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A generalized quantum field theory

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An expression in quantum-field-theoretic language of the four-space formulation (FSF), especially the FSF group properties, is derived by generalizing Schwinger's formulation of Lagrangian quantum field theory (LQFT). The resulting theoretical framework includes a mass operator in addition to the energy-momentum and angular momentum operators. It also contains LQFT as a special case. Broad conclusions regarding conservation laws (of rest mass, energy-momentum, and angular momentum) are obtained from the general formalism. Many mathematical details concerning the FSF group and FSF transformations are presented.

INTRODUCTION

The most widely used formulation of Lagrangian quantum field theory (LQFT) is that by Schwinger.[1] His formulation began with the definition of an action operator W_S in terms of a function $\mathcal{L}_S$ of field operators $\phi_\alpha(x)$ and their first derivatives such that

$$W_S = \int_{R_S} \mathcal{L}_S(\phi_\alpha(x), \partial_\mu \phi_\alpha(x))dx\,. \tag{1}$$

The quantity R_S denotes an infinite four-volume in space-time that is bounded by the spacelike surfaces σ_1, σ_2 as in Fig. 1, and the invariant measure dx is defined as

$$dx \equiv dx^1 dx^2 dx^3 dx^4\,. \tag{2}$$

The notation used here is that of Roman,[2] namely

$$\partial_\mu = \partial/\partial x^\mu \tag{3a}$$

and

$$\partial^\mu = g^{\mu\lambda}\partial_\lambda = \partial/\partial x_\mu\,, \tag{3b}$$

where the nonzero elements of the fundamental metric tensor $g_{\mu\nu}$ are

$$g_{00} = 1 = -g_{11} = -g_{22} = -g_{33}\,. \tag{4}$$

By performing the variation δW_S and then postulating that δW_S is equal to the difference between the generators of canonical transformations F_S at σ_1 and σ_2, i.e.,

$$\delta W_S = F_S[\sigma_2] - F_S[\sigma_1]\,, \tag{5}$$

Schwinger developed LQFT. A correspondence between LQFT and classical field theory (CFT) can be drawn as follows.

Classically[3] the invariant parameter used to trace the evolution of a system point in configuration space is the proper time $\tau_{\rm cl}$. Furthermore, the Lagrangian $L_{\rm cl}$ used in a covariant formulation of Hamilton's principle must satisfy certain specified transformation properties, e.g., Lorentz invariance. The subsequent classical action integral has the form

$$I_{\rm cl} = \int_{\tau_{\rm cl1}}^{\tau_{\rm cl2}} L_{\rm cl}\, d\tau\,, \tag{6a}$$

and Hamilton's principle requires that

$$\delta I_{\rm cl} = 0 \tag{6b}$$

where both $I_{\rm cl}$ and $L_{\rm cl}$ are invariant scalars, and the proper times $\tau_{\rm cl1}, \tau_{\rm cl2}$ are kept fixed. In a relativistic CFT the quantity $L_{\rm cl}$ becomes an integral of a classical Lagrangian density $\mathcal{L}_{\rm cl}$ such that

$$L_{\rm cl} = \int_{R_{\rm cl}} \mathcal{L}_{\rm cl}\, dx\,, \tag{7}$$

where $R_{\rm cl}$ is the appropriate four-volume.

The point of interest here is the similarity of Eqs. (1) and (7). Schwinger's procedure is classically analogous to varying $L_{\rm cl}$ rather than $I_{\rm cl}$. This raises the question: What would QFT look like if the action operator was redefined to more closely parallel CFT? The answer to this question will be obtained by defining an action operator A analogous to $I_{\rm cl}$ and then evaluating δA. The resulting formalism is aesthetically appealing for two reasons: It closely parallels CFT; and it retains all of LQFT as a special case. Beyond aesthetics, however, is a physically significant motivation for performing the ensuing derivation.

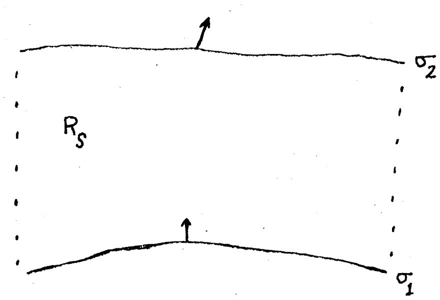

FIG. 1. Schwinger's infinite four-volume R_S. The arrows denote unit four-vectors normal to the spacelike surfaces σ_1, σ_2.

In order to develop a quantum-field-theoretic analog of the classical relativistic action integral I_{cl}, one must introduce the concept of a quantum-mechanical proper time. The concept of a quantum-mechanical proper time that depends on the coordinates $\{x^\mu\}$ was first introduced by Fock[4] in 1937, and later discussed by Stückelberg,[5] Nambu,[6] Schwinger,[7] and Feynman.[8] At the time the usefulness of this concept was not established and interest in it waned.

A revival of interest in the notion of quantum-mechanical proper time, now independent of the coordinates $\{x^\mu\}$, began as early as 1968 with Cooke's[9] work. In 1973 Horwitz and Piron[10] postulated a similar quantum-mechanical formalism for spinless particles that was based on a correspondence principle relating classical and quantum mechanics. Horwitz, Piron, and Reuse[11-12] later extended Horwitz and Piron's[10] work, called "relativistic dynamics," to spin-$\frac{1}{2}$ particles and an interesting calculation was made[13] to justify their formulation. All of this work[9-13] included a mass-operator concept in addition to the quantum-mechanical proper-time concept.

The usefulness of a mass operator has been recognized for years. For example, Feynman, Kislinger, and Ravndal[14] defined mass operators in their efforts to derive observed mass spectra. Despite the usefulness of mass operators, existing theories of elementary particles include mass operators in an *ad hoc* manner only. This situation has recently changed.

A probabilistic foundation for a relativistic theory of spinless particles which incorporates quantum-mechanical proper time and mass operators was recently provided by Fanchi and Collins.[15-17] This theory, referred to as the four-space formulation (FSF), formalized the notion of a quantum-mechanical proper time which does *not* depend on the coordinates $\{x^\mu\}$, and also provided a solid basis for the mass-operator concept applied to spinless particles. The FSF is a consistent single-particle theory of relativistic spinless particles in the presence of an arbitrary four-vector potential. As mentioned in Ref. 16, the FSF is capable of describing particles with nonzero spin and nonelectromagnetic interactions, such as the field equation for spin-$\frac{1}{2}$ particles postulated by Reuse.[13] A field-theoretic formulation of the FSF concepts should enhance the progress of research in this area by making it easier to design and evaluate experimental tests of the FSF.

Three features are present in the FSF which suggest that Schwinger's formulation of LQFT should be extended. The first feature is the appearance of a fifth independent variable that has properties analogous to τ_{cl}. This variable is therefore identified as quantum-mechanical proper time and denoted by the symbol τ. The quantity τ can be introduced into a field theory by defining an action integral similar to I_{cl} and letting the integrand be a function of τ.

The second feature is the invariance of the FSF with respect to a symmetry group which is an 11-parameter continuous group that includes the Poincaré group as a subgroup. The 11-parameter group is called the FSF group and is discussed in Appendix B. Invariance of a field theory with respect to the FSF group is achieved by requiring that the integrand of the action integral, i.e., the Lagrangian, be invariant with respect to the FSF group. This procedure is developed in detail below.

The third feature is perhaps the most important physically. In particular, an eleventh operator has appeared in the FSF which is interpreted as a mass operator. By generalizing Schwinger's formulation of LQFT as prescribed here, a field-theoretic framework including a mass operator is obtained. The result is a formalism which can be used to describe particles with arbitrary spin and the fundamental interactions. This claim is substantiated by showing that LQFT is a special case of the general formalism. Broad conclusions, particularly with regard to conservation laws, are obtained from the general formalism. Detailed applications, other than the derivation of LQFT, are deferred.

Before leaving this section it should be stated that another reason for presenting the ensuing derivation is to express the FSF concepts in the language of the mainstream of elementary-particle physicists, namely field theory. Consequently this paper should make many of the concepts of the FSF more readily accessible to a broad audience of research physicists.

FORMULATION OF THE VARIATIONAL INTEGRAL

As usual the Heisenberg picture is assumed, however, its meaning in the present context needs to be made explicit. In particular, a state vector is specified as the simultaneous eigenket of a complete set of commuting observables in space-time at a fixed value of an independent scalar identified as the proper time and denoted by τ. The quantity τ is a scalar invariant which is used to parametrize the evolution of a physical system[16,17] (see Fig. 2). Attributing a physical interpretation to the scalar τ is not required by the formalism developed below. The mathematics is unchanged if τ is simply thought of as an independent scalar that parametrizes the fields. The quantity τ is identified as the proper time here, however, because

FIG. 2. Proper-time evolution of a physical system.

this identification is consistent with the interpretation of τ given in Refs. 15–17. Furthermore, identifying τ as proper time lets us parallel classical field theory by starting with an action integral that is analogous to Eq. (6a).

By choosing the Heisenberg picture it is now possible to describe the dynamical development of the physical system in proper time. This is achieved by determining the transformation law that connects the observables at proper times τ_1, τ_2 separated by the interval

$$\Delta\tau = \tau_2 - \tau_1 > 0 . \tag{8}$$

The quantity $\Delta\tau$ can be thought of as the invariant interval during which a physical system is observed. It is desired to describe the evolution of the physical system in a space-time region R during the interval $\Delta\tau$.

Let ζ_1, ζ_2 denote complete sets of commuting operators at proper times τ_1, τ_2, respectively, and denote the corresponding state vectors as $\xi(\zeta_1), \xi(\zeta_2)$. The transformation law that connects these state vectors and operators also expresses the dynamical properties—the evolution—of the physical system. This law may be written in terms of the transformation matrix which transforms the family of state vectors $\xi(\zeta_1)$ into the family $\xi(\zeta_2)$. The transformation matrix is just the inner product of $\xi(\zeta_1)$ and $\xi(\zeta_2)$, i.e.,

$$\langle \zeta_1 | \zeta_2 \rangle = \langle \xi(\zeta_1) | \xi(\zeta_2) \rangle . \tag{9}$$

The transformation matrix $\langle \zeta_1 | \zeta_2 \rangle$ represents one path of evolution of the system from τ_1 to τ_2. Alternative evolution paths are obtained by canonically varying $\langle \zeta_1 | \zeta_2 \rangle$. This variation is induced by the infinitesimal canonical transformation discussed in Appendix A:

$$\delta\psi_\alpha = i[G, \psi_\alpha] , \tag{10}$$

where ψ_α is an operator and G is the generator of the infinitesimal canonical transformation. The new operators at τ_2 can be written as $\zeta_2 + \delta\zeta_2$, where

$$\delta\zeta_2 = i[G, \zeta_2] . \tag{11}$$

The eigenvalues of the new operators are the same as the eigenvalues of the old operators since the variation is canonical, but the corresponding state vector is changed by an amount

$$\delta\xi(\zeta_2) = iG(\tau_2)\xi(\zeta_2) . \tag{12}$$

A similar variation of the operators at τ_1 changes the eigenvectors at τ_1 by a similar amount:

$$\delta\xi(\zeta_1) = iG(\tau_1)\xi(\zeta_1) . \tag{13}$$

The new transformation matrix corresponding to the inner product of the new eigenvectors can be written to first order in the variations as $\langle \zeta_1 | \zeta_2 \rangle + \delta\langle \zeta_1 | \zeta_2 \rangle$, where

$$\delta\langle \zeta_1 | \zeta_2 \rangle = i\langle \zeta_1 | \delta A | \zeta_2 \rangle \tag{14}$$

and the operator δA has the form

$$\delta A = G(\tau_2) - G(\tau_1) . \tag{15}$$

Equations (14) and (15) show the effect of the canonical transformation on the transformation matrix and thus on the evolution path. This effect is determined by evaluating δA.

It is now assumed that there exists an invariant scalar function L, subsequently called the Lagrangian, such that δA can be obtained by varying the action integral

$$A = \int_1^2 L d\tau , \tag{16}$$

where the points τ_1, τ_2 are fixed and A is called the action operator. Substituting Eq. (16) into Eq. (15) yields the variational integral

$$\delta A = \delta \int_1^2 L d\tau = G(\tau_2) - G(\tau_1) . \tag{17}$$

In words, the variational integral asserts that the variation of the action operator is equal to the difference between the generators of infinitesimal canonical transformations at τ_1 and τ_2. An explicit evaluation of δA and the generators depends on the functional form of L which, in turn, depends on the symmetry properties of the physical system.

The field-theoretic specification of the functional form of L is in terms of field operators $\psi_\alpha(x,\tau)$ ($\alpha = 1, 2, \ldots, N$) which represent the N degrees of freedom of the physical system. Thus define the Lagrangian operator L as a four-volume integral of a Lagrangian density operator $\mathfrak{L}$ such that

$$L \equiv \int_R \mathfrak{L}(\psi_\alpha, \partial_\mu\psi_\alpha, \dot{\psi}_\alpha) dx , \tag{18}$$

where R is an arbitrary four-volume that contains the entire physical system and

$$\dot{\psi}_\alpha(x,\tau) \equiv \frac{\partial\psi_\alpha(x,\tau)}{\partial\tau} . \tag{19}$$

The field operators ψ_α must be defined such that

$\mathcal{L}$ is integrable. Furthermore, the neglect of higher-order derivatives of ψ_α in $\mathcal{L}$ is an arbitrary assumption designed to simplify the subsequent mathematical manipulations. It should also be emphasized that the invariant scalar τ and all of the components of the four-position x are independent variables as in Refs. 15–17.

An important conclusion can be drawn based upon a comparison of Eqs. (16)–(18). In particular, the operators L and $\mathcal{L}$ must be self-adjoint since the generators G are self-adjoint (see Appendix A). This is a significant restriction on the form the operators L and $\mathcal{L}$ may take. Additional restrictions are imposed by the symmetry properties of the physical system.

GENERALIZED QUANTUM FIELD THEORY

It was stressed in Ref. 16 that the symmetry properties of the single-particle theory of relativistic spinless particles are those of the FSF group (see Appendix B). These same symmetry properties are assumed applicable to the generalized quantum field theory (GQFT) being developed here. Thus an equivalent description of the physical system presently described in the coordinate frame $\{x^\mu\}$ at the proper time τ is achieved by an infinitesimal transformation to a new frame at τ' given by

$$x'^\mu = x^\mu + \delta x^\mu , \tag{20}$$

$$\tau' = \tau + \delta\tau , \tag{21}$$

and by making an infinitesimal transformation of the field operators such that

$$\psi'_\alpha(x,\tau) = \psi_\alpha(x,\tau) + \delta_0\psi_\alpha(x,\tau) . \tag{22}$$

Equation (22) defines the total variation $\delta_0\psi_\alpha(x,\tau)$. Observe that the argument of ψ'_α has the same numerical value as the argument of ψ_α, therefore ψ'_α and ψ_α refer to different physical points.

The total variation of the field operator $\psi_\alpha(x,\tau)$ due to δA is given by Eq. (A11), namely

$$\delta_0\psi_\alpha(x,\tau) = i[\delta A, \psi_\alpha(x,\tau)] , \tag{23}$$

where x must be a point in the region R since the generator δA depends on R. It will be shown below that δA is the generator of the infinitesimal canonical transformation Eqs. (20)–(22). With the symmetry properties of the physical system specified, the explicit evaluation of δA can be made.

The form of the initial action operator A_i is

$$A_i = \int_1^2 \int_R \mathcal{L}(x,\tau)\,dx\,d\tau , \tag{24}$$

where

$$\mathcal{L}(x,\tau) \equiv \mathcal{L}(\psi_\alpha(x,\tau), \partial_\mu\psi_\alpha(x,\tau), \dot\psi_\alpha(x,\tau)) . \tag{24a}$$

The canonical transformation induces a change in A_i. The form of the final, canonically transformed action operator A_f is

$$A_f = \int_1^2 \int_{R'} \mathcal{L}'(x',\tau')\,dx'\,d\tau , \tag{25}$$

where

$$\mathcal{L}'(x',\tau') \equiv \mathcal{L}(\psi'_\alpha(x',\tau'), \partial_\mu\psi'_\alpha(x',\tau'), \dot\psi'_\alpha(x',\tau')) , \tag{25a}$$

and the change in the boundaries of R due to Eq. (20) is denoted by R'. Notice that $d\tau$ is unchanged because the τ end points are fixed and $d\tau' = d\tau$. The variation δA is just the difference between A_f and A_i, i.e.,

$$\delta A = A_f - A_i , \tag{26}$$

or

$$\delta A = \int_1^2 \int_{R'} \mathcal{L}'(x',\tau')\,dx'\,d\tau - \int_1^2 \int_R \mathcal{L}(x,\tau)\,dx\,d\tau . \tag{26a}$$

Equation (26a) is evaluated by first expressing A_f in the unprimed system. The following procedure is analogous to that used in evaluating δW_s.[1,18]

The four-volume element dx' is related to dx by the Jacobian $\partial(x')/\partial(x)$ of the coordinate transformation, namely

$$dx' = \frac{\partial(x')}{\partial(x)}dx \approx \left[1 + \frac{\partial(\delta x^\mu)}{\partial x^\mu}\right]dx , \tag{27}$$

where only terms up to first order in the variations are kept. The quantity $\mathcal{L}'(x',\tau')$ is written in the unprimed system by using Eqs. (20)–(22) and Taylor's series, thus

$$\mathcal{L}'(x',\tau') = \mathcal{L}'(x,\tau) + \frac{\partial\mathcal{L}'(x,\tau)}{\partial x^\mu}\delta x^\mu + \frac{\partial\mathcal{L}'(x,\tau)}{\partial\tau}\delta\tau$$

$$= \mathcal{L}(x,\tau) + \bar\delta\mathcal{L}(x,\tau) + \left\{\frac{\partial}{\partial x^\mu}[\mathcal{L}(x,\tau) + \bar\delta\mathcal{L}(x,\tau)]\right\}\delta x^\mu + \left\{\frac{\partial}{\partial\tau}[\mathcal{L}(x,\tau) + \bar\delta\mathcal{L}(x,\tau)]\right\}\delta\tau , \tag{28}$$

where

$$\mathcal{L}'(x,\tau) \equiv \mathcal{L}(\psi'_\alpha(x,\tau), \partial_\mu\psi'_\alpha(x,\tau), \dot\psi'_\alpha(x,\tau)) \tag{28a}$$

and

$$\bar{\delta}\mathcal{L}(x,\tau)=\frac{\partial\mathcal{L}(x,\tau)}{\partial\psi_\alpha}\,\delta_0\psi_\alpha+\frac{\partial\mathcal{L}(x,\tau)}{\partial\partial_\mu\psi_\alpha}\,\delta_0(\partial_\mu\psi_\alpha)+\frac{\partial\mathcal{L}}{\partial\dot{\psi}_\alpha}\,\delta_0\dot{\psi}_\alpha\,. \tag{28b}$$

Again dropping terms that are second order in the variations lets us simplify Eq. (28):

$$\mathcal{L}'(x',\tau')=\mathcal{L}(x,\tau)+\bar{\delta}\mathcal{L}(x,\tau)+\frac{\partial\mathcal{L}(x,\tau)}{\partial x^\mu}\delta x^\mu+\frac{\partial\mathcal{L}(x,\tau)}{\partial\tau}\delta\tau\,. \tag{29}$$

Substituting Eqs. (27) and (29) into Eq. (26a) yields

$$\delta A=\int_1^2\int_R\left\{\left[\mathcal{L}(x,\tau)+\bar{\delta}\mathcal{L}(x,\tau)+\frac{\partial\mathcal{L}(x,\tau)}{\partial x^\mu}\delta x^\mu+\frac{\partial\mathcal{L}(x,\tau)}{\partial\tau}\delta\tau\right]\left[1+\frac{\partial(\delta x^\mu)}{\partial x^\mu}\right]\right\}dx\,d\tau-\int_1^2\int_R\mathcal{L}(x,\tau)\,dx\,d\tau\,. \tag{30}$$

Expanding and dropping terms that are second order in the variation gives

$$\delta A=\int_1^2\int_R\left\{\bar{\delta}\mathcal{L}(x,\tau)+\frac{\partial\mathcal{L}(x,\tau)}{\partial x^\mu}\delta x^\mu+\frac{\partial\mathcal{L}(x,\tau)}{\partial\tau}\delta\tau+\mathcal{L}(x,\tau)\frac{\partial(\delta x^\mu)}{\partial x^\mu}\right\}dx\,d\tau\,. \tag{30a}$$

The integral

$$\int_1^2\int_R\frac{\partial\mathcal{L}(x,\tau)}{\partial\tau}\delta\tau\,dx\,d\tau$$

vanishes since the τ end points are fixed (as can be shown by an integration by parts). Thus Eq. (30a) becomes

$$\delta A=\int_1^2\int_R\left\{\bar{\delta}\mathcal{L}(x,\tau)+\frac{\partial\mathcal{L}(x,\tau)}{\partial x^\mu}\delta x^\mu+\mathcal{L}(x,\tau)\frac{\partial(\delta x^\mu)}{\partial x^\mu}\right\}dx\,d\tau \tag{30b}$$

or

$$\delta A=\int_1^2\int_R\left\{\frac{\partial\mathcal{L}(x,\tau)}{\partial\psi_\alpha}\delta_0\psi_\alpha+\frac{\partial\mathcal{L}(x,\tau)}{\partial\partial_\mu\psi_\alpha}\partial_\mu(\delta_0\psi_\alpha)+\frac{\partial\mathcal{L}(x,\tau)}{\partial\dot{\psi}_\alpha}\frac{\partial}{\partial\tau}(\delta_0\psi_\alpha)+\frac{\partial}{\partial x^\mu}[\mathcal{L}(x,\tau)\delta x^\mu]\right\}dx\,d\tau\,, \tag{31}$$

where the relations

$$\delta_0\partial_\mu\psi_\alpha=\partial_\mu\delta_0\psi_\alpha\,,\qquad \delta_0\dot{\psi}_\alpha=\frac{\partial}{\partial\tau}(\delta_0\psi_\alpha) \tag{32}$$

have been used. The following manipulations are designed to simplify Eq. (31).

Integrating the term

$$\int_1^2\int_R\frac{\partial\mathcal{L}}{\partial\dot{\psi}_\alpha}\delta_0\dot{\psi}_\alpha\,dx\,d\tau$$

by parts, making use of Eq. (32), and postulating the constraint that $\delta_0\psi_\alpha$ vanishes at the end points τ_1,τ_2 gives

$$\int_1^2\int_R\frac{\partial\mathcal{L}}{\partial\dot{\psi}_\alpha}\delta_0\dot{\psi}_\alpha\,dx\,d\tau=-\int_1^2\int_R\left[\frac{d}{d\tau}\left(\frac{\partial\mathcal{L}}{\partial\dot{\psi}_\alpha}\right)\right]\delta_0\psi_\alpha\,dx\,d\tau\,. \tag{33}$$

The rule for differentiating a product is used to rearrange the second term in Eq. (31) such that

$$\frac{\partial\mathcal{L}}{\partial\partial_\mu\psi_\alpha}\partial_\mu(\delta_0\psi_\alpha)=\partial_\mu\left[\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\psi_\alpha}\right)\delta_0\psi_\alpha\right]-\left[\partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\psi_\alpha}\right]\delta_0\psi_\alpha\,. \tag{34}$$

Substituting Eqs. (33) and (34) into Eq. (31) and then rearranging the result gives

$$\delta A=\int_1^2\int_R\left\{\left[\frac{\partial\mathcal{L}}{\partial\psi_\alpha}-\partial_\mu\pi^{\alpha\mu}-\frac{d}{d\tau}\frac{\partial\mathcal{L}}{\partial\dot{\psi}_\alpha}\right]\delta_0\psi_\alpha\right\}dx\,d\tau+\int_1^2\int_R\{\partial_\mu[\pi^{\alpha\mu}\delta_0\psi_\alpha+\mathcal{L}\delta x^\mu]\}\,dx\,d\tau\,, \tag{35}$$

where

$$\pi^{\alpha\mu}(x,\tau)\equiv\frac{\partial\mathcal{L}(x,\tau)}{\partial\partial_\mu\psi_\alpha(x,\tau)}\,. \tag{36}$$

Equation (36) associates a four-vector $\pi^{\alpha\mu}$ with every field operator ψ^α.

The integrand of the second integral in Eq. (35) is the divergence of a four-vector $\mathcal{J}^\mu$ defined by

$$\mathcal{J}^\mu\equiv\pi^{\alpha\mu}\delta_0\psi_\alpha+\mathcal{L}\delta x^\mu\,. \tag{37}$$

Equation (37) is put into a physically more interesting form by first introducing the local variation

$$\delta\psi_\alpha \equiv \psi'_\alpha(x',\tau') - \psi_\alpha(x,\tau)\,, \tag{38}$$

where x, τ and x', τ' refer to the same physical point as viewed from canonically transformed reference frames. Performing a Taylor series expansion of $\psi'_\alpha(x',\tau')$ and keeping only first-order terms gives

$$\delta\psi_\alpha = \psi'_\alpha(x,\tau) + \partial^\nu \psi'_\alpha(x,\tau)\delta x_\nu + \dot{\psi}'_\alpha(x,\tau)\delta\tau - \psi_\alpha(x,\tau)\,. \tag{39}$$

Equation (22) is now used to express $\delta\psi_\alpha$ in terms of the original fields ψ_α. The result is

$$\delta\psi_\alpha = \delta_0\psi_\alpha + \partial^\nu \psi_\alpha(x,\tau)\delta x_\nu + \dot{\psi}_\alpha(x,\tau)\delta\tau\,, \tag{40}$$

where second-order terms, i.e., $[\partial^\nu \delta_0\psi_\alpha(x,\tau)]\delta x^\nu$ and $[\partial\delta_0\psi_\alpha(x,\tau)/\partial\tau]\delta\tau$ have again been neglected. The quantity $\mathcal{J}^\mu$ now has the form

$$\mathcal{J}^\mu = \pi^{\alpha\mu}\delta\psi_\alpha - (\pi^{\alpha\mu}\partial^\nu\psi_\alpha - \delta^{\nu\mu}\mathcal{L})\delta x_\nu - \pi^{\alpha\mu}\dot{\psi}_\alpha\delta\tau\,, \tag{41}$$

where $\delta^{\nu\mu}$ is the Kronecker δ. Substituting the above results into δA yields

$$\delta A = G(\tau_2) - G(\tau_1) + \int_1^2 \int_R \left(\frac{\partial\mathcal{L}}{\partial\psi_\alpha} - \partial_\mu \pi^{\alpha\mu} - \frac{d}{d\tau}\frac{\partial\mathcal{L}}{\partial\dot{\psi}_\alpha}\right)\delta_0\psi_\alpha\, dx d\tau\,, \tag{42}$$

where $\pi^{\alpha\mu}$ and $\mathcal{J}^\mu$ are defined by Eqs. (36) and (41), respectively, and the notation

$$G(\tau_2) - G(\tau_1) \equiv \int_1^2 \int_R \partial_\mu \mathcal{J}^\mu\, dx d\tau \tag{43}$$

has been used.

At this point the mathematical formalism is essentially complete. Readers familiar with Schwinger's work will undoubtedly notice the similarity of the above formalism to LQFT. The principal differences are due to the incorporation of a fifth independent variable τ and this, in turn, arises because the symmetry properties of the FSF group have been imposed on the physical system.

GENERALIZED FIELD EQUATIONS

The notation used in Eq. (42) is intended to suggest the identification of $G(\tau)$. Equation (42) is consistent with the postulate, Eq. (17), only if the integral

$$I_2 = \int_1^2 \int_R \left(\frac{\partial\mathcal{L}}{\partial\psi_\alpha} - \partial_\mu \pi^{\alpha\mu} - \frac{d}{d\tau}\frac{\partial\mathcal{L}}{\partial\dot{\psi}_\alpha}\right)\delta_0\psi_\alpha dx d\tau \tag{44}$$

is always zero. Observing that $\delta_0\psi_\alpha$ and R are arbitrary implies that I_2 vanishes identically only if the bracketed quantity vanishes. Therefore, Eq. (43) is valid only if

$$\frac{\partial\mathcal{L}}{\partial\psi_\alpha} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\psi_\alpha}\right) - \frac{d}{d\tau}\frac{\partial\mathcal{L}}{\partial\dot{\psi}_\alpha} = 0 \tag{45}$$

is true for all values of α. This relation is the Euler-Lagrange equation and is the quantum-mechanical analog of the relativistic classical field-theoretic result. Equation (45) is the field equation for the field operator $\psi_\alpha(x,\tau)$ and differs from the corresponding field equation of LQFT in that the field operator can be a function of τ. If the field operators are independent of τ, then $\mathcal{L}$ is independent of $\dot{\psi}_\alpha$ and Eq. (45) reduces to the LQFT result.

It is clear from the above that Hamilton's principle yields the field equations for the operators ψ_α. Furthermore, if the Lagrangian density $\mathcal{L}$ is invariant under some symmetry group, such as the FSF group, then the operator field equations are covariant with respect to the same symmetry group.

LAGRANGIAN QUANTUM FIELD THEORY

In order to be a valid physical theory, GQFT must describe all of the phenomena described by LQFT. It will be shown in this section that LQFT is, in fact, a special case of GQFT. This will prove that GQFT describes at least as much phenomena as LQFT.

Equation (42) can be written as

$$\delta A = \int_1^2 \delta L\, d\tau = \int_1^2 \int_R \partial_\mu \mathcal{J}^\mu\, dx d\tau\,, \tag{46}$$

where Eqs. (17) and (45) have been used. If the field operators are independent of τ, as all field operators of LQFT are, then Eq. (46) becomes

$$\delta A_S = \Delta\tau\,\delta L_S = \Delta\tau \int_R \partial_\mu \mathcal{J}^\mu\, dx\,, \tag{47}$$

where the subscript S indicates the operator is independent of τ. Dividing Eq. (47) by $\Delta\tau$ yields

$$\delta L_S = \frac{\delta A_S}{\Delta\tau} = \int_R \partial_\mu \mathcal{J}^\mu\, dx\,. \tag{48}$$

Applying the divergence theorem to Eq. (48) yields

$$\delta L_S = \int_{\sigma_2} \mathcal{J}^\mu\, d\sigma_\mu - \int_{\sigma_1} \mathcal{J}^\mu\, d\sigma_\mu\,, \tag{49}$$

where $d\sigma_\mu$ is the μth surface element defined by

$$\begin{aligned} d\sigma_0 &= dx^1 dx^2 dx^3\,, \\ d\sigma_1 &= dx^0 dx^2 dx^3\,, \\ d\sigma_2 &= dx^0 dx^1 dx^3\,, \\ d\sigma_3 &= dx^0 dx^1 dx^2\,, \end{aligned} \tag{50}$$

and it has also been assumed that the field operators vanish at spatial infinity. The variation δL_S can be written as the difference of generators of observables on the spacelike surfaces σ_1, σ_2 such that

$$\delta L_S = F_S[\sigma_2] - F_S[\sigma_1], \tag{51}$$

where

$$F_S[\sigma] \equiv \int_\sigma \mathcal{J}^\mu d\sigma_\mu , \tag{52}$$

which correspond exactly to the relations used by Schwinger.[1] The formulation of LQFT, including the usual commutation rules, can now proceed as in Refs. 1 or 2.

GENERATORS OF THE FSF GROUP

A physical interpretation can be assigned to specific mathematical constructs of the GQFT by obtaining the field-theoretic form of the FSF group generators. These forms are developed from the equation

$$\delta A = \int_1^2 \int_R \partial_\mu \mathcal{J}^\mu dx d\tau , \tag{53}$$

which was obtained by combining Eqs. (42) and (45). The four-volume integral in Eq. (53) can be transformed into a hypersurface integral by using the divergence theorem. The result is

$$\delta A = \int_1^2 \oint \mathcal{J}^\mu d\mathcal{S}_\mu d\tau , \tag{54}$$

where $d\mathcal{S}_\mu$ is an infinitesimal, covariant four-vector characterizing the hypersurface elements of the closed boundary of R. If the particular four-volume R_S is used as the region of the integration, then $d\mathcal{S}_\mu$ becomes $d\sigma_\mu$ on σ_2 and $-d\sigma_\mu$ on σ_1. (See Fig. 1).

Equation (54) is used to determine the total variation of ψ_α due to δA as in Eq. (23), thus

$$\delta_0\psi_\alpha(x', \tau') = i\left[\int_1^2 \oint \mathcal{J}^\mu d\mathcal{S}_\mu d\tau, \psi_\alpha(x', \tau')\right]. \tag{55}$$

In order to evaluate $\delta_0\psi_\alpha$ for an infinitesimal FSF transformation, it is necessary to determine δx^ν, $\delta\tau$, and $\delta\psi_\alpha$. These increments are developed in Appendix C for the FSF transformation and may be written as

$$\delta x_\nu = \epsilon_{\nu\rho} x^\rho + \epsilon_\nu , \tag{56a}$$

$$\delta\tau = \epsilon , \tag{56b}$$

and

$$\delta\psi_\alpha = \tfrac{1}{2}\Sigma_\alpha^{\beta\nu\rho}\psi_\beta\epsilon_{\nu\rho} , \tag{56c}$$

where $\epsilon_{\nu\rho}$, ϵ_ν, and ϵ are first-order infinitesimals. By Eqs. (56), Eq. (41) now has the form

$$\mathcal{J}^\mu = \tfrac{1}{2}M^{\mu\nu\rho}\epsilon_{\nu\rho} + \tilde{t}^{\mu\nu}\epsilon_\nu + H^\mu\epsilon \tag{57}$$

where $M^{\mu\nu\rho}$ is the covariant angular momentum tensor density defined by

$$M^{\mu\nu\rho} \equiv \tilde{t}^{\mu\rho}x^\nu - \tilde{t}^{\mu\nu}x^\rho + \pi^{\alpha\mu}\Sigma_\alpha^{\beta\nu\rho}\psi_\beta , \tag{58}$$

$\tilde{t}^{\mu\nu}$ is the canonical energy-momentum tensor defined by

$$\tilde{t}^{\mu\nu} \equiv \delta^{\mu\nu}\mathcal{L} - \pi^{\alpha\mu}\partial^\nu\psi_\alpha , \tag{59}$$

and the notation

$$H^\mu \equiv -\pi^{\alpha\mu}\dot{\psi}_\alpha \tag{60}$$

has been used. Observe that $M^{\mu\nu\rho}$ is antisymmetric, i.e.,

$$M^{\mu\nu\rho} = -M^{\mu\rho\nu} . \tag{61}$$

Substituting Eq. (57) into Eq. (55) yields

$$\begin{aligned}\delta_0\psi_\alpha(x', \tau') = \frac{i}{2}&\left[\int_1^2 \oint M^{\mu\nu\rho}d\mathcal{S}_\mu d\tau, \psi_\alpha(x', \tau')\right]\epsilon_{\nu\rho} \\ &+ i\left[\int_1^2 \oint \tilde{t}^{\mu\nu}d\mathcal{S}_\mu d\tau, \psi_\alpha(x', \tau')\right]\epsilon_\nu \\ &+ i\left[\int_1^2 \oint H^\mu d\mathcal{S}_\mu d\tau, \psi_\alpha(x', \tau')\right]\epsilon . \end{aligned}\tag{62}$$

A comparison of Eq. (62) and Eq. (C14) implies that the generators of the FSF transformation are given by

$$J^{\nu\rho} = \int_1^2 \oint M^{\mu\nu\rho}d\mathcal{S}_\mu d\tau , \tag{63}$$

$$P^\nu = \int_1^2 \oint \tilde{t}^{\mu\nu}d\mathcal{S}_\mu d\tau , \tag{64}$$

and

$$H_{\rm op} = \int_1^2 \oint H^\mu d\mathcal{S}_\mu d\tau \tag{65}$$

since the coefficients of the arbitrary infinitesimals $\epsilon_{\nu\rho}$, ϵ_ν, and ϵ must be equal.

If it is assumed that ψ_α vanishes at spatial infinity and the four-volume R_S is chosen as the region of integration, then Eqs. (63)–(65) become

$$J^{\nu\rho} = J^{\nu\rho}(\sigma_2) - J^{\nu\rho}(\sigma_1) , \tag{63a}$$

$$P^\nu = P^\nu(\sigma_2) - P^\nu(\sigma_1) , \tag{64a}$$

and

$$H_{\rm op} = H_{\rm op}(\sigma_2) - H_{\rm op}(\sigma_1) , \tag{65a}$$

where

$$J^{\nu\rho}(\sigma) \equiv \int_1^2 \int_\sigma M^{\mu\nu\rho}d\sigma_\mu d\tau , \tag{63b}$$

$$P^\nu(\sigma) \equiv \int_1^2 \int_\sigma \tilde{t}^{\mu\nu}d\sigma_\mu d\tau , \tag{64b}$$

and

$$H_{op}(\sigma) \equiv \int_1^2 \int_\sigma H^\mu d\sigma_\mu d\tau . \tag{65b}$$

Notice that if the eigenvalues of $J^{\nu\rho}$, P^ν, and H_{op} are zero, then the respective eigenvalues of these generators evaluated at σ_2 equal those evaluated at σ_1, i.e., the eigenvalues are conserved. This situation is more fully discussed in the next section.

The physical meaning of the generators $J^{\nu\rho}(\sigma)$ and $P^\nu(\sigma)$ is obtained by reducing the GQFT formalism to the LQFT formalism and comparing the results. This reduction is achieved by requiring that ψ_α be independent of τ in Eqs. (63b)–(65b). In this case $\dot{\psi}_\alpha$ is zero which, by Eqs. (60) and (65), implies H_{op} vanishes. The resulting equations are

$$J_S^{\nu\rho}(\sigma) = \Delta\tau \int_\sigma M^{\mu\nu\rho} d\sigma_\mu \tag{66a}$$

and

$$P_S^\nu(\sigma) = \Delta\tau \int_\sigma \tilde{t}^{\mu\nu} d\sigma_\mu , \tag{66b}$$

where the subscript S indicates ψ_α is independent of τ as in Schwinger's formulation of LQFT. The generators $J_S^{\nu\rho}(\sigma)$ and $P^\nu_S(\sigma)$ are proportional to Schwinger's field-theoretic expressions for the total angular momentum tensor and the energy-momentum four-vector, respectively. The proportionality constant is the c-number $\Delta\tau$. Thus $J^{\nu\rho}(\sigma)$ and $P^\nu(\sigma)$ are interpreted as the GQFT expressions for the total angular momentum tensor and the energy-momentum four-vector, respectively.

The physical interpretation of H^μ is based on the fact that H^μ is related to the generator H_{op} of infinitesimal translations along τ by Eq. (65). According to Ref. 16, H_{op} has as eigenvalues the inner product of the canonical momentum four-vector for spinless particles. In the case of noninteracting spinless particles the eigenvalues of H_{op} are the same as the eigenvalues of the Casimir operator $P^\mu P_\mu$, which are just the square of the rest mass of the particles. Thus H_{op} is identified as a mass operator and H^μ is called the mass flux operator. The latter terminology depends on the observation that H^μ may be thought of as a flux density and this observation is verified by using the divergence theorem to rewrite Eq. (65) as

$$H_{op} = \int_1^2 \int_R \partial_\mu H^\mu dx d\tau . \tag{67}$$

THE PHYSICAL MEANING OF H_{op}

The interpretation of H_{op} as a mass operator is verified by evaluating the eigenvalue of H_{op} for a free particle. Begin by defining a vacuum state $|0\rangle$ such that

$$H_{op}(\sigma)|0\rangle = 0 , \tag{68a}$$

$$P^\mu(\sigma)|0\rangle = 0 , \tag{68b}$$

$$J^{\mu\nu}(\sigma)|0\rangle = 0 , \tag{68c}$$

and

$$\langle 0|0\rangle = 1 \tag{68d}$$

in a manner analogous to LQFT. The specification of σ is intended to indicate that the eigenvalues of the respective operators belong to a state vector at the spacelike surface σ.

The field equation for a free spinless particle is[16]

$$2\bar{m}i\dot{\psi} = \nabla^2\psi - \frac{\partial^2\psi}{\partial x_0^{\,2}} \tag{69}$$

which can be derived from a Lagrangian density having the form

$$\mathcal{L}_0 = \frac{i}{2}(\dot{\psi}^*\psi - \psi^*\dot{\psi}) + \frac{1}{2\bar{m}}\partial^\mu\psi\partial_\mu\psi^* . \tag{70}$$

The constants $\hbar$ and c have been set equal to one, and $\bar{m}$ is the average observable rest mass.[16] Equation (69) has the general solution

$$\psi(x,\tau) = \int A(k)\exp\left(-iq(k)\frac{\tau}{2\bar{m}} + ikx\right)d^4k , \tag{71}$$

where

$$q(k) = k_0^{\,2} - \vec{k}\cdot\vec{k} \tag{72a}$$

and

$$kx = k_0x_0 - \vec{k}\cdot\vec{x} . \tag{72b}$$

If Eq. (71) is used in Eq. (C15c) and the result applied to the vacuum state, then the result is

$$H_{op}(\sigma)A(k)|0\rangle - A(k)H_{op}(\sigma)|0\rangle = \frac{q(k)}{2\bar{m}}A(k)|0\rangle . \tag{73}$$

Equation (73) is simplified by employing Eq. (68a) such that

$$H_{op}(\sigma)|1\rangle = \frac{q(k)}{2\bar{m}}|1\rangle , \tag{74}$$

where

$$|1\rangle \equiv A(k)|0\rangle . \tag{74a}$$

In words, the eigenvalues of $H_{op}(\sigma)$ are dimensionally proportional to the rest mass of the state $|1\rangle$. Repetition of the above procedure can be used to build up a multiparticle state. Equation (74) further justifies the interpretation of H_{op} as a mass operator.

CONSERVATION LAWS

The scope of GQFT is illustrated by using GQFT to derive conservation laws with broad physical validity. Conservation laws are a result of canonical transformations which do not change the action operator A, i.e.,

$$\delta A = 0 = \int_1^2 \int_R \partial_\mu \mathcal{J}^\mu \, dx d\tau \,, \tag{75}$$

where R is again arbitrary. The necessary and sufficient condition for Eq. (75) to hold is the continuity equation

$$\partial_\mu \mathcal{J}^\mu = 0 \tag{76}$$

which is the differential form of a conservation law. The form of $\mathcal{J}^\mu$ depends on the type of continuous symmetry transformation that leaves A invariant. For the case of FSF transformation Eq. (76) becomes

$$\partial_\mu \mathcal{J}^\mu = 0 = \partial_\mu \left(\tfrac{1}{2} M^{\mu\nu\rho}\epsilon_{\nu\rho} + \tilde{t}^{\mu\nu}\epsilon_\nu + H^\mu \epsilon\right) , \tag{77}$$

where Eq. (57) has been used. By expressing $\partial_\mu \mathcal{J}^\mu$ as in Eq. (77), it is easy to determine the form of the conservation laws for three particularly important transformations.

(i) For a coordinate translation only:

$$\epsilon_{\nu\rho} = \epsilon = 0 \tag{78}$$

and Eq. (77) gives

$$\partial_\mu \tilde{t}^{\mu\nu} = 0 \,. \tag{79}$$

(ii) For a τ translation only:

$$\epsilon_{\nu\rho} = \epsilon_\nu = 0 \tag{80}$$

and Eq. (77) gives

$$\partial_\mu H^\mu = 0 \,. \tag{81}$$

(iii) For a proper homogeneous Lorentz transformation only:

$$\epsilon_\nu = \epsilon = 0 \tag{82}$$

and Eq. (77) gives

$$\partial_\mu M^{\mu\nu\rho} = 0 \,. \tag{83}$$

Equations (79) and (83) represent conservation of energy-momentum and angular momentum, respectively, as in LQFT. These two relations are clearly independent of all spacelike surfaces since the space-time four-volume R is arbitrary. The remaining result, Eq. (81), is a new conservation law that has appeared because the symmetry properties of the FSF group have been imposed on the physical system. Equation (81) is one form of the assertion that the eigenvalues of $H_{\rm op}$ are conserved when Eq. (75) is valid.

Equation (81) is the differential form of the conservation of $H_{\rm op}$. Another form is obtained by substituting Eq. (81) into Eq. (65a) with the result that

$$H_{\rm op}(\sigma_2) = H_{\rm op}(\sigma_1) \,. \tag{84}$$

Equation (84) asserts that $H_{\rm op}$ on the spacelike surface σ_2 is the same as $H_{\rm op}$ on the spacelike surface σ_1. Results similar to Eq. (84) can be obtained for P^ν and $J^{\nu\rho}$ by applying the divergence theorem to Eqs. (63) and (64) such that

$$J^{\nu\rho} = \int_1^2 \int_R \partial_\mu M^{\mu\nu\rho} dx d\tau \tag{85}$$

and

$$P^\nu = \int_1^2 \int_R \partial_\mu \tilde{t}^{\mu\nu} dx d\tau \,. \tag{86}$$

Substituting Eqs. (83) and (79) into Eqs. (85) and (86), respectively, then yields

$$J^{\nu\rho} = 0 = J^{\nu\rho}(\sigma_2) - J^{\nu\rho}(\sigma_1) \tag{87a}$$

and

$$P^\nu = 0 = P^\nu(\sigma_2) - P^\nu(\sigma_1) \,, \tag{87b}$$

where Eqs. (63a) and (64a) have also been used.

As a final example, let us apply the concept of rest mass conservation to the free spinless particle example. This is done by applying Eq. (84) to $|1\rangle$ and then using Eq. (74) to show that the rest mass of the single particle has not changed during the evolution of the state from σ_1 to σ_2. In other words, the rest mass of the noninteracting single-particle state $|1\rangle$ is conserved as it should be. Thus it can be inferred from this simple case that if Eqs. (81) and (84) hold then the rest mass of a state is conserved. It should be noted that if the free-particle state $|1\rangle$ is allowed to interact, then the form of $q(k)$ will change and the conservation law expressed by (84) will no longer be valid. The detailed effects of specific interactions will be considered elsewhere.

SUMMARY

The primary goals of this paper have been to derive a field-theoretic framework (GQFT) which includes a mass operator, to show the consistency of GQFT with LQFT, and to express the concepts of the FSF in the language of the mainstream of elementary-particle physicists, namely field theory. These goals have been achieved. In addition, the field-theoretic formulation of the usual conservation laws and a new one (rest mass conservation) has been presented. Thus the GQFT not only contains LQFT as a special case, but it also employs the concepts of the FSF and provides a field-theoretic framework upon which to build.

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APPENDIX A: CANONICAL TRANSFORMATIONS

The canonical transformation of a field $\psi_\alpha(x,\tau)$ to $\psi'_\alpha(x,\tau)$ by a unitary operator U is defined by

$$\psi'_\alpha(x,\tau)=U\psi_\alpha(x,\tau)U^{-1}\,, \tag{A1}$$

where, for U^* the Hermitian conjugate of U^{-1},

$$UU^*=U^*U=1\,. \tag{A2}$$

All physical properties of the system described by the original field $\psi_\alpha(x,\tau)$ are unchanged in the new system described by the transformed field $\psi'_\alpha(x,\tau)$. A quantity which is invariant with respect to a canonical transformation is considered physically significant.

An infinitesimal canonical transformation can be obtained by writing U as

$$U=1+iF\,, \tag{A3}$$

where F is considered a first-order infinitesimal. The unitarity condition Eq. (A2) when applied to Eq. (A3) and upon neglecting second-order infinitesimals yields

$$U^*U=(1-iF^*)(1+iF)\approx 1-i(F^*-F)=1\,, \tag{A4}$$

i.e., F must be Hermitian such that

$$F^*=F\,. \tag{A5}$$

The operator F is called the generator of the infinitesimal canonical transformation U. The finite canonical transformation is obtained by iterating Eq. (A3), thus

$$U=\lim_{n\to\infty}\left(1+\frac{i}{n}F\right)^n=e^{iF} \tag{A6}$$

and

$$U^*=\lim_{n\to\infty}\left(1-\frac{i}{n}F\right)^n=e^{-iF}\,, \tag{A7}$$

where F is now considered a finite operator.

Substituting Eq. (A3) into Eq. (A1) gives

$$\psi'_\alpha(x,\tau)=(1+iF)\psi_\alpha(x,\tau)(1-iF) \tag{A8}$$

or, to first order,

$$\psi'_\alpha(x,\tau)=\psi_\alpha(x,\tau)+i[F,\psi_\alpha(x,\tau)]\,. \tag{A9}$$

Thus if the change (total variation) of the field operator $\delta_0\psi_\alpha(x,\tau)$ by an infinitesimal canonical transformation is defined by

$$\delta_0\psi_\alpha(x,\tau)=\psi'_\alpha(x,\tau)-\psi_\alpha(x,\tau)\,, \tag{A10}$$

then the substitution of Eq. (A9) into Eq. (A10) yields

$$\delta_0\psi_\alpha(x,\tau)=i[F,\psi_\alpha(x,\tau)]\,. \tag{A11}$$

A similar expression can be obtained for any operator Ω satisfying Eq. (A1), hence

$$\delta_0\Omega=i[F,\Omega]\,. \tag{A12}$$

The corresponding finite transformation is found using Eqs. (A1), (A6), and (A7):

$$\psi'_\alpha(x,\tau)=e^{iF}\psi_\alpha(x,\tau)e^{-iF}\,. \tag{A13}$$

A similar result is obtained for Ω:

$$\Omega'=e^{iF}\Omega e^{-iF}\,. \tag{A14}$$

APPENDIX B: THE FSF GROUP

The FSF group represents the linear transformation

$$x'_\mu=\Lambda_\mu{}^\nu x_\nu+a_\mu\,, \tag{B1a}$$

$$\tau'=\tau+\Delta\tau\,. \tag{B1b}$$

The quantities $\{\Lambda_\mu{}^\nu\}$ represent a homogeneous Lorentz transformation,[19,20] and the quantities $\{a_\mu\},\Delta\tau$ represent translations along the $\{x_\mu\},\tau$ axes, respectively. Equations (B1) can be denoted by $\{a,\Lambda,\Delta\tau\}$ and are referred to as the FSF transformation. The product of two FSF transformations $\{a_1,\Lambda_1,\Delta\tau_1\}$ and $\{a_2,\Lambda_2,\Delta\tau_2\}$ is given by

$$\{a_1,\Lambda_1,\Delta\tau_1\}\,\{a_2,\Lambda_2,\Delta\tau_2\}=\{a_1+\Lambda_1 a_2,\Lambda_1\Lambda_2,\Delta\tau_1+\Delta\tau_2\}\,. \tag{B2}$$

The set containing as elements all FSF transformations together with the binary operation specified in Eq. (B2) comprises the FSF group. Observe that if $\Delta\tau\to 0$, then the Poincaré group is obtained as a subgroup of the FSF group. Furthermore, denoting the Poincaré group by P and the translation group along τ by T_τ, it can be shown that the FSF group, denoted by $\mathfrak{F}$, is the direct product of T_τ and P, i.e.,

$$\mathfrak{F}=T_\tau\otimes P\,. \tag{B3}$$

This fact makes it easy to evaluate the Lie algebra of the FSF group.

The FSF group is an eleven-parameter continuous group; therefore, it has eleven generators which satisfy 55 commutation relations.[21] Of the eleven generators, ten are those of the Poincaré group and satisfy the usual commutation relations[2,21]:

$$[J_{\mu\nu},J_{\lambda\sigma}]=i(g_{\mu\sigma}J_{\lambda\mu}+g_{\mu\lambda}J_{\nu\sigma}+g_{\nu\sigma}J_{\mu\lambda}+g_{\lambda\nu}J_{\sigma\mu})\,, \tag{B4}$$

$$[J_{\mu\nu},P_\lambda]=i(P_\nu g_{\mu\lambda}-P_\mu g_{\nu\lambda})\,, \tag{B5}$$

and

$$[P_\mu,P_\nu]=0\,. \tag{B6}$$

Equations (B4)-(B6) represent 45 of the 55 commutation relations. The remaining 10 are evaluated by taking the commutator of the eleventh generator of the FSF group, call it H_{op}, with each of the ten generators of the Poincaré group. The operator H_{op} is the generator of infinitesimal translations along τ. As such, H_{op} is the generator of the group T_τ and, as a consequence of Eq. (B3), it is known that H_{op} must commute with all of the generators of the Poincaré group P. Thus the remaining ten commutation relations are

$$[J_{\mu\nu}, H_{op}] = 0 \tag{B7}$$

and

$$[P_\mu, H_{op}] = 0 . \tag{B8}$$

Equations (B4)–(B8) completely define the Lie algebra of the FSF groups. It is of interest to note that the structure constants of the Lie algebra of the FSF group are the same as those of the Lie algebra of the Poincaré group.

APPENDIX C: THE FSF TRANSFORMATION AS A CANONICAL TRANSFORMATION

An infinitesimal FSF transformation can be written as

$$\Lambda_\mu{}^\nu = g_\mu{}^\nu + \epsilon_\mu{}^\nu , \tag{C1a}$$

$$a_\mu = \epsilon_\mu , \tag{C1b}$$

and

$$\Delta\tau = \epsilon , \tag{C1c}$$

where $\epsilon_\mu{}^\nu$, ϵ_μ, and ϵ are first-order infinitesimals. The above infinitesimal FSF transformation has the unitary operator representation

$$U_K = 1 + iK , \tag{C2}$$

where

$$K = \tfrac{1}{2} J^{\mu\nu}\epsilon_{\mu\nu} + P^\nu \epsilon_\nu + H_{op}\epsilon . \tag{C3}$$

It is easy to prove that U_K is unitary to first order in infinitesimals by showing that

$$U_K U_K^* = U_K^* U_K = 1 \tag{C4}$$

as in Appendix A. A finite FSF transformation can be obtained by iterating Eq. (C2). Since U_K is unitary, the FSF transformation can be interpreted as a canonical transformation.

A set of commutation relations of the operators $J^{\mu\nu}$, P^ν, and H_{op} with the field operators ψ_α are obtained by replacing F of Eq. (A11) with K such that

$$\delta_0 \psi_\alpha(x,\tau) = i[K, \psi_\alpha(x,\tau)] . \tag{C5}$$

Each side of Eq. (C5) is developed separately in terms of the infinitesimals $\epsilon_{\mu\nu}$, ϵ_μ, and ϵ. The coefficients of the linear terms on both sides are then equated.

The term $\delta_0\psi_\alpha$ is given by Eq. (40), namely

$$\delta_0\psi_\alpha = \delta\psi_\alpha - \partial^\nu\psi_\alpha \delta x_\nu - \dot{\psi}_\alpha \delta\tau . \tag{C6}$$

The increments $\delta\psi_\alpha$, δx_ν, $\delta\tau$ are specified in terms of the infinitesimal FSF transformation:

$$\delta x_\nu = \epsilon_{\nu\mu} x^\mu + \epsilon_\nu , \tag{C7a}$$

$$\delta\tau = \epsilon , \tag{C7b}$$

and

$$\delta\psi_\alpha = \tfrac{1}{2}\Sigma_\alpha{}^{\beta\nu\mu}\psi_\beta(x,\tau)\epsilon_{\nu\mu} , \tag{C7c}$$

where $\Sigma_\alpha{}^{\beta\nu\mu}$ are the infinitesimal operators of the proper homogeneous Lorentz group. Observe that Eq. (C7c) is valid because translations along x, τ do not cause a variation $\delta\psi_\alpha$ although they do cause a variation $\delta_0\psi_\alpha$. This is shown by performing an infinitesimal translation along x, τ such that

$$\psi'_\alpha(x^\mu, \tau) = \psi_\alpha(x^\mu - \epsilon^\mu, \tau - \epsilon) . \tag{C8}$$

Substituting Eq. (C8) into Eq. (22) yields

$$\begin{aligned} \delta_0\psi_\alpha &= \psi_\alpha(x^\mu - \epsilon^\mu, \tau - \epsilon) - \psi_\alpha(x^\mu, \tau) \\ &= -\partial^\nu\psi_\alpha\epsilon_\nu - \dot{\psi}_\alpha\epsilon . \end{aligned} \tag{C9}$$

Hence it follows from Eqs. (C6) and (C9) that

$$\delta\psi_\alpha = \delta_0\psi_\alpha + \partial^\nu\psi_\alpha\epsilon_\nu + \dot{\psi}_\alpha\epsilon = 0 . \tag{C10}$$

Combining Eqs. (C6) and (C7) gives

$$\begin{aligned} \delta_0\psi_\alpha = &\tfrac{1}{2}[\Sigma_\alpha{}^{\beta\nu\mu}\psi_\beta + (x^\nu\partial^\mu - x^\mu\partial^\nu)\psi_\alpha]\epsilon_{\nu\mu} \\ &- \partial^\nu\psi_\alpha\epsilon_\nu - \dot{\psi}_\alpha\epsilon , \end{aligned} \tag{C11}$$

where $\partial^\nu\psi_\alpha\epsilon_{\nu\mu}x^\mu$ has been rewritten as

$$\partial^\nu\psi_\alpha\epsilon_{\nu\mu}x^\mu = \tfrac{1}{2}[(x^\nu\partial^\mu - x^\mu\partial^\nu)\psi_\alpha]\epsilon_{\nu\mu} \tag{C12}$$

and the antisymmetry relation

$$\epsilon_{\nu\mu} = -\epsilon_{\mu\nu} \tag{C13}$$

has been used. The commutator $i[K, \psi_\alpha]$ is obtained by substituting Eq. (C3) into Eq. (C5), thus

$$\begin{aligned} i[K, \psi_\alpha] = &\tfrac{1}{2} i[J^{\nu\mu}, \psi_\alpha]\epsilon_{\nu\mu} + i[P^\nu, \psi_\alpha]\epsilon_\nu \\ &+ i[H_{op}, \psi_\alpha]\epsilon . \end{aligned} \tag{C14}$$

Comparing Eqs. (C11) and (C14) yields the commutation relations

$$i[J^{\nu\mu}, \psi_\alpha] = \Sigma_\alpha{}^{\beta\nu\mu}\psi_\beta + (x^\nu\partial^\mu - x^\mu\partial^\nu)\psi_\alpha , \tag{C15a}$$

$$i[P^\nu, \psi_\alpha] = -\partial^\nu\psi_\alpha , \tag{C15b}$$

and

$$i[H_{op}, \psi_\alpha] = -\dot{\psi}_\alpha . \tag{C15c}$$

Equations (C15) can also be considered the defining relations for the operators $J^{\nu\mu}$, P^ν, H_{op}.

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