PHYSICAL REVIEW D VOLUME 20, NUMBER 12 15 DECEMBER 1979

Soliton structures in $P(\phi)_2$

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(Received 11 July 1979)

We investigate general soliton features of a scalar field theory in two dimensions with polynomial self-interactions. The static solution and its linear fluctuations are discussed in general and detailed calculations carried out for ϕ^6 and ϕ^8 models. For ϕ^6 the linear fluctuations can be analyzed explicitly, enabling quantum effects to be evaluated. As a result we are able to show that the first-order soliton mass correction is finite.

I. INTRODUCTION

Solitonlike objects in a realistic space-time dimension are difficult to investigate because of the complexity of the classical field equations which determine their detailed properties. It seems worthwhile therefore to investigate much simpler models in two dimensions as fully as possible, classically and quantum mechanically, to obtain clues on soliton behavior. The sine-Gordon model has been particularly fruitful in this way, and the ϕ^4 model has also been useful. A range of two-dimensional models with intermediate properties, much less investigated, is provided by $P(\phi)$, i.e., a scalar field with a polynomial self-interaction. Whereas the sine-Gordon and ϕ^4 models possess an infinite set of vacuums, and two vacuums, respectively, $P(\phi)$ can be chosen to have any given finite number of vacuums, which will not in general be connected by some symmetry transformation, as occurs for sine-Gordon and ϕ^4 models. In $P(\phi)$ we then have several distinct meson sectors, corresponding to the separate vacuums, between which the solitons interpolate. Accordingly, there will also be several distinct soliton sectors, characterized among other things by different soliton masses.

We consider Lagrangians of the form

$$\mathcal{L}=\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\,.$$

Let us repeat some well-known facts about solitons in such a theory,[1] mainly to set the notation. $V(\phi)$ must have at least two zeros for solitons to exist; the time-independent soliton will interpolate smoothly and monotonically between adjacent zeros. The field equation is

$$\partial^\mu\partial_\mu\phi + \frac{\delta V(\phi)}{\delta\phi}=0 \qquad (1)$$

and for static solutions with finite energy is solved by

$$d\phi/dx = \pm[2V(\phi)]^{1/2}\,. \qquad (2)$$

The energy (mass) of the soliton is given by

$$E=\int dx\left[\frac{1}{2}\left(\frac{d\phi}{dx}\right)^2 + V(\phi)\right] = \int d\phi[2V(\phi)]^{1/2}\,. \qquad (3)$$

For a static solution $\phi_c(x)$ the field equation is equivalent to a stationary energy,

$$\delta E(\phi)/\delta\phi = 0$$

By regarding x as "time" we can picture the solution as the motion of a "particle" in a potential $-V$, passing from one maximum of $-V$ to an adjacent maximum. This picture is particularly helpful when V possesses several local minima between two adjacent absolute minima, as occurs in polynomial models.

Stability of the soliton is ensured by the existence of the different vacuum values attained at spatial infinity, since these values cannot be altered under perturbation. Alternatively, we can examine small perturbations around the classical solution in the form

$$\phi(x,t)=\phi_c(x)+\eta(x,t)\,. \qquad (4)$$

We find that η can be written

$$\eta(x,t)=e^{i\omega_k t}\eta_k(x)\,,$$

where $\eta_k(x)$ are eigenfunctions of the Schrödinger operator K:

$$K\eta_k(x)=\omega_k^{\,2}\eta_k(x)\,,\qquad K\equiv\frac{-d^2}{dx^2}+V''(\phi_c(x))\,. \qquad (5)$$

Stability follows from the fact that $\omega_k^{\,2}\geq 0$.

The quantum theory can be investigated by perturbing around the classical solution which is valid for weak coupling. Let us define the coupling constant λ such that $V(\phi,\lambda)=(1/\lambda)V(\sqrt{\lambda}\phi,1)$, then ϕ_c is of order $\lambda^{-1/2}$ and contributes a large scale effect for small λ. We shift the quantum field operator $\phi(x,t)$ by the classical solution, as in Eq. (4), to define a new field operator $\eta(x,t)$, which is treated perturbatively in powers of λ. Although we do not discuss it here, strictly one should take into account the zero-mode problem.[1] Quantum corrections of $O(\lambda^0)$ can be described in terms of the eigenfunctions of the Schrödinger operator K [Eq. (5)]. In general the Hamiltonian can be separated into parts,[1]

$$H = E_c + H_0 + H_I ,$$

where $E_c \sim O(1/\lambda)$ is the classical contribution of the soliton energy, H_0 is $O(\lambda^0)$, and H_I the interaction Hamiltonian is $O(\lambda)$ and can be treated perturbatively to take account of higher-order effects.

II. POLYNOMIAL SELF-INTERACTIONS

Now consider the case where $V(\phi)$ is a polynomial in ϕ. We require that $V(\phi)$ be non-negative and will sometimes assume $V(\phi) = P(\phi)^2$ for some polynomial $P(\phi)$. In general we impose the symmetry $V(\phi) = V(-\phi)$. The simplest such example is ϕ^4:

$$V(\phi) = \lambda(\phi^2 - v^2)^2 . \tag{6}$$

Other polynomial self-interactions have also been considered,[2] including[3] the ϕ^6 model with potential:

$$V(\phi) = \lambda^2(\phi^2 + \epsilon^2)(\phi^2 - v^2)^2 , \quad \epsilon > 0 . \tag{7}$$

However, as in most polynomial models, analytic progress is difficult because first-order quantum corrections are determined, in this case, by Heun functions. The potential (7) admits a static solution which has an energy density peaked in two separate locations.[3] This model of a "bag" containing two "kinks" can be generalized, by taking higher-degree polynomials $V(\phi)$, so as to incorporate an arbitrary number of kinks confined within a bag.

We will investigate in detail the ϕ^6 model

$$V(\phi) = \tfrac{1}{2}\lambda^2\phi^2(\phi^2 - v^2)^2 \tag{8}$$

which possesses three minima (Fig. 1). Analytic progress is possible here because first-order quantum fluctuations can be expressed in terms of hypergeometric functions.

A general class of models of some interest are those which approximate the sine-Gordon potential. This approximation can be achieved by truncating the infinite-product formula for $\cos\phi$ to obtain either of the following two polynomial potentials:

$$V^{(n)} = \lambda^{(n)} \prod_{k=1}^{n} [\phi^2 - v^2(k - \tfrac{1}{2})^2]^2 , \tag{9}$$

$$V^{(n)} = \lambda^{(n)} \phi^2 \prod_{k=1}^{n} (\phi^2 - k^2 v^2)^2 . \tag{10}$$

Here the first potential reduces to ϕ^4 [Eq. (6)] and the second to ϕ^6 [Eq. (8)] for $n=1$ and both approach the sine-Gordon potential for $n \to \infty$ (with appropriate n dependence in $\lambda^{(n)}$). One is adding more minima for higher n and, correspondingly, more soliton sectors and only in the limit $n \to \infty$ does the symmetry operation $\phi \to \phi + v$ appear to connect these sectors. As a result, these polynomial theories are in general more difficult to handle than sine-Gordon theory, although for large n one would expect the soliton properties to approach those of sine-Gordon solitons. Whether the interacting solitons for potentials (9) and (10) can, to any extent, be described by analytic formulas as in sine-Gordon theory is a question which we do not consider here. We will be satisfied to extract from the classical system the static solution as given in Eq. (2) and investigate quantum fluctuations around this solution.

To obtain the static solution $\phi_c(x)$, we need merely evaluate the integral [from Eq. (2)]

$$\pm(x - x_0) = \int \frac{d\phi}{[2V(\phi)]^{1/2}} \tag{11}$$

to obtain $x = f(\phi)$ for some function f. For potentials of the form $V(\phi) = P(\phi)^2$ we can always obtain f explicitly. Since f is monotonic we obtain $\phi_c(x)$ as a (generally implicit) smooth monotonic function of x, $\phi_c = f^{-1}(x)$. Different branches of the inverse function f^{-1} correspond to solitons of different sectors. ϕ_c interpolates between adjacent minima a, b of $V(\phi)$, i.e., $\phi_c \to a (x \to -\infty)$ and $\phi_c \to b (x \to \infty)$. The general shape of $\phi_c(x)$ can be deduced as explained in Sec. I. As an explicit example consider the potential given by Eq. (10) with

$$\lambda^{(n)} = \alpha/[2(n!)^4 v^{4n}] , \quad \alpha = \text{constant} .$$

Using the identity

$$\frac{(n!)^2}{t \prod_{k=1}^{n} (k^2 - t^2)} = \frac{1}{t} - \sum_{k=1}^{n} n_k \left(\frac{1}{k-t} - \frac{1}{k+t} \right),$$

where

$$n_k = \frac{(-)^k (n!)^2}{(n-k)!\,(n+k)!} ,$$

we can evaluate the integral (11):

$$\exp[\pm\sqrt{\alpha}(x - x_0)] = \frac{\phi}{v} \prod_{k=1}^{n} \left(k^2 - \frac{\phi^2}{v^2} \right)^{n_k} . \tag{12}$$

In the limit $n \to \infty$ we regain the sine-Gordon soli-

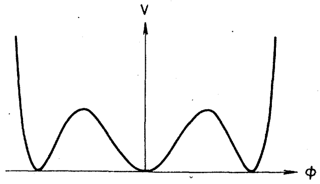

FIG. 1. ϕ^6 potential V with three minima.

ton,

$$\exp[\pm\sqrt{\alpha}(x-x_0)]=\tan\left(\frac{\phi\pi}{2v}\right).$$

For finite n Eq. (12) reduces to a polynomial equation in ϕ, which can be solved explicitly for $n=1$ (Sec. III) and $n=2$, by solving a quartic equation. Similarly for the potential given by Eq. (9) we can obtain the explicit static solution for $n=1$ (ϕ^4) and also $n=2$ (ϕ^8) by solving a quartic. Explicit solutions, although useful, are not necessary and quantum fluctuations, for example, can be studied to some extent with only implicit solutions, as will be seen.

Let us remark on the soliton stability. The vacuum structure of $V(\phi)$ distinguishes between different soliton sectors according to the vacuum values attained by $\phi_c(x)$. The infinite energy required to alter these vacuum values prevents soliton decay. This is evident from the following argument,[4] rewriting Eq. (3):

$$E=\int dx\left[\frac{1}{2}\left(\frac{d\phi}{dx}\pm[2V(\phi)]^{1/2}\right)^2\mp\frac{d\phi}{dx}[2V(\phi)]^{1/2}\right].$$

The second term of the integrand can be written $dF(\phi)/dx$, where

$$dF/d\phi=[2V(\phi)]^{1/2}, \tag{13}$$

and can therefore be integrated to yield $F(b)-F(a)$, where a and b are the relevant adjacent vacuum values. Any perturbation in ϕ, which must preserve asymptotic values, can only increase the energy and therefore the soliton is stable. One can, to some extent, also view stability as a consequence of the identically conserved current $j_\mu=\epsilon_{\mu\nu}\partial^\nu\phi$ and the corresponding conserved charge $Q=\phi(\infty)-\phi(-\infty)=b-a$, carried by each soliton. However, we shall see (Sec. V) that in potentials such as (9) and (10) solitons of different mass can carry the same charge Q, but cannot decay the heavier to the lighter, because each occupies a different sector with different vacuum values. We can take account of this phenomenon by defining instead a conserved current

$$j_\mu=\epsilon_{\mu\nu}\partial^\nu F(\phi), \tag{14}$$

where $F(\phi)$ is defined by Eq. (13), and the soliton charge is then proportional to its mass.

Alternatively, to demonstrate stability we can examine perturbations around the classical solution as shown in Eq. (4), and this entails a study of the Schrödinger equation (5). Let us consider this equation in detail as it arises also in the quantum theory. We have

$$-\eta''+U(x)\eta=\omega^2\eta, \tag{15}$$

where $\eta=\eta(x)$, $U(x)=V''(\phi_c(x))$. Since ϕ_c is an implicit function of x in general, it is convenient to transform to $\phi_c=\phi$ as the independent variable, via Eq. (2). Define also $g(\phi)=[V(\phi)]^{1/2}\eta(\phi)$, then (15) becomes

$$2Vg''+3V'g'+\omega^2g=0, \tag{16}$$

where V, g are functions of ϕ, $g'=dg/d\phi$, etc. The boundary conditions on $\eta(x)$ are, as usual, those of square integrability:

$$\int|\eta(x)|^2dx<\infty. \tag{17}$$

The L_2 inner produce for functions $\eta(x)$ translates to the inner product for solutions of (16):

$$(f,g)=\int_a^b d\phi[V(\phi)]^{1/2}\overline{f(\phi)}g(\phi). \tag{18}$$

Define

$$L=-2Vd^2/d\phi^2-3V'd/d\phi,$$

then we find

$$(f,Lf)=\omega^2(f,f)$$
$$=2\int_a^b d\phi\left|\frac{df}{d\phi}\right|^2V^{3/2}(\phi)\geq 0, \tag{19}$$

showing that always $\omega^2\geq 0$, and stability is guaranteed.

Equation (16) is a useful form of Eq. (15), with all coefficients known explicitly. The nature of Eq. (16) is determined by the singular points which evidently occur at the zeros of $V(\phi)$, real or complex. Since the coefficient of g' is V' these singularities will be regular if the zeros of V are distinct, but otherwise will, in general, be irregular. Let us take specifically the form $V(\phi)=P(\phi)^2$, i.e.,

$$V=\lambda\prod_i(\phi-a_i)^2, \tag{20}$$

where the a_i are real or complex, V is real, and we can also impose $V(\phi)=V(-\phi)$. Then

$$V'=\sum_i\frac{2V}{\phi-a_i}$$

and Eq. (16) becomes

$$g''+\sum_i\frac{3}{\phi-a_i}g'+\frac{\omega^2}{2V}g=0. \tag{21}$$

If the a_i are distinct all singular points are regular, and this is true for the point at infinity also. In fact, for an arbitrary polynomial $V(\phi)$ the point at infinity is a regular singularity. To see this, put $\phi\to 1/\phi$ and Eq. (16) becomes

$$g''+\frac{1}{\phi}\left(2-\frac{3V'(1/\phi)}{2\phi V(1/\phi)}\right)g'+\frac{\omega^2g}{2\phi^4V(1/\phi)}=0, \tag{22}$$

and $\phi=0$ is a regular singularity for V of degree ≥ 2. For potentials such that $V(\phi)=V(-\phi)$ the number of regular singularities can be reduced by the substitution $Z=\phi^2$. For ϕ^4 [Eq. (6)] and ϕ^6 [Eq. (8)] we then have equations with three regular singularities, i.e., the hypergeometric equation or a special case thereof. For some ϕ^8 models and for the ϕ^6 potential (7) the equation will have four regular singularities (Heun's equation) and exact results are harder to obtain.

Returning to Eq. (21) for the case of regular singularities, we can expand $g(\phi)$ near $\phi=a_i$ according to $g\sim(\phi-a_i)^{\sigma_i}$, where σ_i satisfies the indicial equation

$$\sigma_i^2+2\sigma_i+\frac{\omega^2}{V''(a_i)}=0\,, \tag{23}$$

with

$$V''(a_i)=2\lambda\prod_{j,j\neq i}(a_i-a_j)^2\,. \tag{24}$$

Hence for each i

$$\sigma_i=-1\pm\left[1-\frac{\omega^2}{V''(a_i)}\right]^{1/2}.$$

For $\omega^2<V''(a_i)$ there are two real values of σ_i, corresponding to two solutions in the vicinity $\phi\sim a_i$, but only one solution is square integrable since we require, from the definition (18), that $Rl(\sigma_i)>-1$. The condition $\omega^2<V''(a_i)$ corresponds to the discrete spectrum, and for $\omega^2\geq V''(a_i)$ we expect ω^2 to attain a continuum of values. This is evident from an examination of the Schrödinger equation (15). For large x (positive or negative), $U(x)$ will attain a constant value $V''(a_i)$ since $\phi_c(x)$ approaches the vacuum value a_i at infinity. For $\omega^2\geq V''(a_i)$ then we expect to obtain plane-wave (continuum) solutions.

The situation can be understood from the quantum-mechanical viewpoint. We regard $\phi(x,t)$ as a quantum field operator which is shifted by the classical solution $\phi_c(x)$ to define a new quantum field operator $\eta(x,t)$ [Eq. (4)]. This field η describes meson oscillations around the soliton, as is apparent for large x when ϕ_c approaches the vacuum value, and η represents the meson states built on that vacuum. The mass m_i of mesons built on the vacuum with expectation value $\langle\phi\rangle=a_i$ is given by, in lowest order,

$$V''(a_i)=m_i^2\,. \tag{25}$$

In general, adjacent vacuums will not be connected by any symmetry transformation and the soliton will interpolate between sectors with different meson masses. The soliton presents a potential $U(x)=V''(\phi_c(x))$ to incoming meson states and the approximate soliton-meson scattering process can be inferred from the shape of $U(x)$. If the mesons of the adjoining vacuums are all massive, the solution $\phi_c(x)$ will approach its vacuum values exponentially and consequently $U(x)$ also approaches its asymptotic values exponentially. For ϕ^6 (Sec. III), $U(x)$ has the shape shown in Fig. 2 and this shape applies generally, except that the central depression could contain various bumps. The asymptotic values of $U(x)$, m_a^2 and m_b^2, need not be equal. If one of the meson masses is zero, $\phi_c(x)$ and $U(x)$ will approach their asymptotic values slowly, according to some power law, as shown in Fig. 3 for ϕ^8 (Sec. V).

From the shape of $U(x)$ we can infer the following properties of the Schrödinger equation (15). The central well could permit several discrete eigenvalues ω^2 corresponding to meson-soliton bound states. Here, we ignore the eigenvalue $\omega^2=0$, with eigenvector $\eta(x)=d\phi_c/dx$, or $g(\phi)=$ constant in Eq. (16), which is merely a manifestation of translation invariance. For $m_a^2\leq\omega^2<m_b^2$ we can have incoming plane-wave continuum solutions (mesons of mass m_a) which are completely reflected from the central potential (the soliton). For $\omega^2\geq m_b^2$ we can have incoming meson states which will be partly reflected and partly transmitted, i.e., a meson of sufficiently high momentum can pass through the soliton and appear on the other side as a meson of different mass appropriate to that vacuum. This phenomenon will be investigated in detail for ϕ^6 theory. For ϕ^4 and sine-Gordon theory there is essentially only one meson sector, and mesons can pass through the soliton without reflection, with only a phase change.

III. ϕ^6 MODEL

We take now for V,

$$V=\tfrac{1}{2}\lambda^2\phi^2\left(\phi^2-\frac{\mu}{\lambda}\right)^2,\quad \mu,\lambda>0\,, \tag{26}$$

which has the shape shown in Fig. 1. There is a central minimum at $\phi=0$, and meson states built on this vacuum have mass μ. There are also minima at $\phi=\pm(\mu/\lambda)^{1/2}$ connected by (and spon-

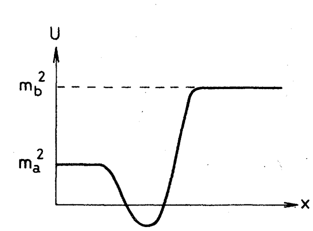

FIG. 2. The Schrödinger potential $U(x)$ relevant to meson-soliton scattering, involving mesons of mass m_a, m_b.

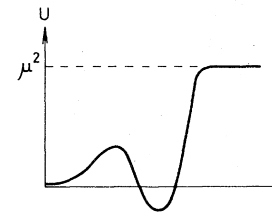

FIG. 3. The potential $U(x)$ for the case of one massless meson; $U(x)$ approaches zero slowly for large negative x.

taneously breaking) the symmetry operation $\phi \to -\phi$. Meson states built on either of these vacuums have mass 2μ. In the absence of solitons the separate meson sectors form distinct worlds, but with the existence of solitons both meson types will appear, always separated by solitons.

We obtain the static solution $\phi_c(x)$ by integrating Eq. (11):

$$\phi_c(x) = \left[\frac{\mu}{2\lambda}(\tanh\mu x + 1)\right]^{1/2}$$

$$= \left[\frac{\mu}{\lambda(1+e^{-2\mu x})}\right]^{1/2}. \qquad (27)$$

The solution has the appearance shown in Fig. 4. For large positive x, $\phi_c \sim (\mu/\lambda)^{1/2} + O(e^{-2\mu x})$ and for large negative x, $\phi_c \sim O(e^{\mu x})$ so that ϕ_c quickly reaches vacuum values exponentially according to the relevant meson mass.

We also obtain other solutions by putting $x \to -x$ (antisolitons) and $\phi_c \to -\phi_c$ in Eq. (27) to give four possibilities, each interpolating between adjacent vacuums. However, we know nothing about possible two-soliton solutions which must be time dependent, and which, unlike ϕ^4, can exist for the potential (26). The solution (27) can of course be translated and Lorentz boosted. The soliton mass, from Eq. (3), comes out to be

$$M = \mu^2/4\lambda. \qquad (28)$$

Next, we try to find solutions in the linearized approximation, i.e., solve Eq. (15). We find

$$U(x) = V''(\phi_c(x))$$

$$= \frac{5\mu^2}{2} + \frac{3\mu^2}{2}\tanh\mu x - \frac{15\mu^2}{4\cosh^2\mu x}. \qquad (29)$$

$U(x)$ has the shape shown in Fig. 2, with asymptotic values $U(x) \to 4\mu^2$ (for $x \to \infty$) and $U(x) \to \mu^2$ $(x \to -\infty)$. The Schrödinger equation for this $U(x)$ has been studied in detail (Morse and Feshbach, Ref. 5, p. 1651) and the solutions expressed in terms of hypergeometric functions. The specific correspondence with Ref. 5 is found by putting

$$z \to \mu x,$$
$$e^{2\mu} \to \tfrac{3}{2},$$
$$v \to \tfrac{18}{5}, \qquad (30)$$
$$\epsilon \to \frac{\omega^2}{\mu^2} + \tfrac{7}{5}.$$

For $\omega^2 < \mu^2$ there can be only discrete levels, and for $U(x)$ given by Eq. (29) there is just one such level, $\epsilon = \frac{7}{5}$ corresponding to $\omega^2 = 0$. This is merely the translational mode with eigenfunction $d\phi_c/dx$; there are no meson-soliton bound states in this model. For $\mu^2 \leq \omega^2 < 4\mu^2$ there is the following continuum solution:

$$\eta(x) = N\frac{e^{-(K_+ + ik_-)\mu x/2}}{(e^{\mu x}+e^{-\mu x})^b}$$
$$\times F\left(b - \tfrac{3}{2},\; b + \tfrac{5}{2},\; K_+ + 1 \,\middle|\, \frac{e^{-\mu x}}{e^{\mu x}+e^{-\mu x}}\right), \qquad (31)$$

where

$$k_- = \left(\frac{\omega^2}{\mu^2} - 1\right)^{1/2}, \quad K_+ = \left(4 - \frac{\omega^2}{\mu^2}\right)^{1/2}, \quad b = \tfrac{1}{2}(K_+ - ik_-),$$

and N is some normalization and F is the hypergeometric function. Asymptotically we have

$$\eta(x) \to 0 \text{ for } x \to \infty,$$
$$\eta(x) \to N[A(k_-)e^{i\mu k_- x} + A^*(k_-)e^{-i\mu k_- x}], \quad x \to \infty, \qquad (32)$$

where

$$A(k_-) = \frac{\Gamma(K_+ + 1)\Gamma(-ik_-)}{\Gamma(b - \frac{3}{2})\Gamma(b + \frac{5}{2})}.$$

This solution represents incoming meson states of mass μ which are completely reflected from the soliton and return to $-\infty$. The mesons cannot penetrate the barrier $U(x) \sim 4\mu^2$ encountered on the right of the soliton.

For $\omega^2 \geq 4\mu^2$ we can have the following solutions and asymptotic behavior:

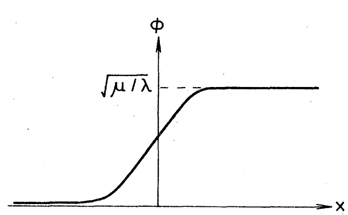

FIG. 4. The static ϕ^6 solution $\phi_c(x)$, interpolating smoothly between vacuums $\phi = 0$, $(\mu/\lambda)^{1/2}$.

(A) $$\eta(x)=N\,\frac{e^{(i/2)(k_+-k_-)\mu x}}{(e^{\mu x}+e^{-\mu x})^b}\times F\left(b-\tfrac{3}{2},\ b+\tfrac{5}{2},\ 1-ik_+\left|\frac{e^{-\mu x}}{e^{\mu x}+e^{-\mu x}}\right.\right), \tag{33a}$$

for which

$$\eta(x)\to Ne^{i\mu k_+x},\quad \text{for } x\to\infty,$$
$$\eta(x)\to N[A(k_-,k_+)e^{i\mu k_-x}+A(-k_-,k_+)e^{-i\mu k_-x}],\quad \text{for } x\to-\infty, \tag{33b}$$

(B) $$\eta(x)=N\,\frac{e^{i/2(k_+-k_-)\mu x}}{(e^{\mu x}+e^{-\mu x})^b}\times F\left(b-\tfrac{3}{2},\ b+\tfrac{5}{2},\ 1-ik_-\left|\frac{e^{\mu x}}{e^{\mu x}+e^{-\mu x}}\right.\right), \tag{34a}$$

for which

$$\eta(x)\to Ne^{-i\mu k_-x},\quad \text{for } x\to-\infty,$$
$$\eta(x)\to N[A(k_+,k_-)e^{-i\mu k_+x}+A(-k_+,k_-)e^{i\mu k_+x}],\quad \text{for } x\to\infty, \tag{34b}$$

where

$$k_-=\left(\frac{\omega^2}{\mu^2}-1\right)^{1/2},\quad k_+=\left(\frac{\omega^2}{\mu^2}-4\right)^{1/2},\quad b=-\frac{i}{2}(k_++k_-),$$
$$A(k_-,k_+)=\frac{\Gamma(1-ik_+)\Gamma(-ik_-)}{\Gamma(-\frac{1}{2}ik_+-\frac{1}{2}ik_-+\frac{5}{2})\Gamma(-\frac{1}{2}ik_+-\frac{1}{2}ik_--\frac{3}{2})}\,. \tag{35}$$

We observe that (31) is obtained from (33) by putting $k_+=iK_+$. Part (A) represents incoming mesons (from the left, of mass μ) which are partly reflected from the soliton, and partly transmitted. Transmission can occur only for sufficiently energetic mesons ($\omega^2\geq 4\mu^2$) and then the mesons reappear on the right side of the soliton with a heavier mass 2μ, appropriate to that vacuum sector. Part (B) represents incoming mesons (from the right, of mass 2μ) which are partly reflected from the soliton, but can also pass through at any energy to emerge on the left with mass μ.

The functions $\eta(x)$ are used to calculate first-order quantum corrections to the classical theory, such as one-meson matrix elements and soliton mass corrections.

IV. SOLITON MASS CORRECTION

We calculate here the quantum correction of $O(\lambda^0)$ to the classical soliton mass $M=\mu^2/4\lambda$, due to quantum fluctuations around the classical solution. Using the explicit functions $\eta(x)$ we can show that the correction is finite, having subtracted the vacuum energy and renormalized. This subtraction has to allow for the fact that two distinct vacuums are involved.

We substitute Eq. (4) into the expression (3) for the total energy to obtain

$$E=\frac{\mu^2}{4\lambda}+\tfrac{1}{2}\int dx\left[\left(\frac{\partial\eta}{\partial t}\right)^2+\eta K\eta\right]+O(\lambda), \tag{36}$$

where K is given by Eq. (5). We diagonalize K by expanding the operator $\eta(x,t)$ in terms of eigenfunctions η_k of K, and the correction of $O(\lambda^0)$ is then merely a harmonic-oscillator term,[1] so that

$$E=\frac{\mu^2}{4\lambda}+\frac{1}{2}\sum_k\omega_k, \tag{37}$$

where

$$\omega_k^{\,2}=\mu^2(k_-^{\,2}+1)=\mu^2(k_+^{\,2}+4). \tag{38}$$

[This expression is correct to $O(\lambda^0)$ even when the zero mode is properly considered.[1]] The correction

$$\tfrac{1}{2}\mu\sum_k(k_-^{\,2}+1)^{1/2}$$

diverges quadratically but subtraction of the vacuum energy should render this divergence logarithmic, to be canceled in turn by renormalization counterterms. To regulate the infrared divergence we place the soliton in a large box of length L, with periodic boundary conditions. In order to evaluate the vacuum contribution, we must consider the situation in which the soliton is removed from the box. Faced with two unrelated vacuums we must decide how to take account of each vacuum in order to subtract properly the vacuum energy. To $O(\lambda^0)$ this vacuum energy will be given by an expression similar to that of Eq. (36),

$$E_{\text{vac}}=\tfrac{1}{2}\int dx\left[\left(\frac{\partial\eta}{\partial t}\right)^2+\eta K_0\eta\right], \tag{39}$$

where $K_0=-d^2/dx^2+U_0(x)$ is the Schrödinger operator in the absence of the soliton. The potential $U(x)$ is shown in Fig. 2; the soliton contributes the central dip, on either side of which $U(x)$ attains constant asymptotic values. Without the soliton $U(x)$ is restricted to those asymptotic (vacuum) values only, i.e.,

$$\begin{aligned}U_0(x)&=4\mu^2\quad\text{for } x>x_0\\&=\mu^2\quad\text{for } x<x_0\,.\end{aligned} \tag{40}$$

By translation we can choose $x_0=0$. The mass correction is then

$$\tfrac{1}{2}\sum_k(\omega_k-\omega_k^{\,0})=\tfrac{1}{2}\mathrm{Tr}(\sqrt{K}-\sqrt{K_0})\,.$$

The expression (40) for $U_0(x)$ is obtained in effect by shifting $\phi(x,t)$ as before [Eq. (4)], but taking

$$\phi_c(x)=\theta(x)(\mu/\lambda)^{1/2}, \quad (41)$$

where $\theta(x)=1$ for $x>0$ and $\theta(x)=0$ for $x<0$. This merely corresponds to taking the asymptotic (vacuum) values of the soliton solution (27). The "classical solution" (41) has a discontinuity at $x=0$ separating the two vacuums and contributes a divergent energy to the "soliton vacuum" of $O(1/\lambda)$. This divergence may be ignored in evaluating $E_{\rm vac}$ since there is no corresponding divergence in the soliton solution to be subtracted.

By placing the soliton in a box with suitable boundary conditions we can render the continuum modes countable and thereby follow them when the soliton is removed. We obtain then, in principle, the relation between the vacuum wave number q_- and the wave number k_- with the soliton included, enabling us to evaluate

$$\tfrac{1}{2}\mu\sum\left[(k_-^2+1)^{1/2}-(q_-^2+1)^{1/2}\right].$$

In practice, this procedure is effective only for situations in which the potential permits tranmission only, as in ϕ^4 and the sine-Gordon model for example. The reason for this can be seen upon imposing periodic boundary conditions on the step potential (40) to obtain a periodic potential, with period L, as shown in Fig. 5. The continuum wave solutions break into a countable number of distinct bands within which q_- varies continuously.[6] Since the operator K_0 is invariant under translation by L, wave functions η_0 satisfy

$$\eta_0(x+L)=e^{i\mu QL}\eta_0(x) \quad (42)$$

for some Q (the propagation constant). Q can be quantized by identifying wave functions after N periods, i.e., bending the line after N periods into a circle of length NL. Each energy band then splits into a finite number of levels, and we have

$$Q=\frac{2\mu n}{\pi NL}, \quad n=0,\pm1,\ldots. \quad (43)$$

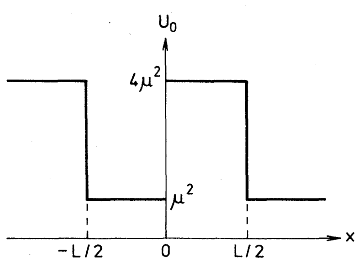

FIG. 5. The Schrödinger potential $U_0(x)$ due to the "soliton vacuum," with period L.

By matching wave functions and first derivatives at all boundaries we obtain the condition which forces the existence of the bands,[6]

$$\cos\mu QL=\frac{(q_++q_-)^2}{4q_-q_+}\cos(q_-+q_+)\tfrac{1}{2}\mu L - \frac{(q_+-q_-)^2}{4q_-q_+}\cos(q_--q_+)\tfrac{1}{2}\mu L\,. \quad (44)$$

The wave numbers q_- and q_+ are defined by

$$(\omega^0)^2=\mu^2(q_-^2+1)^{1/2}=\mu^2(q_+^2+4)^{1/2}, \quad (45)$$

where $(\omega^0)^2$ is an eigenvalue of K_0. Equation (44) applies even if $(\omega^0)^2<4\mu^2$ by analytic continuation, setting $q_+=iQ_+$ where Q_+ is real.

Similarly, for the more general case of potentials for which the asymptotic behavior of the wave functions is as given in Eqs. (33) and (34), imposition of periodic boundary conditions produces solutions with a band structure. Because of this, it is difficult to determine exactly how the countable levels change when the soliton is introduced into the box since there will not be an equal shift for each level. One obtains $q_-^{(n)}$ as a function of $k_-^{(n)}$ in the form

$$q_-^{(n)}=k_-^{(n)}+\frac{1}{\mu L}\gamma^{(n)}(k_-^{(n)}), \quad (46)$$

where the shift $\gamma^{(n)}$ depends explicitly on the level n. Only for the situation of no reflection does the band structure disappear, and with it the explicit dependence of γ on the nth level. For the potential under consideration this situation, in fact, arises at high energies (large k_-) when the back scattering disappears. This is evident from the coefficient $A(k_-,k_+)$ in Eq. (35). Writing

$$A(k_-,k_+)=N(k_-,k_+)e^{i\delta(k_-,k_+)}, \quad (47)$$

we have

$$N(k_-,k_+)^2=\frac{k_+\cosh^2[\frac{1}{2}\pi(k_++k_-)]}{k_-\sinh\pi k_-\sinh\pi k_+}. \quad (48)$$

Current conservation implies

$$N(k_-,k_+)^2-N(-k_-,k_+)^2=k_+/k_-\,. \quad (49)$$

For large $k_-=k$ we find

$$N(-k_-,k_+)=O(e^{-\pi k}), \quad (50)$$

$$N(k_-,k_+)^2=k_+/k_-+O(e^{-2\pi k}) \quad (51)$$

[setting $k_+=k-3/2k+O(k^{-3})$]. Therefore, the coefficient of reflection falls off exponentially for large k.

The mass correction for a reflectionless potential is[1]

$$\tfrac{1}{2}\mu\int\frac{dk}{2\pi}\,\delta(k)\,\frac{d}{dk}\,(k^2+1)^{1/2}, \quad (52)$$

where $\delta(k)$ is the phase shift, and this applies in our situation only for large k. This is precisely the region in which divergences appear and we can check that upon inclusion of renormalization counterterms, the ultraviolet divergence cancels out. The phase shift $\delta(k)$ for high energies is given by the asymptotic value of $\delta(k_-, k_+)$ as defined in Eq. (47). $\delta(k_-, k_+)$ is given by

$$\delta(k_-, k_+) = -\frac{i}{2}\ln\left[\frac{k_-\sinh\pi k_-\sinh\pi k_+}{k_+\cosh^2[\frac{1}{2}\pi(k_+ + k_-)]}\right] - i\ln\left[\frac{\Gamma(1-ik_+)\Gamma(-ik_-)}{\Gamma(-\frac{1}{2}ik_+ - \frac{1}{2}ik_- + \frac{5}{2})\Gamma(-\frac{1}{2}ik_+ - \frac{1}{2}ik_- - \frac{3}{2})}\right]. \tag{53}$$

To find the asymptotic value $\delta(k)$ we use known asymptotic formulas for $\ln\Gamma(z)$,[7] essentially Stirling's formula. We find for example that (for constants a)

$$\ln\Gamma\left(-\frac{ik_+}{2} - \frac{ik_-}{2} + a\right) = k\left(i - \frac{\pi}{2}\right) + \frac{1}{2}\ln 2\pi + \frac{i\pi}{4} - \frac{ia\pi}{2} + \left(a - \frac{1}{2} + \frac{3i}{4k} - ik\right)\ln k + \frac{1}{k}\left(\frac{i}{12} - \frac{ia}{2} + \frac{ia^2}{2} + \frac{3\pi}{8}\right) + O(k^{-2}).$$

We obtain

$$\delta(k) = -\frac{15}{4k} + O(k^{-2}), \tag{54}$$

and so the integral (52) is logarithmically divergent.

Now we must consider the effect of renormalization. The Lagrangian $\mathcal{L}$ will be renormalized by a normal ordering of the polynomial self-interaction. Since

$$\phi^6 = :\phi^6: + a_1:\phi^4: + a_2:\phi^2: + a_3, \qquad \phi^4 = :\phi^4: + b_1:\phi^2: + b_2 \tag{55}$$

for coefficients a_i, b_i, we must add ϕ^4 and ϕ^2 counterterms to $\mathcal{L}$. The counterterms of $O(\lambda)$ are required to cancel the one-loop diagrams (Fig. 6). Taking into account combinatorial factors, we obtain

$$\mathcal{L} \to \mathcal{L} - \tfrac{1}{2}\delta\mu^2\phi^2 + \tfrac{5}{8}(\lambda/\mu)\delta\mu^2\phi^4, \tag{56}$$

where, to $O(\lambda)$,

$$\delta\mu^2 = \frac{3\mu\lambda}{\pi}\int\frac{dk}{(k^2+\mu^2)^{1/2}}. \tag{57}$$

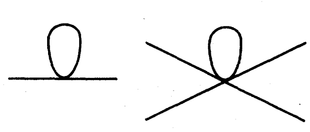

FIG. 6. One-loop diagrams for ϕ^6, logarithmically divergent.

The counterterms render finite the physical quantities calculated in the meson sector based on the vacuum $\phi = 0$, but also apply to the $\phi = (\mu/\lambda)^{1/2}$ vacuum sector, by shifting the field suitably, $\phi \to \phi - (\mu/\lambda)^{1/2}$. As we shall see, the same counterterms also work for the soliton sector.

The counterterms contribute to the soliton mass additional terms of $O(\lambda^0)$:

$$\tfrac{1}{2}\delta\mu^2\int\phi_c^2 dx - (5\lambda/8\mu)\delta\mu^2\int\phi_c^4 dx, \tag{58}$$

where ϕ_c is given by Eq. (27). (The divergent integrals are regulated by integrating over $|x| \le L/2$.) The counterterms also contribute to the vacuum an amount to be subtracted from (58). This contribution can be calculated from (58) but substituting instead ϕ_c as given by Eq. (41), the vacuum contribution of the soliton solution. The vacuum counterterm is then $(-\mu L/16\lambda)\delta\mu^2$, and when subtracted from the soliton counterterm (58) gives a total additional term $(5/16\lambda)\delta\mu^2$, in which we have let $L \to \infty$.

The soliton mass, including only high-energy fluctuations, is now

$$E = \frac{\mu^2}{4\lambda} + \frac{\mu}{2}\int\frac{dk}{2\pi}\delta(k)\frac{k}{(k^2+1)^{1/2}} + \frac{15\mu}{16\pi}\int\frac{dk}{(k^2+1)^{1/2}}. \tag{59}$$

The integrand of the counterterm is asympotically $15\mu/16\pi k$, which by Eq. (54) nicely cancels the logarithmic divergence due to the function $\delta(k)$, and we conclude that the first-order soliton mass correction is finite.

Before leaving ϕ^6, let us remark on the fact that there are four static solutions [namely Eq. (27)] and those obtained by putting $x \to -x$, $\phi_c \to -\phi_c$, and that quantum mechanically these can be distinguished. This is unlike ϕ^4 where the operation $x \to -x$ (to produce the antisoliton) is equivalent to $\phi_c \to -\phi_c$ and both solutions correspond to the same quantum particle.[1] For ϕ^6 the antisoliton is different because each side of the soliton is distinguished by the different meson mass allowed i.e., the operation $x \to -x$ cannot be reversed by putting $\phi_c \to -\phi_c$. The soliton differing from another by the operation $\phi_c \to -\phi_c$ can also be distinguished by adjoining to it another soliton, with the solutions suitably patched together.[1] The soliton charge $Q = \int (d\phi_c/dx)dx$ will be either zero, or double that of a single charge, and the two possibilities can be distinguished. Here we are using the fact that ϕ^6 allows double solitons.

V. ϕ^8 MODELS

Models with ϕ^8 self-interactions offer a variety of soliton phenomena, but exact calculations cannot be carried far and only general statements are possible. One new feature for ϕ^8 is the possibility of massless mesons. We consider potentials of the form

$$V=\lambda^3(\phi^2-v^2)^2(\phi^2-\alpha v^2)^2\,, \tag{60}$$

as shown in Fig. 7 for $\alpha>1$. Special cases occur for $\alpha<0$, $\alpha=0$, and $\alpha=1$, but consider first $\alpha>1$. There are four vacuums, two being connected by the symmetry $\phi\to-\phi$, and these support mesons of mass given by

$$\begin{aligned} V''(v)&=8\lambda^3v^6(\alpha-1)^2=\mu^2\,,\\ V''(\sqrt{\alpha}v)&=8\lambda^3v^6\alpha(\alpha-1)^2=\alpha\mu^2\,. \end{aligned} \tag{61}$$

Two distinct soliton types are possible, one denoted S_1, interpolating between $\phi=-v$ and $\phi=v$, the other S_2 between $\phi=v$ and $\phi=\sqrt{\alpha}v$. The soliton masses, of $O(1/\lambda)$, are

$$\begin{aligned} M_1&=(2\lambda^3)^{1/2}\frac{4v^5}{15}(5\alpha-1)\,,\\ M_2&=(2\lambda^3)^{1/2}\frac{2v^5}{15}(\sqrt{\alpha}-1)^3(\alpha+3\sqrt{\alpha}+1)\,, \end{aligned} \tag{62}$$

and are equal only for $\alpha-3\sqrt{\alpha}+1=0$, i.e., $\alpha\simeq 6.85$. The static solutions are given by the formula

$$e^{\pm\mu x}=\left(\frac{v+\phi}{v-\phi}\right)\left(\frac{v\sqrt{\alpha}-\phi}{v\sqrt{\alpha}+\phi}\right)^{1/\sqrt{\alpha}}\,, \tag{63}$$

which can be explicitly solved for ϕ only in special cases, for example, $\alpha=4$ and $\alpha=9$ [the $n=2$ case of the potential (9)].

The Schrödinger equation (15) describing linear oscillations around the static solution can be converted to Heun's equation, for which little can be said. However, the general shape of $U(x)$ is as shown in Fig. 8 for the soliton S_1. $U(x)$ is symmetric and, for $1<\alpha<5$, has local maxima on either side of the central depression. One expects that incoming plane waves will be partly scattered and partly reflected. Similar statements can be made for $U(x)$ applicable to S_2, except that $U(x)$ will be asymmetric.

We can show that for α large enough mesons can form bound states with S_1, i.e., there can be discrete levels in addition to $\omega^2=0$. We take the eigenvalue equation in the form Eq. (16). An upper bound on a second discrete eigenvalue $\omega_1{}^2$ is given by

$$\omega_1{}^2\leqslant\frac{(f,Lf)}{(f,f)}\,, \tag{64}$$

where

$$L=-2V\frac{d^2}{d\phi^2}-3V'\frac{d}{d\phi}\,,$$

the inner product(,) is defined by Eq. (18) and f is a trial function orthogonal to $f_0=$ constant, the eigenfunction of the lowest eigenvalue $\omega_0{}^2=0$. A suitable trial function is found by noting that for $\alpha\to\infty$ the potential (60) (with suitable dependence of λ on α) approaches the ϕ^4 potential (6) for which the exact eigenfunction of the second discrete eigenvalue $\omega_1{}^2/\mu^2=\frac{3}{4}$ is[1] $f(\phi)=\phi(v^2-\phi^2)^{-1/2}$. Using this as a trial function and with the result (19) we find

$$\frac{\omega_1{}^2}{\mu^2}\leqslant\frac{\alpha^3-\alpha^2+\frac{3}{5}\alpha-\frac{1}{7}}{4(\alpha-1)^2(\frac{1}{3}\alpha-\frac{1}{5})}\,, \tag{65}$$

which must be <1 for a discrete eigenvalue $\omega_1{}^2$ to exist, since the continuum begins at $\omega_1{}^2=\mu^2$. We find that for $\alpha>\sim 6.35$, at least one discrete level will exist besides $\omega^2=0$, i.e., a meson-soliton bound state can form.

If the soliton charge is defined by $Q=\int(d\phi_c/dx)dx$, then the ratio of charges for S_1 to S_2 is $2/(\sqrt{\alpha}-1)$, and we note that this need not be rational. For $\alpha=9$ the charges are equal but, although S_2 is heavier than S_1, no decay can take place because each soliton occupies a different

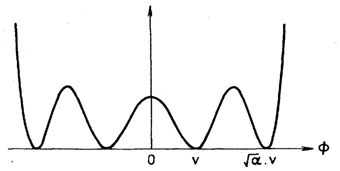

FIG. 7. ϕ^8 potential V with four minima.

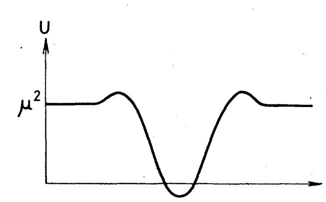

FIG. 8. Approximate potential $U(x)$ for ϕ^8 soliton interpolating between related vacuums supporting mesons of mass μ.

sector of the solution space, with distinct vacuum values. Let us also point out that S_1, but not S_2, is self-conjugate in the sense that the symmetry $\phi_c \to -\phi_c$ is equivalent to $x \to -x$, as for ϕ^4.[1]

Finally, let us consider the potential (60) with $\alpha = 0$,

$$V(\phi) = \lambda^3\phi^4(\phi^2 - v^2)^2 . \tag{66}$$

V is similar to the ϕ^6 potential (Fig. 1) with three minima, but the mesons built on the vacuum $\phi = 0$ are now massless, leading to long-range interactions. Meson states built on the vacuums $\phi^2 = v^2$ have mass

$$\mu = (2\lambda^3)^{1/2} 2 v^3 .$$

There is one soliton (together with those obtained by symmetry) with mass $M = (2\lambda^3)^{1/2}2v^5/15$. The static solutions are given by

$$\mu x = -\frac{v}{\phi} + \ln\frac{v+\phi}{v-\phi} , \tag{67}$$

and have an appearance similar to the ϕ^6 solutions (Fig. 4), except that the falloff on the side with massless mesons is very slow: $\phi \sim v/\mu|x|$ for x large and negative. The soliton charge density $d\phi_c/dx$ similarly has a slow falloff and the soliton might therefore be expected to have long-range interactions, like a one-dimensional "magnetic" monopole. The long-range interaction is indeed apparent from the shape of $U(x)$ (Fig. 3), since for x large and negative $U(x) \sim 6/x^2$. Massless mesons will therefore experience a repulsive force at long range, but can penetrate the soliton at energies such that $\omega^2 > \mu^2$, and acquire a mass μ on the right. The equation determining the scattering [Eq. (16)] can be converted to the following by substituting $Z = v^2/\phi^2$:

$$4Z(Z-1)^2 g''(Z) - 6(Z-1)(Z-3)g'(Z) + \frac{4\omega^2}{\mu^2} Z^2 g(Z) = 0 . \tag{68}$$

This equation has regular singularities at $Z = 0, 1$ and an irregular singularity at $Z = \infty$, but cannot be identified with any well-known equation. For large negative x, $U(x) \sim 6/x^2$ and we find

$$\eta(x) = e^{i\omega x}\left(\frac{3}{\omega^2 x^2} - \frac{3i}{\omega x} - 1\right) , \tag{69}$$

which, together with $\eta^*(x)$, represents incoming and outgoing plane waves (massless mesons). Meson-soliton bound states ("dyons") cannot form because the continuum begins at $\omega^2 > 0$.

For the potential (60) with $\alpha = 1$ there are also massless mesons, but in this case no massive mesons at all, yet the soliton will nevertheless have a mass of $O(1/\lambda)$. Again, the soliton has long-range interactions.

VI. CONCLUSION

We have examined $P(\phi)_2$ models and seen that they can support a rich variety of soliton phenomena. The possibility of many minima of the potential allows for many meson and soliton sectors, which need not be connected by any symmetry, so that both meson and soliton masses will differ from sector to sector. For a potential with n minima there will be n^2 sectors of the solution space, between which transitions cannot occur. Although we have been able to say nothing about multisoliton solutions, we have investigated linear oscillations around the static soliton and so described meson-soliton scattering at least approximately. The exact solution obtained for ϕ^6 theory has enabled the soliton mass correction due to these oscillations to be studied; the finite correction which eventuates confirms that, as for ϕ^4 and sine-Gordon models, renormalizability of the theory is not affected by the shift to a different sector, be it a meson or soliton sector. The subtraction of the vacuum energy involving disparate vacuums leads to complicated expressions, but this will be the general situation.

Many of the results for $P(\phi)_2$ are very special to two dimensions, although the quantization procedure and the renormalization properties are general. One restrictive feature in one space dimension is the patching procedure, which requires that adjacent solitons must correspond to smoothly joined solutions. This means that, like mesons of different sectors, the solitons of different sectors can interact only via adjoining solitons. Therefore, although the theory may be very rich in particle content, the possible interactions of these particles are highly restricted.

ACKNOWLEDGMENT

I am grateful to other members of the department for invaluable discussions and advice.

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