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Conformal conservation laws in action-at-a-distance theory

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As a consequence of invariance under the special conformal transformation, a new conserved four-vector is derived for two massive charged point particles interacting through classical relativistic action-at-a-distance scalar and vector potentials. This represents a generalization of the conservation law discovered for sourceless electromagnetic fields by Bessel-Hagen in 1921. The conserved four-vector is explicitly evaluated for Schild's solution of the equation of motion, corresponding to uniform circular motion. All 15 of the conserved quantities related to the 15-parameter conformal group are now known for Schild's solution which has been extended to cover scalar as well as vector potentials.

I. INTRODUCTION

Recently, there have been many investigations of the full conformal group of transformations, especially as applied in field theories.[1] This 15-parameter group transforms coordinates such that the space-time interval is invariant to within a coordinate-dependent factor. In particular, a spherical surface expanding with speed c transforms into a spherical surface again expanding with speed c.[2,3] The familiar 10-parameter Lorentz group is that subset which generates translations and rotations of space and time coordinates, and leads to conservation of the momentum four-vector and the angular momentum tensor.

The one-parameter dilation transformation changes the scale of the interval and leads to a conserved dilation scalar D for the free electromagnetic field[4] and for a massless scalar field.[5] If one takes D for a point particle in a field to have the same form as that for the free field itself

$$D=\underline{S}\cdot\underline{P}\,, \tag{1.1}$$

then D is "partially" conserved[6] in the limit as $m\to 0$. Here $\underline{S}=(\vec{r}, ict)$ is the position of the particle and $\underline{P}=(\vec{p}, iw/c)$ is its momentum. The inclusion of a mass term[7,8]

$$D=\underline{S}\cdot\underline{P}+mc^2\tau \tag{1.2}$$

allows the dilation to be conserved for any mass m. Here τ is the proper time of the particle. Using action-at-a-distance Wheeler-Feynman interactions, Andersen and von Baeyer[9] have shown that the conserved dilation for two interacting point masses is of the form of Eq. (1.2) plus an interaction term.

The four-parameter special conformal transformation is an inversion which has been variously interpreted as a transformation from an inertial frame to one with constant acceleration,[10,11] or of a generalized coordinate-dependent dilation.[6] For a free electromagnetic field, this transformation leads to a conserved conformal four-vector[4] $\underline{K}$. If one takes $\underline{K}$ for a point particle in a field to have the same form as that for the free field itself

$$\underline{K}=2\underline{S}\,\underline{S}\cdot\underline{P}-\underline{P}\,\underline{S}\cdot\underline{S}\,, \tag{1.3}$$

then $\underline{K}$ is also "partially" conserved[6] in the limit as $m\to 0$.

Using action-at-a-distance scalar and vector interactions without radiation reaction between two point particles, we will show in Sec. II and III that a conformal vector can be constructed which is relativistically conserved for any nonzero mass. As with all relativistically conserved quantities, $\underline{K}$ includes both kinematic terms and interaction terms with double integrals over the world lines of the particles.[12] This is in accord with the "no-interaction" theorems[13] which state that if a conserved quantity is composed entirely of kinematic terms, then only "trivial" constant velocity motion is possible. In Sec. IV we calculate the value of $\underline{K}$ for two point particles moving in concentric circular orbits, and in Sec. V for one particle in field-free linear motion.

II. SCALAR POTENTIAL

The momentum $\underline{P}_i$ of a point particle moving with four-velocity $\underline{U}_i$ through a scalar potential ϕ_i is

$$\underline{P}_i=(m_i+g_i\phi_i/c^2)\underline{U}_i\,, \tag{2.1}$$

where m_i is the free-particle mass and g_i is the "strength" of the interaction of that particle with the potential. For attractive interactions between two particles, $g_i > 0$. The equation of motion for the particle is

$$d\underline{P}_i/d\tau_i = -g_i \underline{\Box}_i\, \phi_i\,, \tag{2.2}$$

where τ_i is its proper time and $\underline{\Box} \equiv (\vec{\nabla}, \partial/ic\partial t)$.

A. Time-symmetric interaction

The time-symmetric massless scalar potential at particle one is half the sum of the advanced plus retarded potentials arising from the positions of particle two:

$$\phi_1(\vec{r}_1, t_1) = -c^{-1} g_2 \int_{-\infty}^{\infty} d\tau_2\, \delta(\xi^2)\,. \tag{2.3}$$

Here $\xi^2 = (t_1 - t_2)^2 - R^2/c^2$, $\vec{R} = \vec{r}_1 - \vec{r}_2$, $R = |\vec{R}|$, and the Dirac δ function $\delta(\xi^2)$ ensures that interactions only occur on the past and future light cones for particle one at $\underline{S}_1 = (\vec{r}_1, ict_1)$. Similarly, the potential at particle two is

$$\phi_2(\vec{r}_2, t_2) = -c^{-1} g_1 \int_{-\infty}^{\infty} d\tau_1\, \delta(\xi^2)\,. \tag{2.4}$$

Combining the equations of motion with the calculation of

$$\frac{d}{d\tau_i}(2\underline{S}_i\,\underline{S}_i\cdot\underline{P}_i - \underline{P}_i\,\underline{S}_i\cdot\underline{S}_i) \tag{2.5}$$

and using the formalism developed in Ref. 14, we find that the conserved conformal vector is

$$\underline{K} = (2\underline{S}_1\underline{S}_1\cdot\underline{P}_1 - \underline{P}_1\underline{S}_1\cdot\underline{S}_1)_{\tau_1} + 2m_1c^2\int_{-\infty}^{\tau_1} d\tau_1'\,\underline{S}_1 + (2\underline{S}_2\,\underline{S}_2\cdot\underline{P}_2 - \underline{P}_2\,\underline{S}_2\cdot\underline{S}_2)_{\tau_2} + 2m_2c^2\int_{-\infty}^{\tau_2} d\tau_2'\,\underline{S}_2 + \underline{K}_I\,, \tag{2.6}$$

$$\underline{K}_I = 2c^{-3}g_1g_2 \times\left(\int_{-\infty}^{\tau_2} d\tau_2' \int_{\tau_1}^{\infty} d\tau_1' - \int_{\tau_2}^{\infty} d\tau_2' \int_{-\infty}^{\tau_1} d\tau_1'\right)\delta'(\xi^2) \times(\underline{S}_1\,\underline{S}_2\cdot\underline{S}_2 - \underline{S}_2\,\underline{S}_1\cdot\underline{S}_1)\,, \tag{2.7}$$

where the prime on the δ function indicates differentiation with respect to its argument.[15] The proper times τ_1, τ_2 are arbitrary points on the world lines of the two particles. This conformal vector is conserved in the sense that $d\underline{K}/d\tau_1 = 0$ and $d\underline{K}/d\tau_2 = 0$, as can be directly verified by carrying out the differentiations.

The semi-infinite integrals in Eq. (2.6) appear to cause problems at times $\tau \to -\infty$. It will be shown in Secs. IV and V that at least in some instances these problems are spurious, as can be seen by manipulating those terms which give rise to these integrals in going from Eq. (2.5) to (2.6) so as to obtain the identity

$$2m_ic^2\int_{-\infty}^{\tau_i} d\tau_i'\,\underline{S}_i = [m_ic^2(1-\beta_i^2)^{1/2}\underline{S}_i t_i]_{\tau_i} + m_ic^2\int_{-\infty}^{\tau_i} d\tau_i'(\underline{S}_i - \underline{u}_i t_i' + \gamma_i^2\underline{S}_i t_i' \vec{\beta}_i\cdot\dot{\vec{\beta}}_i)\,, \tag{2.8}$$

where $\underline{U}_i = \gamma_i\underline{u}_i = \gamma_i(\vec{u}, ic)$, $\gamma_i = (1-\beta_i^2)^{-1/2}$, and $\vec{\beta}_i = \vec{u}_i/c$. The double integrals in the interaction term $\underline{K}_I$ give nonzero values only over finite segments of the world-lines of the two particles. This is shown explicitly in a convenient calculational form by integrating Eq. (2.7) first over the world line of particle one, then over that of particle two:

$$\begin{aligned}\underline{K}_I = (2c)^{-1}g_1g_2\Bigg\{ &\frac{\underline{S}_1\,\underline{S}_2\cdot\underline{S}_2 - \underline{S}_2\,\underline{S}_1\cdot\underline{S}_1}{\gamma_1\gamma_2(R+\vec{R}\cdot\vec{\beta}_1)(R+\vec{R}\cdot\vec{\beta}_2)}\Bigg|_{t_1,t_2^+} - \frac{\underline{S}_1\,\underline{S}_2\cdot\underline{S}_2 - \underline{S}_2\,\underline{S}_1\cdot\underline{S}_1}{\gamma_1\gamma_2(R-\vec{R}\cdot\vec{\beta}_1)(R-\vec{R}\cdot\vec{\beta}_2)}\Bigg|_{t_1,t_2^-} \\ &+\int_{t_2}^{t_2^+} dt_2'\left[\frac{1}{R+\vec{R}\cdot\vec{\beta}_1}\frac{d}{dt_1'}\frac{\underline{S}_1\,\underline{S}_2\cdot\underline{S}_2 - \underline{S}_2\,\underline{S}_1\cdot\underline{S}_1}{\gamma_1\gamma_2[c(t_1'-t_2')-\vec{R}\cdot\vec{\beta}_1]}\right]_{t_1'=t_2'-R/c} \\ &-\int_{t_2^-}^{t_2} dt_2'\left[\frac{1}{R-\vec{R}\cdot\vec{\beta}_1}\frac{d}{dt_1'}\frac{\underline{S}_1\,\underline{S}_2\cdot\underline{S}_2 - \underline{S}_2\,\underline{S}_1\cdot\underline{S}_1}{\gamma_1\gamma_2[c(t_1'-t_2')-\vec{R}\cdot\vec{\beta}_1]}\right]_{t_1'=t_2'+R/c}\Bigg\}\end{aligned} \tag{2.9}$$

where $t_2^{\pm} = t_1 \pm R/c$.

B. Time-asymmetric interaction

Rudd and Hill[16] and Bruhns[17] obtained solutions for two particles interacting through time-asymmetric vector potentials. Their integral-free conserved momentum vectors and angular momentum tensors were evaluated for the particular choice of reference times $t_1, t_2 = t_1 - R/c$. If these conserved quantitites are evaluated for any arbitrary reference times t_1, t_2 they will in general contain integrals of the world lines.[14]

Particle one interacts with the retarded massless scalar potential arising from particle two on its past light cone,

$$\phi_1(\vec{r}_1, t_1) \equiv \phi_1^-(\vec{r}_1, t_1) = -g_2\int_{-\infty}^{\infty} d\tau_2'\,\delta(\xi)/R\,, \tag{2.10}$$

and particle two interacts with the advanced potential arising from particle one on its future light cone,

$$\phi_2(\vec{r}_2, t_2) \equiv \phi_2^+(\vec{r}_2, t_2) = -g_1 \int_{-\infty}^{\infty} d\tau_1' \delta(\xi)/R \,, \tag{2.11}$$

where $\xi = t_1' - t_2' - R/c$. Using the same procedure as before, we again obtain Eqs. (2.6) and (2.8) for $\underline{K}$, but now the interaction term is

$$\underline{K}_I = -c^{-1} g_1 g_2 \Bigg\{ \frac{\underline{S}_1 \underline{S}_2 \cdot \underline{S}_2 - \underline{S}_2 \underline{S}_1 \cdot \underline{S}_1}{\gamma_1\gamma_2(R - \vec{R}\cdot\vec{\beta}_1)(R - \vec{R}\cdot\vec{\beta}_2)}\bigg|_{t_1, t_2^- = t_1 - R/c} + \int_{t_2^-}^{t_2} dt_2' \bigg[\frac{1}{R - \vec{R}\cdot\vec{\beta}_1} \frac{d}{dt_1'} \frac{\underline{S}_1 \underline{S}_2 \cdot \underline{S}_2 - \underline{S}_2 \underline{S}_1 \cdot \underline{S}_1}{\gamma_1\gamma_2[c(t_1' - t_2') - \vec{R}\cdot\vec{\beta}_1]} \bigg]_{t_1' = t_2' + R/c} \Bigg\} \,. \tag{2.12}$$

The reference times t_1, t_2 can be chosen arbitrarily; note however that the integral in $\underline{K}_I$ will vanish identically for the choice of $t_1, t_2 = t_1 - R/c$.

III. VECTOR POTENTIAL

The equation of motion for a point particle with rest mass m_i moving with four-velocity $\underline{U}_i$ through a massless vector (electromagnetic) potential $\underline{A}_i$ without radiation reaction is

$$d(m_i \underline{U}_i)/d\tau_i = q_i(\underline{\Box}_i \times \underline{A}_i)\cdot \underline{U}_i \tag{3.1}$$

or

$$d\underline{P}_i/d\tau_i = q_i(\underline{\Box}_i \underline{A}_i)\cdot \underline{U}_i \,, \tag{3.2}$$

where the generalized momentum is

$$\underline{P}_i = m_i \underline{U}_i + q_i \underline{A}_i \,, \tag{3.3}$$

and the cross product for four-vectors is defined by the antisymmetric dyadic $\underline{F}\times\underline{G} = \underline{F}\,\underline{G} - \underline{G}\,\underline{F}$. The electric charge of the particle is q_i.

A. Time-symmetric interaction

The time-symmetric (Wheeler-Feynman) interaction is for potentials

$$\underline{A}_1(\vec{r}_1, t_1) = c^{-1} k q_2 \int_{-\infty}^{\infty} d\tau_2' \delta(\xi^2) \underline{U}_2 \,, \tag{3.4}$$

$$\underline{A}_2(\vec{r}_2, t_2) = c^{-1} k q_1 \int_{-\infty}^{\infty} d\tau_1' \delta(\xi^2) \underline{U}_1 \,, \tag{3.5}$$

where $k = \mu_0/4\pi$ for the MKSA system of units. We again obtain a conserved conformal vector by combining the equations of motion in the form of Eq. (3.2) with the derivative Eq. (2.5) to again obtain Eqs. (2.6) and (2.8), with the interaction term given by

$$\begin{aligned}\underline{K}_I = {} & c^{-1} k q_1 q_2 \left(\int_{-\infty}^{\tau_2} d\tau_2' \int_{\tau_1}^{\infty} d\tau_1' - \int_{\tau_2}^{\infty} d\tau_2' \int_{-\infty}^{\tau_1} d\tau_1' \right) [\, \delta(\xi^2) \underline{U}_1 \times \underline{U}_2 \cdot (\underline{S}_1 + \underline{S}_2) \\ & + 2c^{-2} \delta'(\xi^2) \underline{U}_1 \cdot \underline{U}_2 (\underline{S}_1 \underline{S}_1 \cdot \underline{S}_2 - \underline{S}_2 \underline{S}_1 \cdot \underline{S}_1)] \\ & - c^{-1} k q_1 q_2 \left(\int_{-\infty}^{\infty} d\tau_2' \int_{-\infty}^{\tau_1} d\tau_1' + \int_{-\infty}^{\tau_2} d\tau_2' \int_{-\infty}^{\infty} d\tau_1' \right) \delta(\xi^2) \underline{U}_1 \times \underline{U}_2 \cdot (\underline{S}_1 - \underline{S}_2) \,.\end{aligned} \tag{3.6}$$

By integrating first over the world line of particle one then over that of particle two, $\underline{K}_I$ can be expressed as

$$\begin{aligned}\underline{K}_I = 2^{-1} k q_1 q_2 \Bigg\{ & \frac{\underline{U}_1 \cdot \underline{U}_2 (\underline{S}_1 \underline{S}_2 \cdot \underline{S}_2 - \underline{S}_2 \underline{S}_1 \cdot \underline{S}_1)}{c\gamma_1\gamma_2(R + \vec{R}\cdot\vec{\beta}_1)(R + \vec{R}\cdot\vec{\beta}_2)}\bigg|_{t_1, t_2^+} - \frac{\underline{U}_1 \cdot \underline{U}_2 (\underline{S}_1 \underline{S}_2 \cdot \underline{S}_2 - \underline{S}_2 \underline{S}_1 \cdot \underline{S}_1)}{c\gamma_1\gamma_2(R - \vec{R}\cdot\vec{\beta}_1)(R - \vec{R}\cdot\vec{\beta}_2)}\bigg|_{t_1, t_2^-} \\ & - \int_{t_2}^{t_2^+} dt_2' \left[\frac{1}{R + \vec{R}\cdot\vec{\beta}_1} \left(\frac{2\underline{U}_1 \times \underline{U}_2 \cdot \underline{S}_1}{\gamma_1\gamma_2} - \frac{d}{dt_1'} \frac{\underline{U}_1 \cdot \underline{U}_2 (\underline{S}_1 \underline{S}_2 \cdot \underline{S}_2 - \underline{S}_2 \underline{S}_1 \cdot \underline{S}_1)}{c\gamma_1\gamma_2[c(t_1' - t_2') - \vec{R}\cdot\vec{\beta}_1]} \right) \right]_{t_1' = t_2' - R/c} \\ & + \int_{t_2^-}^{t_2} dt_2' \left[\frac{1}{R - \vec{R}\cdot\vec{\beta}_1} \left(\frac{2\underline{U}_1 \times \underline{U}_2 \cdot \underline{S}_1}{\gamma_1\gamma_2} - \frac{d}{dt_1'} \frac{\underline{U}_1 \cdot \underline{U}_2 (\underline{S}_1 \underline{S}_2 \cdot \underline{S}_2 - \underline{S}_2 \underline{S}_1 \cdot \underline{S}_1)}{c\gamma_1\gamma_2[c(t_1' - t_2') - \vec{R}\cdot\vec{\beta}_1]} \right) \right]_{t_1' = t_2' + R/c} \\ & - 2 \int_{-\infty}^{t_2} dt_2' \left[\frac{\underline{U}_1 \times \underline{U}_2 (\underline{S}_1 - \underline{S}_2)}{\gamma_1\gamma_2(R + \vec{R}\cdot\vec{\beta}_1)}\bigg|_{t_1' = t_2' - R/c} + \frac{\underline{U}_1 \times \underline{U}_2 \cdot (\underline{S}_1 - \underline{S}_2)}{\gamma_1\gamma_2(R - \vec{R}\cdot\vec{\beta}_1)}\bigg|_{t_1' = t_2' + R/c} \right] \Bigg\} \,.\end{aligned} \tag{3.7}$$

Unlike the scalar interaction, $\underline{K}_I$ for the vector interaction contains semi-infinite integrals. We will show in Sec. IV that these integrals are well behaved for at least one bound state.

B. Time-asymmetric interaction

The time-asymmetric interaction is for vector potentials

$$\underline{A}_1(\vec{r}_1, t_1) \equiv \underline{A}_1^-(\vec{r}_1, t_1) = kq_2 \int_{-\infty}^{\infty} d\tau_2' \, \underline{U}_2 \delta(\xi)/R \,, \tag{3.8}$$

$$\underline{A}_2(\vec{r}_2, t_2) \equiv \underline{A}_2^+(\vec{r}_2, t_2) = kq_1 \int_{-\infty}^{\infty} d\tau_1' \, \underline{U}_1 \delta(\xi)/R \,. \tag{3.9}$$

With the same $\underline{K}$ as before, $\underline{K}_I$ is given by

$$\underline{K}_I = -kq_1q_2 \left\{ \frac{\underline{U}_1 \cdot \underline{U}_2 (\underline{S}_1 \underline{S}_2 \cdot \underline{S}_2 - \underline{S}_2 \underline{S}_1 \cdot \underline{S}_1)}{c\gamma_1\gamma_2(R - \vec{R}\cdot\vec{\beta}_1)(R - \vec{R}\cdot\vec{\beta}_2)} \right|_{t_1, t_2^- = t_1 - R/c}$$
$$- \int_{t_2^-}^{t_2} dt_2' \left[\frac{1}{R - \vec{R}\cdot\vec{\beta}_1} \left(\frac{2\underline{U}_1 \times \underline{U}_2 \cdot \underline{S}_1}{\gamma_1\gamma_2} - \frac{d}{dt_1'} \frac{\underline{U}_1 \cdot \underline{U}_2 (\underline{S}_1 \underline{S}_2 \cdot \underline{S}_2 - \underline{S}_2 \underline{S}_1 \cdot \underline{S}_1)}{c\gamma_1\gamma_2[c(t_1' - t_2') - \vec{R}\cdot\vec{\beta}_1]} \right) \right]_{t_1' = t_2' + R/c}$$
$$\left. + 2 \int_{-\infty}^{t_2} dt_2' \frac{\underline{U}_1 \times \underline{U}_2 \cdot (\underline{S}_1 - \underline{S}_2)}{\gamma_1\gamma_2 (R - \vec{R}\cdot\vec{\beta}_1)} \right|_{t_1' = t_2' + R/c} \Bigg\} \,. \tag{3.10}$$

The first integral, but not the second, will vanish for the choice of reference times t_1, $t_2 = t_1 - R/c$.

IV. UNIFORM CIRCULAR MOTION

We demonstrate that solutions for the conserved conformal vector $\underline{K} = (\vec{K}, iK_t)$ do exist by considering two point particles moving in concentric circles with uniform angular speed ω in the X-Y plane of an inertial frame and interacting through time-symmetric scalar or vector potentials, as shown in Fig. 1. Particle one interacts with the potentials arising from particle two in the retarded (−) and advanced (+) positions, $\vec{r}_2(t_2^\pm) = \vec{r}_2(t_1 \pm T)$, where T is the time required for the interaction to propagate the distance $R = |\vec{r}_1(t_1) - \vec{r}_2(t_2^\pm)|$ at the speed of light, and $\theta = \omega T$. The energy, momentum, and angular momentum tensor for such motion were calculated by Schild[18] for a vector potential, and by Andersen and von Baeyer[19] for a scalar potential. In both cases the conserved dilation is calculated to be $D = 0$.

The semi-infinite integrals which appear in the spatial components $\vec{K}$ in Eqs. (2.6), (3.7), and (3.10) can be evaluated for uniform circular trajectories by using convergence factors to calculate terms such as

$$\int_{-\infty}^{t} dt' \cos\omega t' = \omega^{-1} \sin\omega t \,. \tag{4.1}$$

The semi-infinite integrals in the time components K_t of Eqs. (3.7) and (3.10) are identically zero for uniform circular motion, while those appearing in Eq. (2.6) are readily evaluated by use of Eq. (2.8) to give

$$2m_i c^2 \int_{-\infty}^{\tau_i} d\tau_i' \, \underline{S}_i = [m_i c^2 (1 - \beta_i^2)^{1/2} \underline{S}_i t_i]_{\tau_i} \,. \tag{4.2}$$

The evaluation of the remaining terms is straightforward, albeit tedious. The result for the scalar interaction is

$$\vec{K} = \vec{0} \,, \tag{4.3}$$

$$K_t = -\frac{g_1 g_2}{\omega\gamma_1\gamma_2(\theta + \beta_1\beta_2 \sin\theta)^2} \left(2\beta_1\beta_2 \sin\theta - 2\theta\beta_1\beta_2 \cos\theta + \tfrac{1}{3}\theta^3 + \frac{1 + \beta_1\beta_2\cos\theta}{\theta + \beta_1\beta_2 \sin\theta} \, \theta^2 (1 + \beta_1\beta_2\cos\theta - \tfrac{1}{3}\theta^2) \right) , \tag{4.4}$$

and for the vector interaction

$$\vec{K} = \vec{0} \,, \tag{4.5}$$

$$K_t = \frac{c^2 kq_1q_2}{\omega(\theta + \beta_1\beta_2\sin\theta)} \left\{ -\theta^2 - 2\theta\beta_1\beta_2 \sin\theta + \frac{1 + \beta_1\beta_2\cos\theta}{\theta + \beta_1\beta_2\sin\theta} (2\beta_1\beta_2\sin\theta - 2\theta\beta_1\beta_2\cos\theta + \tfrac{1}{3}\theta^3) \right.$$
$$\left. + \left[\frac{\beta_1\beta_2\sin\theta}{\theta + \beta_1\beta_2\sin\theta} + \left(\frac{1 + \beta_1\beta_2\cos\theta}{\theta + \beta_1\beta_2\sin\theta} \right)^2 \right] \theta^2 (1 + \beta_1\beta_2\cos\theta - \tfrac{1}{3}\theta^2) \right\} . \tag{4.6}$$

V. ONE-BODY FIELD-FREE MOTION

Consider a particle moving in field-free space with constant momentum $\underline{P}=(\vec{p}, iw/c)$ along the straight line

$$\underline{S}=(\vec{r}_0+\vec{u}t, ict), \quad -\infty \leqslant t \leqslant \infty . \tag{5.1}$$

Its conserved dilation is

$$D=\underline{S}\cdot\underline{P}+mc^2\tau \tag{5.2}$$

$$=\vec{r}_0\cdot\vec{p}. \tag{5.3}$$

The conserved conformal vector is obtained by extending the method used to obtain Eq. (2.8) for a constant velocity trajectory:

$$\underline{K}=2\underline{S}\,\underline{S}\cdot\underline{P}-\underline{P}\,\underline{S}\cdot\underline{S}+mc^2(1-\beta^2)^{1/2}\underline{S}t+mc^2\underline{S}_0\tau \tag{5.4}$$

$$=(\vec{r}_0\vec{r}_0\cdot\vec{p}+\vec{r}_0\times[\vec{r}_0\times\vec{p}], -ic\vec{r}_0\cdot\vec{r}_0). \tag{5.5}$$

The spatial vector $\vec{K}$ has components parallel to and perpendicular to $\vec{r}_0$; $\underline{K}=\underline{0}$ if and only if $\vec{r}_0=\vec{0}$. This is a more stringent condition than on the dilation since $D=0$ for either $\vec{r}_0=\vec{0}$ or for $\vec{r}_0$ perpendicular to $\vec{p}$.

VI. DISCUSSION

Relativistic action-at-a-distance dynamics for particles with nonzero masses is invariant under the full conformal group of transformations. This implies the conservation of five quantities beyond the well-known ten from Lorentz invariance. As Bessel-Hagen[4] put it in 1921: To what extent they will be useful to physicists, the future will have to decide.

Dynamically, we can consider the angular momentum tensor $\vec{L}=\underline{S}\times\underline{P}$ and the dilation scalar $D=\underline{S}\cdot\underline{P}+mc^2\tau$ as a statement of the conservation of the first moment of momentum, and the conformal vector, written as

$$\underline{K}=\underline{S}\,\underline{S}\cdot\underline{P}+\underline{S}\times\underline{P}\cdot\underline{S}+mc^2\int_{-\infty}^{\tau} d\tau'\underline{S}, \tag{6.1}$$

as a statement of the conservation of the second moment of momentum.

It is intriguing to speculate about the physical meanings of the mass terms in D and $\underline{K}$. The proper time τ is related to the lifetime of elementary particles. We must note, however, that in quantum mechanics the lifetime is a quantity independent of the history of an individual particle. Only in a statistical sense do unstable particles represent clocks which measure the proper time elapsed since their creation.

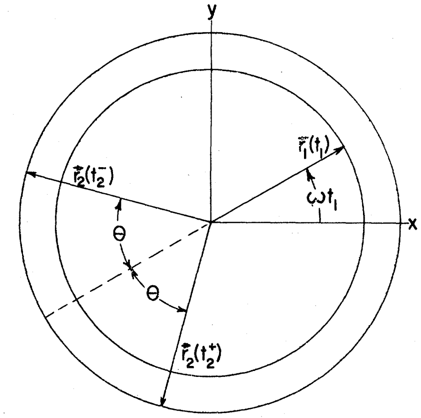

FIG. 1. For time-symmetric interactions, particle one interacts with the potentials arising from particle two in the retarded position $\vec{r}_2(t_2^-)$ and in the advanced position $\vec{r}_2(t_2^+)$.

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