\not{D}$, there exists for every positive (nonvanishing) eigenvalue of $\hat{\not{D}}$ a corresponding negative one. Hence, we formally define, up to a sign,

$$\det{}' i\hat{\not{D}} = \prod_{\lambda_i > 0} \lambda_i^2 \,, \tag{2.5}$$

where the prime indicates that zero eigenvalues have been omitted in the product. It is this (truncated) formally divergent product which enters the Feynman path integral after fermion integration (see Sec. III); its "value" may be calculated by various methods. The results obtained by different regularization procedures are all expressible in terms of the associated $\zeta$ function defined by

$$\zeta_A(s, R) = \sum_{\lambda_i > 0} \frac{1}{(\lambda_i^2)^s} \,, \tag{2.6}$$

where the subscript $A$ refers to the dependence of the eigenvalues on the gauge field $A_\mu$. The label $R$ in the argument of $\zeta$ indicates that $\varphi$ will, in general, depend on the choice radius of the stereographic sphere.[7]

$\zeta(S, R)$ is a meromorphic function of $s$, with $s = 0$ a regular point. In terms of it, the normalized determinant is given by

$$\frac{\det{}' i\hat{\not{D}}}{\det i\hat{\not{\partial}}} = \exp\{-[\zeta_A'(0,R) - \zeta_0'(0,R)] + [\zeta_A(0,R) - \zeta_0(0,R)]\ln M\} \,, \tag{2.7}$$

where

$$\hat{\not{\partial}} = \frac{R^2 + x^2}{2R} \not{\partial}$$

and $M^2$ is an arbitrary dimensionless parameter reflecting the freedom one has in the choice of regularization procedure for computing the product (2.5). For a Pauli-Villars regularization, $\ln M$ is just a particular linear combination of the Pauli-Villars regulators $\ln M_j$.[11] For our problem in question, $\zeta_A(O, R)$ is conveniently calculated by expressing $\zeta_A(O, R)$ in terms of the Mellin transform of the heat kernel[6,9]

$$h(t; x, y) = \sum_i e^{-\lambda_i^2 t} u_i(x) u_i^\dagger(y) \,. \tag{2.8}$$

Recalling (2.6) and (2.4), one evidently has

$$\zeta_A(O, R) = \lim_{s \to 0} \frac{1}{\Gamma(s)} \int_0^\infty dt\, t^{s-1} \int d^2x \frac{2R}{R^2 + x^2} \operatorname{tr}\left[h(t; x, x) - \sum_i u_i^{(0)}(x) u_i^{\dagger(0)}(x)\right] \,, \tag{2.9}$$

where $\{u_i^{(0)}\}$ denotes the eigenfunctions (2.3b) corresponding to $\{\lambda_i\}=0$. The right-hand side may be evaluated by making use of the well-known asymptotic property of the heat-equation kernel, which reads, for the diagonal part,

$$\frac{2R}{R^2+x^2}\,(h(t;x,x))\underset{t\to 0}{\sim}\frac{1}{(4\pi t)^{\dim/2}}\times(a_0+a_1t+a_2t^2+\cdots)\,. \tag{2.10}$$

All the $a_i$'s are computable and for the first few coefficients explicit formulas have been given in the literature.[12] Substituting (2.10) into (2.9) we obtain[12]

$$\zeta_A(O,R)-\zeta_0(O,R)=\frac{1}{(4\pi)^{\dim/2}}\int d^2x\,\mathrm{tr}\,\{a_{\dim/2}[A]-a_{\dim/2}[0]\}-N\,, \tag{2.11}$$

where $N$ = number of zero modes.

Note that the result (2.11) is not model specific; it holds for any geometric elliptic differential operator on a compact manifold without boundary which can be written in the Dirac form, the dimensions of space being arbitrary.[13] The form of $a_{\dim/2}$ is, however, complicated, in general. The dependence on the underlying metric drops out, however, in the trace of $a_{\dim/2}$ if the corresponding vacuum value ($A_\mu=0$) is subtracted. In two dimensions this simplification occurs even for the coefficient itself. One has

$$a_1[A]-a_1[0]=\tfrac{1}{2}\,e\gamma_5\epsilon_{\mu\nu}F_{\mu\nu}\,. \tag{2.12}$$

Hence, the first term in (2.11) vanishes. This corresponds to the vanishing of the trace anomaly in $D=2$. Moreover, in two dimensions one has a vanishing theorem: For winding number $n\gtrless 0$, there are no normalizable negative (positive) chirality "zero-energy" eigenfunctions. The proof is by explicit computation,[6] but there exists (perhaps) a more elegant structural proof. As shown by Atiyah, Hitchin, and Singer,[14] such a theorem definitely fails in four dimensions, where it is only true for multi-instanton configurations which minimize the pure gauge part of the interaction. We have thus

$$\zeta_A(O,R)-\zeta_0(O,R)=-n\,, \tag{2.13}$$

where $n$ is the winding number associated with the field configuration $A_\mu$ in question. $\zeta_A(O,R)$ is thus *independent* of the particular representative $A_\mu$ within a given Chern class. This is a property specific of two dimensions. The difference (2.13) is seen to be independent of $R$. This is in fact a general property, which is however not shared by $\zeta_A'(O,R)-\zeta_0'(O,R)$. Hence, the normalized determinant (2.7) will depend in general on the choice of metric, that is $R$. It is, in fact, easy to show by methods similar to those leading to (2.11) that the normalized determinant (2.7) will be independent of $R$ if, and only if, the trace of the energy-momentum tensor $\Theta_{\mu\nu}$ vanishes, that is, in the absence of a trace anomaly. Such an anomaly will always occur if zero modes are present. This follows from the fact that, independent of the dimension $D$ of space,

$$\tfrac{1}{2}\int d^Dx\,\Theta_{\mu\mu}(x)=\zeta_A(O,R)=-N+\frac{1}{(4\pi)^{D/2}}\int d^Dx\,\mathrm{tr}\,a_{D/2}(x)\,,$$

where $N$ is again the number of zero modes. In four dimensions $\Theta_{\mu\mu}\neq 0$ even in the absence of zero modes. The situation is simpler in two dimensions, where no trace anomaly exists if zero modes are absent. In two dimensions a dependence on $R$ will thus only enter through the zero modes. However, as will become evident from our discussion in Sec. III, this remaining dependence on $R$ also cancels in the correlation functions for the problem in question. Hence, in two dimensions we may choose any value of $R$ as far as the correlation functions are concerned, whereas in four dimensions the limit $R\to\infty$ will have to be taken after proper renormalization.

We proceed now to derive a functional-differential equation for the logarithm of the determinant. It will be convenient to introduce the notation

$$\Gamma=-\ln\left(\frac{\det i\hat{D\!\!\!/}}{\det i\hat{\partial\!\!\!/}}\right). \tag{2.14}$$

Noting from (2.13) that $\zeta_A(O,R)$ is independent of $A_\mu$, we have

$$\frac{\delta\Gamma}{\delta A_\mu(x)}=-\lim_{s\to 0}\left[\sum{}'\,\frac{\lambda_i}{(\lambda_i^2)^{s+1}}\,\frac{\delta\lambda_i}{\delta A_\mu(x)}+O(s)\right]. \tag{2.15}$$

The variation of the Dirac eigenvalue can be easily calculated from the eigenvalue equation:

$$\frac{\delta\lambda_i}{\delta A_\mu(x)}=eu_i^\dagger(x)\gamma_\mu u_i(x)\,. \tag{2.16}$$

Substitution of (2.16) into (2.15) yields

$$\frac{\delta\Gamma}{\delta A_\mu(x)}=-\,ej_\mu(x)\,, \tag{2.17}$$

where

$$j_\mu(x)=\lim_{s\to 0}\sum{}'\,\frac{\lambda_i}{(\lambda_i^2)^{s+1}}\,u_i^\dagger(x)\gamma_\mu u_i(x)\,. \tag{2.18}$$

Making use of the spectral representation of the external-field Green's function $G(x,y;A)$, we recognize (2.18) as the generalization of the familiar definition

$$j_\mu(x) = \mathrm{tr}\,\gamma_\mu G(x,y;A)$$

of the induced current to the case where zero modes are present: The limit $s\to 0$ in the $\zeta$-regularized version (2.18) replaces the "old fashioned," gauge-invariant, "separated-point" technique for computing $\mathrm{tr}\gamma_\mu G(x,x;A)$. The $\zeta$-function definition (2.7) of the determinant thus satisfies the formal properties of the Schwinger functional calculus, the induced current being given by the functional derivative of $\ln(\det i\not{D})$ with respect to $A_\mu$.

Now, in two dimensions one has the simple identity $i\gamma_\mu\gamma_5 = \epsilon_{\mu\nu}\gamma_\nu$, so that we have from (2.18)

$$\epsilon_{\mu\nu} j_\nu(x) = \lim_{s\to 0} {\sum}' \frac{\lambda_i}{(\lambda_i^2)^{s+1}} u_i^\dagger(x) i\gamma_\mu\gamma_5 u_i(x) \quad (2.19)$$

or

$$\tilde\partial_\mu j_\mu(x) = \lim_{s\to 0} 2\frac{2R}{R^2+x^2} {\sum}' \frac{1}{(\lambda_i^2)^s} u_i^\dagger(x)\gamma_5 u_i(x)\,, \quad (2.20)$$

where

$$\tilde\partial_\mu = \epsilon_{\mu\nu}\partial_\nu\,. \quad (2.21)$$

We may again rewrite this expression in terms of the heat-equation kernel (2.10):

$$\begin{aligned}\tilde\partial_\mu j_\mu(x) &= \lim_{s\to 0}\frac{4R}{R^2+x^2}\frac{1}{\Gamma(s)}\int_0^\infty dt\, t^{s-1}\,\mathrm{tr}\gamma_5\left[h(t;x,x) - \sum_i u_i^{(0)}(x)u_i^{(0)\dagger}(x)\right]\\ &= \frac{1}{2\pi}\mathrm{tr}\gamma_5 a_1 - 2\frac{2R}{R^2+x^2}\sum_i u_i^{(0)\dagger}(x)\gamma_5 u_i^{(0)}(x)\,. \end{aligned} \quad (2.22)$$

Recalling (2.12) and noting that $\mathrm{tr}\gamma_5 a_1[0]=0$ we thus obtain from (2.22) and (2.17) the desired functional-differential equation for $\Gamma'$:

$$\tilde\partial_\mu \frac{\delta\Gamma'}{\delta A_\mu(x)} = -\frac{e^2}{2\pi}\epsilon_{\mu\nu}F_{\mu\nu}(x) + 2\frac{2R}{R^2+x^2} e\sum_i u_i^{(0)\dagger}(x)\gamma_5 u_i^{(0)}(x)\,. \quad (2.23)$$

This equation will provide the basis in Sec. III for calculating $\Gamma'$ for generic external field configurations. As already noted, the $R$ dependence coming from the zero modes actually cancels in the correlation functions as we shall see in Sec. III.

## III. DETERMINANT FROM FUNCTIONAL INTEGRATION AND CLUSTER ARGUMENTS

For zero-winding field configuration $A_\mu^{[0]}$ the generic form of the fermion determinant in the Schwinger model has been known since Schwinger's pioneering work.[15] It is conveniently written in the form

$$\Gamma[A] = \frac{e^2}{8\pi}\int d^2x \int d^2x'\epsilon_{\mu\lambda}F_{\mu\lambda}(x)D(x-x')\epsilon_{\nu\rho}F_{\nu\rho}(x')\,, \quad (3.1)$$

where $\Gamma$ is defined as in (2.14);

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\mu A_\mu \quad (3.2)$$

and

$$D(z) = -\frac{1}{4\pi}\ln\mu^2 z^2\,, \quad \Box D(z) = -\delta^2(z)\,. \quad (3.3)$$

In this section we determine its form for generic field configurations of arbitrary winding $n$ by obtaining a formal representation for the $\theta$-vacuum fermion correlation function in terms of the determinant and comparing the expression with the exact result as obtained in Ref. 5 from the well-known cluster properties of the theory. Our starting point is the functional integral

$$\begin{aligned}&\langle\theta|\psi_{\alpha_1}(x_1)\cdots\psi_{\alpha_m}(x_m)\bar\psi_{\beta_1}(y_1)\cdots\bar\psi_{\beta_m}(y_m)|\theta\rangle_A\\ &\quad = Z^{-1}\exp\left[i\theta\int\frac{d^2z}{4\pi}e\epsilon_{\mu\nu}F_{\mu\nu}(z)\right]\int d\bar\psi\, d\psi\,\psi_{\alpha_1}(x_1)\cdots\psi_{\alpha_m}(x_m)\bar\psi_{\beta_1}(y_1)\cdots\bar\psi_{\beta_m}(y_m)\exp\left[-\int d^2z\,\bar\psi(z)i\not{D}\psi(z)\right],\end{aligned} \quad (3.4)$$

where

$$Z = \det(i\not{\partial})$$

and where the functional integral on the right-hand side defines the correlation function on the left. In order to perform the functional integration in (3.4), we expand the $\psi_\alpha(x)$'s in terms of a complete set of orthonormal functions $u_{i\alpha}(x)$:

$$\psi_\alpha(x) = \sum_i b_{i\alpha} u_{i\alpha}(x)\,, \tag{3.5}$$

where the expansion coefficients $b_{i\alpha}$ are Gaussmann variables satisfying the usual rules

$$\int db\, db^\dagger \begin{bmatrix} 1 \\ b \\ \bar{b} \\ \bar{b}b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\,, \quad b^2 = \bar{b}^2 = 0\,. \tag{3.6}$$

The $u_i$'s are conveniently chosen to be solutions of the eigenvalue problem (2.3) and (2.4).

Replacing $d\bar{\psi}\, d\psi$ by $\prod d\bar{b}_i db_i$ and recalling that according to our discussion in Sec. II there exist, in two dimensions, precisely $n$ normalizable "zero-energy" eigenfunctions $u_{i\alpha}^{(0)}$ [with chirality = sgn($n$)] in a winding field $A_\mu^{[n]}$, we obtain after integration (we set $m=n$ for convenience)

$$\langle\theta\,|\,\psi_{\alpha_1}(x_1)\cdots\psi_{\alpha_n}(x_n)\bar{\psi}_{\beta_1}(y_1)\cdots\bar{\psi}_{\beta_n}(y_n)\,|\,\theta\rangle = e^{in\theta}\frac{\prod'\lambda_2[A^{[n]}]}{\prod\lambda_i[0]}\Big(\sum_i \epsilon_{i_1\cdots i_n} u_{i_1\alpha_1}^{(0)}(x_1)\cdots u_{i_n\alpha_n}^{(0)}(x_n)\Big)\Big(\sum_j \epsilon_{j_1\cdots j_n}\bar{u}_{j_1\beta_1}^{(0)}(y_1)\cdots\bar{u}_{j_n\beta_n}^{(0)}(y_n)\Big)\,, \tag{3.7}$$

where the product only runs over the nonvanishing eigenvalues. In order to compare this result with the result (2.20) obtained in Ref. 5, we expand the orthonormal "zero-energy" eigenfunctions $u_{i\alpha}^{(0)}(x)$ in terms of the nonorthonormal set of "zero-energy" eigenfunctions,

$$\psi_l^{(0)}(x) = \left(\frac{\mu}{2\pi}\right)^{1/2}[\mu(x_1+ix_2)]^{l-1}\begin{bmatrix} e^{\phi(x;A^{[n]})} \\ 0\end{bmatrix}\,, \quad l=1,\ldots,n\,, \tag{3.8a}$$

where[16]

$$\phi(x;A) = e\int d^2z\, D(x-z)\epsilon_{\lambda\mu}\partial_\lambda A_\mu(z) \tag{3.8b}$$

and $\mu$ is conveniently chosen to be the same arbitrary parameter as appearing in the definition (3.3) of $D(z)$. As already advertised, all "zero-energy" eigenfunctions have the same chirality (in two dimensions). Hence, among all correlation functions (3.4) only that one is nonvanishing, for which $\alpha_i=\beta_i=1$, $i=1,\ldots,n$. This is of course entirely in agreement with what we know from the operator solution of Lowenstein and Swieca.[17] We shall thus omit from here on the indices $\alpha_i$ and $\beta_i$, it being understood that $\alpha_i=\beta_i=1$.

Expanding now the $u_i^{(0)}$'s in terms of the eigenfunctions (3.9)

$$u_i^{(0)}(x) = \sum_l a_{il}\psi_l^{(0)}(x)\,, \tag{3.9}$$

we may cast (3.7) into the form

$$\langle\theta\,|\,\psi_1(x_1)\cdots\psi_1(x_n)\bar{\psi}_1(y_1)\cdots\bar{\psi}_1(y_n)\,|\,\theta\rangle = \frac{\prod'\lambda_k[A^{[n]}]}{\prod\lambda_k[0]}\det(a^\dagger a)e^{in\theta}\Big(\sum_l \epsilon_{l_1\cdots l_n}\psi_{l_1}^{(0)}(x_1)\cdots\psi_{l_n}^{(0)}(x_n)\Big)\Big(\sum_l \epsilon_{l_1\cdots l_n}\bar{\psi}_{l_1}^{(0)}(y_1)\cdots\bar{\psi}_{l_n}^{(0)}(y_n)\Big)\,. \tag{3.10}$$

The totally antisymmetric product of eigenfunctions in (3.10) reproduces the translationally invariant polynomial in Eq. (2.20) of Ref. 5. (See also Ref. 6.) It is useful to introduce the "normalization matrix" $N$ defined by the overlap integrals

$$(N^{-1})_{l'l}=\int d^2x\,\frac{2R}{R^2+x^2}\,\psi_{l'}^{(0)}(x)\psi_l^{(0)*}(x) \qquad (3.11)$$

in terms of which

$$a^\dagger a=N\,. \qquad (3.12)$$

Hence, comparing with the "clustering" result of Ref. 5, we conclude that

$$\frac{\prod\lambda_i[A]}{\prod\lambda_i[0]}=e^{-\Gamma[A]}e^{-\mathrm{tr}(\ln N[A])}. \qquad (3.13)$$

This generalizes the result (3.8) of Ref. 5 to arbitrary winding.

As was already remarked, only the first factor in Eq. (3.13) contributes to zero-winding fields. The second factor, characteristic for fields $A_\mu$ of nonzero winding, is highly nonlinear in $A_\mu$. It is however, to be noted that only the combination $\det(i\not{D})\times\exp(\mathrm{tr}\ln N)$ actually occurs, that is, the zero-mode part in $\det'(i\not{D})$ drops out in the correlation function (3.10). According to our discussion in Sec. II, this means that the correlation function (3.10) is in fact independent of the size of the stereographic sphere. We shall thus set $R=1$ from here on.

The calculation above is unsatisfactory in the sense that it relies on the knowledge of the tunneling correlations functions, which in general involves the solution of an even harder problem than the calculation of the determinant itself. In the following section we shall present a more self-contained method which essentially relies only on the knowledge of the axial-vector current anomaly relation in the presence of zero modes which is also known for $\mathrm{QCD}_2$ and $\mathrm{QCD}_4$.

## IV. DETERMINANT FROM ANOMALY RELATION

In this section we obtain a generic expression for $\det(i\not{D})$ using a method[18] which does not require an *a priori* knowledge of fermion correlation functions in an external field, but merely relies on the modified anomaly relation (2.23). We now choose $R=1$.

In order to integrate Eq. (2.23), we need to determine the zero-mode contribution as a function of $A_\mu$; this in turn would require the construction of the orthonormal set of "zero-energy" eigenfunctions $u_i^{(0)}(x)$. We shall get around the orthonormalization problem by working instead on the particular set of eigenfunctions (3.8) and expanding the $u_i^{(0)}(x)$'s in terms of these, as done in Eq. (3.9).

Defining the matrix $\rho$ by

$$\rho_{ll'}(x)\equiv\psi_{l'}^{(0)\dagger}(x)\gamma_5\psi_l^{(0)}(x) \qquad (4.1)$$

and using (3.12), we may cast the anomaly relation (2.23) into the form

$$\tilde{\partial}_\mu\frac{\delta\Gamma'}{\delta A_\mu(x)}=-\frac{e^2}{2\pi}\epsilon_{\mu\nu}F_{\mu\nu}(x)+\frac{4e}{1+x^2}\,\mathrm{tr}\,(N[A]\rho(x))\,, \qquad (4.2)$$

where the "normalization matrix" $N$ has been defined in Eq. (3.11).

Integration of (4.2) yields

$$\frac{\delta\Gamma'}{\delta A_\mu(x)}=-\frac{e^2}{2\pi}\int d^2y\,\tilde{\partial}_\mu D(x-y)\epsilon_{\mu\nu}F_{\mu\nu}(y)-2e\int d^2y\,\frac{2}{1+y^2}\,\tilde{\partial}_\mu D(x-y)\times\mathrm{tr}\,(N[A]\rho(y))+\partial_\mu F(x)\,. \qquad (4.3)$$

The first term will give rise to the "zero-winding" form of the determinant, Eq. (3.1), after integration. Note, however, that $\Gamma[A]$ is finite also for nonzero-winding configurations; the second term gives a new contribution to configurations of nonzero winding; the third term is the gradient of an arbitrary function $F(x)$. Recalling that $\partial_\mu j_\mu(x)=0$ and using (2.17), we have

$$\Box F(x)=0\,.$$

Only the trivial (constant) solution is consistent with a normalizable current; hence, the third term in (4.3′) is actually absent.

Now, a simple computation starting from the definition (3.11), and using (3.8), gives

$$\frac{\delta}{\delta A_\mu(x)}\,\mathrm{tr}\ln N[A]=-2e\int d^2y\,\frac{2}{1+y^2}\,\tilde{\partial}_\mu D(x-y)\times\mathrm{tr}\,(N[A]\rho(x))\,. \qquad (4.4)$$

This allows us to integrate Eq. (4.3) with the result

$$\Gamma'[A]=\Gamma[A]+\mathrm{tr}\ln N[A]+c\,, \qquad (4.5)$$

where $c$ is an arbitrary integration constant whose value can depend at most on the Chern class to which $A_\mu$ belongs, but not on the particular configuration within that class.

Comparison with the previously obtained result (3.13) shows that $c=0$. This is, however, not a very satisfying way of determining this constant since the method presented here should be self-contained. The constant $c$ may, however, also be determined if one succeeds in calculating the determinant for some conveniently chosen representative within each Chern class. This will be done in the following section.

## V. DETERMINATION OF THE ARBITRARY CONSTANT $c$

In the following we shall fix the arbitrary constant appearing in (4.5) by evaluating (4.5) for a particular representative in each Chern class and then comparing this result with the result one obtains from the definition of the determinant in terms of the $\zeta$ function, Eq. (2.6). The particular representatives we shall consider are

$$A_\mu^{[n]}(x) = \frac{n}{e}\,\epsilon_{\lambda\mu}\,\frac{x_\lambda}{1+x^2}\,. \tag{5.1}$$

The corresponding zero-energy eigenfunctions (3.9) are easily calculated to be

$$\psi_l^{(0)}(x) = \left(\frac{\mu}{2\pi}\right)^{1/2}[\mu(x_1+ix_2)]^{l-1} \times \frac{u^{-n}}{(1+x^2)^{n/2}}\,,\quad l=1,\ldots,n\,.$$

Substituting this result into (4.9) we obtain for the normalization matrix $N$ [Eq. (3.12)]

$$N_{l'l} = \delta_{l'l}(\mu^2)^{n-l+1/2}\,\frac{\Gamma(n+1)}{\Gamma(l)\Gamma(n-l+1)}\,,$$

which yields

$$\operatorname{tr}(\ln N) = -\sum_{l=1}^{n}(n-2l)\ln l + \tfrac{1}{2}n^2\ln\mu^2\,. \tag{5.2}$$

On the other hand, substitution of (5.1) into (3.1) yields

$$\Gamma = -\tfrac{1}{2}n^2 - \tfrac{1}{2}n^2\ln\mu^2\,.$$

Hence, combining everything in (4.5) we have for the special field configuration (5.1)

$$\Gamma' = -\tfrac{1}{2}n^2 - \sum_{l=1}^{n}(n-2l)\ln l + c\,. \tag{5.3}$$

We now proceed to determine the so-far unknown constant $c$ by calculating $\Gamma'$ from its definition in terms of the $\zeta$ function, Eqs. (2.7) and (2.14).

The calculation of $\Gamma$ thus involves two steps: (a) finding the eigenvalue spectrum of the operator $i\hat{\not{D}}$ in the external field (5:1) and (b) calculation of $\zeta'(0,1)$.

(a) *Eigenvalue spectrum*. Introducing the variables

$$z = x_1 + ix_2\,,\quad \bar{z} = x_1 - ix_2\,,$$

Eq. (3.7) in the external vortex field (5.1) reads

$$\begin{pmatrix} \dfrac{-\lambda}{1+\bar{z}z} & -\left(2\partial_z - n\dfrac{\bar{z}}{1+\bar{z}z}\right) \\ \left(2\partial_{\bar{z}} + n\dfrac{z}{1+\bar{z}z}\right) & -\dfrac{\lambda}{1+\bar{z}z} \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0\,. \tag{5.4}$$

It is convenient to introduce a new spinor $\varphi_\alpha$ by

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} (1+\bar{z}z)^{-n/2} & \varphi_1 \\ (1+\bar{z}z)^{n/2} & \varphi_2 \end{pmatrix}\,. \tag{5.5}$$

$\varphi_1$ then satisfies the differential equation

$$[L^2 - (1-n)(1+\bar{z}z)\bar{z}\partial_{\bar{z}}]\,\varphi_1 = \lambda^2\varphi_1\,, \tag{5.6}$$

where

$$L^2 = -(1+\bar{z}z)^2\partial_{\bar{z}}\partial_z\,.$$

$L^2$ can be identified with the square of the angular momentum operator $\vec{L}^2$

$$L^2 = L_+L_- + L_3{}^2 - L_3\,,$$

where the operators

$$L_+ = -i(z^2\partial_z + \partial_{\bar{z}})\,,$$
$$L_- = -i(\bar{z}^2\partial_{\bar{z}} + \partial_z)\,,$$
$$L_3 = z\partial_z - \bar{z}\partial_{\bar{z}}$$

satisfy the usual angular momentum commutation relations. Thus, for $n=1$ we immediately have

$$\lambda^2 = l(l+1)\,,\quad d_l = 2l+1\,, \tag{5.7}$$

where $l=0,1,\ldots$ and $d_l$ denotes the degeneracy of the square of the eigenvalues. For the general case, we have two sets of solutions corresponding to the ansatz

$$\varphi_1 = \bar{z}^m f(y) \tag{5.8a}$$

and

$$\varphi_1 = z^m g(y)\,, \tag{5.8b}$$

where we have found it convenient to introduce the variable $y = 1+\bar{z}z$.

Substitution of the ansatz (5.8) into (5.6) yields the differential equations

$$\begin{aligned} &y^2(1-y)f'' - [(n-1)y(1-y) + (m+1)y^2]f' + [(n-1)my - \lambda^2]f = 0\,, \\ &y^2(1-y)g'' - [(n-1)y(1-y) + (m+1)y^2]g' - \lambda^2 g = 0\,. \end{aligned} \tag{5.9}$$

We expand $f$ and $g$ in a power series,

$$f(y)=y^{\alpha}\sum_{k=0} f_k y^{\kappa},$$
$$g(y)=y^{\beta}\sum_{k=0} g_k y^{\kappa}. \tag{5.10}$$

Substituting this ansatz into the above equations and requiring the series to terminate for some integer $k_{\max}=N$ yields, respectively,

$\alpha=-(m+N)$ with $\lambda^2=(N+m)(N+m+n)$, $m=(0,\infty)$

or

$\alpha=-(N+1-n)$ with $\lambda^2=(N+1)(N+1-n)$, $m=0$, $n=1$,

and

$\beta=-(N+m+1-n)$ with $\lambda^2=(N+m+1)(N+m+1-n)$, $m\geq n-1$

or

$\beta=-N$ with $\lambda^2=N(N+n)$, $m<N$,

where the allowed values of $m$ are determined by the condition that the solution (5.5) be normalizable in the sense of (2.4). However, not all of the corresponding solutions are found to be independent. Selecting only the independent ones, we may summarize the results by the simple formula

$$\lambda_l{}^2=l(l+n),\quad d_l=2l+n, \tag{5.11}$$

where $d_l$ refers again to the degeneracy of $\lambda^2$. The result (5.11) also includes the case of zero external field $n=0$. For $n=1$ we recover the previous result (5.7).

(b) *Calculation of* $\zeta'(0)$. For the eigenvalue spectrum (5.11) the $\zeta$ function (2.6) is conveniently written in the form. [we set $\zeta_A(s,l)\equiv\zeta(s)$]

$$\zeta(s)=2\sum_{l=1}^{\infty}\left(l+\frac{n}{2}\right)^{-2s+1}\left[1-\frac{n^2}{4(l+n/2)^2}\right]^{-s}. \tag{5.12}$$

Performing a binomial expansion and differentiating with respect to $s$ we obtain, for $s=0$,

$$\zeta'(0)=4\zeta_R'\left(-1,1+\frac{n}{2}\right)+\frac{n^2}{2}\left[\zeta_R\left(2s+1,1+\frac{n}{2}\right)-\frac{1}{2s}\right]_{s=0}+R^{(n)}, \tag{5.13}$$

where $\zeta_R$ denotes the usual (modified) Riemann $\zeta$ function, and

$$R^{(n)}=\lim_{\lambda\to\infty}R_\lambda^{(n)}$$

with

$$R_\Lambda^{(n)}=-\sum_{l=1}^{\Lambda}\left\{(2l+n)\left[\ln l(l+n)-\ln\left(l+\frac{n}{2}\right)^2\right]-\frac{n^2}{2}\frac{1}{l+n/2}\right\} \tag{5.14}$$

and $\Lambda$ some (large) integer.

The sum (5.14) is conveniently evaluated by making use of the asymptotic formula

$$\sum_{\nu=0}^{\Lambda}\frac{1}{(\nu+\alpha)^s}\simeq\zeta_R(s,\alpha)+\frac{(\alpha+\Lambda)^{-s+1}}{1-s}+\tfrac{1}{2}(\alpha+\Lambda)^{-s}-\frac{s}{12}(\alpha+\Lambda)^{-1-s}+O(\Lambda^{-s-3}).$$

We obtain after some calculation

$$R^{(n)}=-\frac{n^2}{2}+2[\zeta_R'(-1,1)+\zeta_R'(-1,1+n)]-n[\zeta_R'(0,1+n)-\zeta_R'(0,1)]-4\zeta_R'\left(-1,1+\frac{n}{2}\right)-\frac{n^2}{2}\gamma-\frac{n^2}{2}\left[\zeta_R\left(s,1+\frac{n}{2}\right)-\zeta_R(s,1)\right]_{s=0}.$$

Introducing this result in (5.13) and combining terms, we are left with

$$\zeta'(0)=-\frac{n^2}{2}+2[\zeta_R'(-1,1+n)-\zeta_R'(-1,1)]-n[\zeta_R'(0,1+n)-\zeta_R'(0,1)]+4\zeta_R'(-1,1),$$

which is readily evaluated to be

$$\zeta'(0)=-\frac{n^2}{2}-\sum_{l=1}^{n}(n-2l)\ln l+4\zeta_R'(-1,1).$$

Subtracting the $n=0$ contribution and recalling (2.13) we obtain for the normalized determinant

$$\hat{\Gamma}=-\ln\left[\frac{\det(i\hat{\not{D}})}{\det(i\not{\partial})}\right]=-\frac{n^2}{2}-\sum_{l=1}^{n}(n-2l)\ln l+n\ln M. \tag{5.15}$$

Comparison with (5.3) shows that the so-far undetermined constant $c$ is equal to $n\ln M$. In particular, the result (3.13) obtained by comparison with the known form of the Schwinger-model correlation functions[5] corresponds to the choice $M=1$.

## VI. CONCLUSION

Our explicit construction of the Schwinger-model determinant in a generic field of arbitrary winding has shown that the presence of Atiyah-Singer zero modes leads to a nonlocal and nonpolynomial modification of the well-known zero-winding result (3.1). Nevertheless, this added complication does not manifest itself in this dramatic way in the fermion correlation functions (3.4) which, after integrating over the fermionic degrees of freedom, are still described in terms of an effective action involving only up to quadratic terms in $A_\mu$, independent of the winding. This reflects, of course, the exact solubility of the model.

It is clear from our discussion in Sec. III how the result (3.11) generalizes to an Abelian gauge theory with flavor. For an external field configuration of winding $n$, there will now exist $n\times N$ zero modes, where $N$ is the dimension of the flavor group SU($N$), so that the first nonvanishing fermion correlation function will be of order $2nN$. This is indeed borne out by explicit operator solutions which have recently been constructed for this case.[2]

One may similarly rederive the results obtained previously from boson representation for the torus of SU($N$) color.[2] This does not, however, include the question of screening versus confinement[1] where the ongoing aspects of boson representation are involved which have not been understood so far within the framework of functional integration.

An interesting model results from combining $QED_2$ with the $CP^n$ model.[19] The fact that the nonpolynomial part in the fermion determinant does not manifest itself in the correlation functions allows one to study "induced instantons" in this model, which, unlike the case of $QED_2$, are expected to consist now of a whole parametric family of configurations minimizing the effective action for every Chern class. In this sense, this hybrid model provides a more tractable working ground resembling the situation one would expect to encounter in $QCD_4$ if configurations were considered which minimize the effective (correlation function dependent) action obtained after fermion integration.

Finally, let us remark that the techniques used in Sec. IV for calculating the $QED_2$ determinant may also be applied to $QCD_2$. This is, however, a much less trivial task despite the fact that the topological aspects are trivial here; the corresponding determinant is no longer a polynomial in $A_\mu$.

After completing this work we received a report by A. Patrascioiu (Ref. 20) where the problem of nonzero-winding fermion determinant for $QED_2$ was studied for a special field configuration. The open questions raised in this paper are answered in our work. If we were to discuss our problem on the sphere as a limit of a boundary-value problem (a sphere with a boundary around the north pole) as Patrascioiu does, great care would have to be taken in choosing the right type of boundary condition which does not lead to "topological obstructions." The correct boundary condition for the Dirac equation is necessarily nonlocal and of the "spectral" kind.[21]

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