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# Gravitational wave pulse in a spatially homogeneous universe

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An exact solution of the Einstein field equations with perfect fluid source is interpreted as a gravitational wave pulse propagating in a spatially homogeneous spacetime. By studying the matter density of the fluid, we reach the conclusion that if the model universe is expanding the perfect fluid loses energy as a result of the passage of the wave pulse, while if the universe is contracting the opposite occurs. We also study the effect of the wave pulse on the kinematics of the fluid and on the curvature and conformal curvature of the hypersurfaces orthogonal to the fluid flow.

## I. INTRODUCTION

The aim of this paper is to construct an explicit example of a gravitational wave pulse of finite duration which propagates in a spatially homogeneous[1] fluid-filled cosmological model. This type of construction was apparently first used by Bondi[2] and Bondi, Pirani, and Robinson,[3] who gave an example of a plane-wave pulse propagating in a vacuum spacetime. On the basis of their geometrical properties,[4] these waves are currently referred to as plane-fronted waves with parallel rays, or more briefly, pp waves. Our example differs from the above example in two respects: (1) The wave pulse is not a pp wave, primarily owing to the expansion of the universe in which it propagates, and (2) the spacetime is fluid filled. The second difference permits us to investigate the interaction of a gravitational wave pulse with a perfect fluid, when the fluid itself acts as a source of the gravitational field and is not simply treated as a test fluid.

The solution that we use is obtained as follows. In a recent paper[5] it was shown that one could generalize a certain one-parameter family of Bianchi type-I spatially homogeneous perfect-fluid solutions by incorporating an arbitrary function, which is constant on null hypersurfaces, in the metric. By restricting this function appropriately, we are able to construct the desired wave-pulse solution. The resulting metric satisfies the Einstein field equations with a perfect-fluid source. The only unsatisfactory feature is that, as regards the source, we are restricted to the equation of state for stiff matter, namely $p=\mu$.

Various authors[2-4] have shown that a distribution of test particles can extract energy from a pp gravitational wave pulse in a vacuum. Here we are considering a gravitational wave pulse that interacts with a distribution of perfect fluid which itself acts as a source of the wave pulse. Our tentative conclusion is that whether or not the fluid extracts energy from the wave depends on whether the universe model is contracting or expanding.

In Sec. II the wave-pulse solution is presented, and in Secs. III–VII, various properties of the spacetime are studied and used to justify the physical interpretation. Section VIII contains the concluding remarks. The complex null tetrad components of the Ricci and Weyl tensors are listed in the Appendix. All the calculations were performed using a library of programs[6,7] written in the algebraic computing language CAMAL.[8] Finally, we note that this paper presupposes a knowledge of some of the terminology associated with orthonormal frames (i.e., tetrads), as described in Ref. 9. A knowledge of the properties of pp gravitational wave pulses[3,4] would also be helpful.

## II. A WAVE PULSE IN A BIANCHI TYPE-I COSMOLOGY

We consider the following exact solution of the Einstein field equations with perfect-fluid source. The line element is[5]

$$ds^2=e^{2k}(-dt^2+dx^2)+t^{1/2}[dy+w(t-x)dz]^2 + t^{3/2}dz^2, \quad (2.1)$$

where

$$e^{2k}=t^{2m}e^{n(t-x)},$$

and $m$ is a constant subject to $m\geq -\frac{3}{16}$. Here $w$ is an arbitrary function of $t-x$, which is assumed to be of at least class $C^1$, and $n(t-x)$ is related to $w$ according to

$$n'=(w')^2, \quad (2.2)$$

where a prime denotes derivative.

The coordinates assume the following values:

$$0<t<+\infty, \quad -\infty<x,y,z<+\infty. \quad (2.3)$$

The fluid velocity is

$$u=e^{-k}\frac{\partial}{\partial t}, \quad (2.4)$$

and the density and pressure are

 

$$16\pi\mu = 16\pi p = (2m + \tfrac{3}{8})t^{-2m-2}e^{-n(t-x)} . \qquad (2.5)$$

If $w' = 0$, Eqs. (2.1)–(2.5) define a one-parameter family of spatially homogeneous solutions of Bianchi type I, and form a subset of the solutions of Jacobs.[10] The nature of the initial singularity at $t = 0$ depends critically on the parameter $m$. If $m < 0$ the singularity is of the cigar type, if $m = 0$ it is of the barrel type, and if $m > 0$ it is of the point type.[10] The rate of expansion of the fluid is positive and so the model universe is expanding overall. However, if $m < 0$ there is a contraction in the $x$ direction as $t$ increases. These facts can be inferred from Eqs. (4.1) and (4.4). If $w' \neq 0$, the spacetime is "plane symmetric" (i.e., the metric tensor admits a two-parameter Abelian group of isometries with spacelike orbits diffeomorphic to $R^2$), but spatially inhomogeneous. The special case $m = 0$ was originally given by Oleson[11] but in a completely different coordinate system. Our solution is more general than the $p = \mu$ solutions of Tabensky and Taub,[12] which admit the three-parameter group of motions of the Euclidean plane.

By suitably choosing $w(t - x)$, we now construct a spacetime which can be interpreted as a gravitational wave pulse propagating in a spatially homogeneous universe of Bianchi type I.

Let $v = t - x$ and let $v_1, v_2$ be two constants with $v_1 < v_2$. We choose

$$w(v) = \begin{cases} A_1, & v \leq v_1 \\ A(v), & v_1 \leq v \leq v_2 \\ A_2, & v \geq v_2 \end{cases} \qquad (2.6)$$

where $A_1, A_2$ are constants and $A$ is a $C^2$ function with

$$A(v_1) = A_1, \quad A(v_2) = A_2,$$
$$A'(v_1) = 0 = A'(v_2).$$

Equation (2.2) determines $n$ as follows:

$$n(v) = \begin{cases} B_1, & v \leq v_1 \\ B_1 + \int_{v_1}^{v} A'(\bar{v})^2 d\bar{v}, & v_1 \leq v \leq v_2 \\ B_2, & v \geq v_2 \end{cases} \qquad (2.7)$$

where $B$ is an arbitrary constant, and

$$B_2 = B_1 + \int_{v_1}^{v_2} A'(v)^2 dv . \qquad (2.8)$$

The resulting line element (2.1) is of class $C^1$ and piecewise $C^2$ over the coordinate range (2.3).

The spacetime, which is covered by a single coordinate chart defined by (2.3), consists of three regions I, II, and III, separated by the

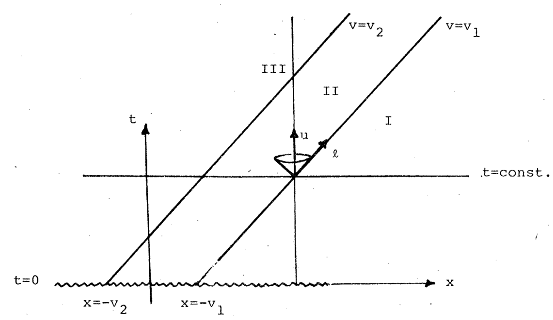


FIG. 1. The dashed line indicates how the $\mu$ = constant hypersurface deviates from the $\Delta\tau$ = constant hypersurface. The events $P_1$, $P_2$, and $P_3$ are located on a hypersurface of constant $\Delta\tau$, and hence are regarded as simultaneous. The diagram shows that $\mu(P_1) = \mu(P_3)$, in accordance with the fact that neither of the fluid particles $x = x_1$, $x = x_3$ has been affected by the wave pulse. On the other hand $\mu(P_2) < \mu(P_1)$, indicating that the wave pulse causes a decrease in density.

hypersurfaces $v = v_1$ and $v = v_2$. It is essential to note that these hypersurfaces are null. In view of our remarks following Eq. (2.5), it follows from (2.6) that the spacetime is spatially homogeneous in regions I and III, and spatially inhomogeneous in region II. We can represent the spacetime by drawing a 2-space $y$ = const, $z$ = const, with $t, x$ taken to be Cartesian coordinates.

We claim that the metric in region II can be regarded as wavelike (i.e., radiative). Lacking an invariant method of locally inferring the presence of gravitational waves in general, one is compelled to use other less well-defined criteria. For example, the presence of an *arbitrary* function, which is constant on a null hypersurface, and which cannot be eliminated by a coordinate transformation, is indicative of the wavelike character of the pp metrics.[3] We were likewise initially led to interpret the metric in region II as being wavelike, by the presence of the arbitrary function $w(t - x)$. An analysis of the Weyl tensor and of the behavior of the perfect-fluid source in region II tends to support this preliminary conclusion. This is described in detail in the next sections.

We will refer to region II as the *gravitational wave pulse*, and to the union of regions I and III as the *background spacetime*. It should be stressed that the wave pulse is not simply a perturbation of a background metric; instead the metric of the whole spacetime is an exact solution of the Einstein field equations.

## III. CHARACTERISTIC SURFACES AND THE WEYL TENSOR

It is well known (see, for example, Ref. 13) that the characteristic surfaces of the Einstein vacuum field equations (i.e., hypersurfaces across which the second derivatives of the metric tensor are discontinuous) are *null hypersurfaces*. In addition the discontinuity in the Weyl tensor (i.e., in vacuum, the Riemann tensor) is of Petrov type $N$. The proof in fact depends only on the Ricci tensor being continuous across the hypersurface in question. Thus if we demand continuity of the energy-momentum tensor, it follows that the characteristic surfaces of the Einstein nonvacuum field equations are null hypersurfaces, and that the discontinuity in the Weyl tensor is of Petrov type $N$.

In our example, the Ricci tensor is continuous across the null hypersurfaces $v=v_1$ and $v=v_2$ (in fact it is of class $C^1$ everywhere), since by Eqs. (2.4) and (2.5), the density, pressure, and fluid velocity are of class $C^1$. The Weyl tensor is, however, discontinuous, and hence the null hypersurfaces $v=v_1$ and $v=v_2$ are characteristic hypersurfaces. As expected, the discontinuity in the Weyl tensor is of type $N$. This can be verified directly by calculating the complex Weyl tensor components relative to a null tetrad $\{l, n, m, \overline{m}\}$, with $l$ tangent to the null geodesic generators of the hypersurfaces $v=\text{const}$ (see the Appendix). One finds that only $\psi_4$ is discontinuous. Unlike in the vacuum case, however, one can construct curvature scalars which exhibit the discontinuity. For example, the scalar

$$I=\frac{C_{apbq}S^{pq}C^{arbs}S_{rs}}{S_{cd}S^{cd}}, \tag{3.1}$$

which is homogeneous of degree 2 in the Weyl tensor and of degree 0 in the trace-free Ricci tensor $S_{ab}$, has a discontinuity across $v=v_1$ and $v=v_2$ given by

$$[I]=\tfrac{2}{3}[w'']t^{-1}e^{-4k}, \tag{3.2}$$

where $[w'']$ denotes the discontinuity in the second derivative of $w(t-x)$ across the null hypersurfaces (see the Appendix). This means, incidentally, that the arbitrary function $w(t-x)$ in the metric is essential, i.e., it cannot be eliminated by a coordinate transformation.

We have thus shown that region II is bounded by characteristic surfaces of the field equations, across which the Weyl tensor discontinuity is necessarily of purely radiative type (i.e., type $N$). This provides some justification for the interpretation of region II as a gravitational wave pulse.

## IV. KINEMATICS OF THE FLUID

The effect of the wave pulse on the motion of the fluid can be studied by calculating the kinematic quantities[9] associated with the fluid congruence. Firstly, the rate of expansion scalar is

$$\theta=[(m+1)t^{-1}+\tfrac{1}{2}(w')^2]e^{-k}. \tag{4.1}$$

Since $\theta>0$, the fluid is expanding throughout spacetime. In addition it follows from one of the contracted Bianchi identities[9] viz. $\dot{\mu}=-(\mu+p)\theta$, that $\dot{\mu}<0$, i.e., that the matter density always decreases into the future along the fluid flow lines. However, Eqs. (4.1) and (2.1) show that $\theta$, and hence $\dot{\mu}$, is affected by the wave pulse.

We will consider the components of the acceleration vector and the rate of expansion tensor relative to the following orthonormal frame:

$$\begin{aligned} &e_{(0)}=e^{-k}\frac{\partial}{\partial t}, \quad e_{(1)}=e^{-k}\frac{\partial}{\partial x},\\ &e_{(2)}=t^{-1/4}\frac{\partial}{\partial y}, \quad e_{(3)}=t^{-3/4}\left(\frac{\partial}{\partial z}-w(t-x)\frac{\partial}{\partial y}\right). \end{aligned} \tag{4.2}$$

The associated one-forms are

$$\begin{aligned} &w^{(0)}=e^{k}dt, \quad w^{(1)}=e^{k}dx,\\ &w^{(2)}=t^{1/4}[dy+w(t-x)dz], \quad w^{(3)}=t^{3/4}dz. \end{aligned}$$

Note that $e_{(0)}$ is the four-velocity of the fluid, and that the frame is invariant under the group of local isometries. The required components can be calculated by using the commutators[9] for the vector fields (4.2).

For the acceleration vector, the only nonzero component is

$$\dot{u}_1=\tfrac{1}{2}(w')^2e^{-k}. \tag{4.3}$$

Thus in regions I and III, where $w'=0$ (i.e., the background spacetime), the fluid flow lines are geodesics, while *within region* II (i.e., *the wave pulse*), *the acceleration of the fluid is nonzero.*

Finally, the nonzero components of the expansion tensor are as follows:

$$\begin{aligned} &\theta_{11}=[mt^{-1}+\tfrac{1}{2}(w')^2]e^{-k},\\ &\theta_{22}=\tfrac{1}{4}t^{-1}e^{-k},\\ &\theta_{23}=\tfrac{1}{2}w't^{-1/2}e^{-k},\\ &\theta_{33}=\tfrac{3}{4}t^{-1}e^{-k}. \end{aligned} \tag{4.4}$$

It is clear that the frame (4.2) is not an eigenframe of the expansion tensor in region II, since $\theta_{23}\neq 0$ there. We can, however, introduce an expansion eigenframe by means of a change of frame of the following form:

$$\begin{aligned} &\tilde{e}_{(2)}=\cos\psi\, e_{(2)}+\sin\psi\, e_{(3)},\\ &\tilde{e}_{(3)}=-\sin\psi\, e_{(2)}+\cos\psi\, e_{(3)}, \end{aligned} \tag{4.5}$$

where $\psi$ is a function of $t$ and $x$. Under (4.5),

$\theta_{23}$ transforms[14] according to

$$\tilde{\theta}_{23} = \cos 2\psi\, \theta_{23} - \tfrac{1}{2}\sin 2\psi\, (\theta_{22} - \theta_{33}).$$

Thus, on account of Eqs. (4.4), the choice

$$\tan 2\psi = -2t^{1/2} w' \tag{4.6}$$

will yield an expansion eigenframe. The rate of expansion in the $e_{(1)}$ direction, $\theta_{11}$, is unchanged and one finds that the principal rates of expansion in the 23 plane (i.e., the eigenvalues of $\theta_{\alpha\beta}$ in this plane) are

$$\tfrac{1}{2}t^{-1}\{1 \pm \tfrac{1}{2}[1 + 4t(w')^2]^{1/2}\}e^{-k}.$$

We can thus see the effect of the wave pulse on the principal expansion rates.

The wave pulse also causes the expansion eigenframe to rotate in the 23 plane, relative to an orthonormal frame which is Fermi propagated along the fluid congruence. Such a frame is interpreted as *nonrotating*.[15] The angular velocity[9] of the expansion eigenframe relative to a nonrotating frame can be calculated by considering the commutators of the frame vectors. One finds that

$$\Omega_1 = \tfrac{1}{2}w' t^{1/2} e^{-k}, \quad \Omega_2 = 0, \quad \Omega_3 = 0. \tag{4.7}$$

Under the frame transformation (4.5), the angular velocity components transform according to

$$\tilde{\Omega}_1 = \Omega_1 + \partial_0 \Omega, \quad \tilde{\Omega}_2 = \Omega_2, \quad \tilde{\Omega}_3 = \Omega_3,$$

where $\partial_0$ denotes the directional derivative in the $e_{(0)}$ direction. It follows, using (4.7), that

$$\tilde{\Omega}_1 = -t^{1/2}\frac{[w'' - 2(w')^3]}{1 + 4t(w')^2}e^{-k}.$$

*Thus, in the background spacetime* (I∪III), *the expansion eigenframe is Fermi propagated* (as expected, on account of the spatial homogeneity[1]), *while within the wave pulse* (II) *the expansion eigenframe rotates about the* $e_{(1)}$ *direction with an angular velocity of magnitude* $|\tilde{\Omega}_1|$.

To summarize, the wave pulse breaks the spatial homogeneity of the pressure (and density) in the direction of propagation (the $e_{(1)}$ direction), thereby giving the fluid a nonzero acceleration in this direction. This in turn affects the rate of expansion $\theta_{11}$ in the $e_{(1)}$ direction. The wave pulse also rotates the expansion eigenframe about the direction of propagation, and affects the principal rates of expansion orthogonal to the direction of propagation. In this respect, the wave pulse behaves like a *transverse* wave.

## V. ENERGY DENSITY OF THE FLUID

Our aim is to determine the effect of the wave pulse on the matter-energy density of the fluid. We wish to compare the density of matter which has interacted with the wave pulse with the density of matter which has not interacted with the pulse. Because the density is affected by the expansion of the universe, however, we must compare the density at events which can be regarded as simultaneous. The timelike coordinate $t$ is a global timelike coordinate but nevertheless cannot be used to define simultaneity globally. The reason is that the time elapsed between $t = t_1$ and $t = t_2$, as measured along the world line of a fluid particle, viz.

$$\int_{t_1}^{t_2} t^m e^{n(t-x)/2} dt, \quad x = \text{const},$$

is dependent on the particular fluid world line (i.e., on the $x$ coordinate) due to the inhomogeneity introduced by the wave pulse. A reasonable choice of time appears to be the (proper) time elapsed since the initial singularity, as measured along the fluid flow lines.

Let $\tau(P)$ denote the time, as measured along a fluid world line $C$, from the initial singularity $t = 0$ to an event $P$ on $C$. It follows from Eqs. (2.1) and (2.4) that

$$\tau(P) = \int_0^{t_P} t^m e^{n(t-x)/2} dt, \quad x = \text{const},$$

where $t_P$ is the value of the $t$ coordinate at $P$. This integral is improper, but converges since $m \geq -\frac{3}{16}$. If we integrate by parts and use Eqs. (2.2) and (2.5), we can express $\tau(P)$ in terms of the density $\mu(P)$ at $P$, and an integral

$$I(P) = \int_0^{t_P} t^{m+1} e^{n(t-x)/2}[w'(t-x)]^2 dt. \tag{5.1}$$

The equation for $\tau(P)$ can be solved to yield the following expression for $\mu(P)$:

$$4\pi\mu(P) = \frac{2m + \frac{3}{8}}{[(2m+2)\tau(P) + I(P)]^2}. \tag{5.2}$$

If there is no interaction with the wave prior to the event $P$, i.e., if the world line in question does not intersect region II prior to reaching $P$, then $I(P) = 0$. Thus for any two such simultaneous events $P_1$, $P_2$, i.e., such that $\tau(P_1) = \tau(P_2)$, it follows from Eq. (5.2) that $\mu(P_1) = \mu(P_2)$, as one would expect in view of the spatial homogeneity of the background spacetime. On the other hand, consider any two events $P_1, P_2$ with $\tau(P_1) = \tau(P_2)$, which are such that the matter at $P_2$ has interacted with the wave, while the matter at $P_1$ has not. Since $I(P_2) > 0$, $I(P_1) = 0$, it follows from Eq. (5.2) that

$$\mu(P_2) < \mu(P_1).$$

Thus the density of the fluid is *decreased* by the

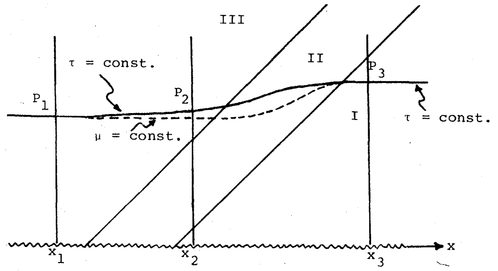


FIG. 2. Diagram of a two-space $y=\text{const}$, $z=\text{const}$. Lines at 45° represent null geodesics, e.g., the lines $v=v_1$ and $v=v_2$, where $v=t-x$, represent the initial and final wave hypersurfaces, respectively. Region II represents the wave pulse. Vertical lines $x=\text{const}$ represent the fluid flow lines, and are geodesics except when they pass through region II. Horizontal lines $t=\text{const}$ represent hypersurfaces orthogonal to the fluid flow lines, and are surfaces of homogeneity, except when they pass through region II. The jagged line $t=0$ represents the initial singularity.

passage of the wave pulse. This is illustrated schematically in Fig. 2.

It is not at all clear how to give an intuitive physical explanation of this phenomenon. A possible explanation has been suggested to me by Dr. D. Eardley, namely that the wave is not purely gravitational, but also contains a sonic component. This would in principle be possible, since for the stiff equation of state $p=\mu$ the speed of sound equals the speed of light.[16] Heuristically a sonic component of the wave, which would be longitudinal, would appear solely in $\theta_{11}$ (since propagation is in the $e_{(1)}$ direction). With this line of reasoning, Eq. (4.4) suggests the presence of a sonic rarefaction wave (in contrast to a compressional wave), with amplitude proportional to $\frac{1}{2}(\omega')^2$, which could possibly be the cause of the decrease in density of the fluid.

It is natural to ask whether this decrease in energy of the fluid is related to the fact that the wave pulse is propagating in an expanding universe. One might expect that if one reverses the time orientation inthe spacetime of Sec. II (i.e., specify that *t decreases* into the future), then the fluid would gain energy from the wave pulse. A detailed analysis supports this conjecture. In this case, however, the physical interpretation is not as attractive, since the wave pulse is created at past infinity (in some sense) in a spacetime which is undergoing gravitational collapse.

With this interpretation, region III is a portion of a spatially homogeneous universe, which has in no way been affected by the wave pulse. It is thus natural to regard events in region III, with $t=\text{const}$, as being simultaneous. This being granted, we select an initial slice $t=t_0$ in region III. Let $\tilde{\tau}(P)$ denote the time, as measured along a fluid world line $C$ from $t=t_0$ to an arbitrary event $P$ on $C$ ($P$ may be in III, II, or I). Let $\mu(P)$ be the energy density of the fluid at $P$, and $\mu_0$ be the (constant) energy density on the slice $t=t_0$. We can repeat the derivation leading to Eq. (5.2) to obtain

$$\mu(P)=\mu_0\{1-[4\pi\mu_0/(2m+\tfrac{3}{8})]^{1/2} \times[2(m+1)\tilde{\tau}(P)+\tilde{I}(P)]\}^{-2}, \qquad (5.3)$$

where

$$\tilde{I}(P)=-\int_{t_0}^{t_P} t^{m+1}e^{n(t-x)/2}[w'(t-x)]^2dt\,.$$

We note that $\tilde{I}(P)\geq 0$ since $t_P<t_0$. Whether or not the fluid particle with world line $C$ interacts with the wave prior to event $P$ depends on whether $\tilde{I}(P)>0$. Consider two fluid world lines $C_1, C_2$ which pass through the slice $t=t_0$ within region III, i.e., the matter density on $C_1$ "initially" (i.e., when $t=t_0$) equals the density on $C_2$. Consider two events $P_1, P_2$ on $C_1, C_2$, respectively, such that $\tilde{I}(P_1)=0$, $\tilde{I}(P_2)>0$. It follows from Eq. (5.3) that

$$\mu(P_2)>\mu(P_1)\,,$$

i.e., *the effect of the wave pulse in the collapsing universe is to increase the density of the fluid.*

## VI. THE DURATION OF THE WAVE PULSE

Let $\Delta\tau$ denote the time taken, as measured along the fluid flow lines, for the wave pulse to pass. We will refer to this as *the duration of the wave*, for the fluid particle in question. This quantity in general depends on the $x$ coordinate, and so we write $\Delta\tau(x)$. It follow from (2.1) and (2.4) that

$$\Delta\tau(x)=\int_{t=v_1+x}^{v_2+x} t^m e^{n(t-x)/2}dt\,,$$

for any fixed $x$ satisfying $-v_1<x<\infty$. A simpler formula can be obtained by using $v=t-x$ as integration variable. One obtains

$$\Delta\tau(x)=\int_{v_1}^{v_2}(v+x)^m e^{n(v)/2}dv\,, \quad \text{with } -v_1<x<\infty\,.$$

We see that the duration depends crucially on the parameter $m$, and that *the duration is constant if and only if* $m=0$. In addition, since $m\geq -\frac{3}{16}$, it follows that

$$\lim_{x\to -v_1^+}\Delta\tau(x) \text{ is finite,}$$

for all values of $m$. On the other hand, as $x\to\infty$,

$$\lim_{x\to\infty}\Delta\tau(x)=\begin{cases}0, & \text{if } -\frac{3}{16}\leq m<0\\ +\infty, & \text{if } m>0.\end{cases}$$

One can also calculate the "spatial thickness" $\Delta l(t)$ of the pulse in the direction of propagation (i.e., $y, z = \text{const}$), as measured along a $t = \text{con-}$stant slice. One finds that

$$\Delta l(t) = t^m \int_{v_1}^{v_2} e^{n(v)/2} dv\,, \quad \text{for fixed } t > 0\,.$$

Thus $\Delta l(t) = \text{const}$ if and only if $m = 0$, and

$$\lim_{t\to\infty} \Delta l(t) = \begin{cases} 0\,, & \text{if } -\frac{3}{16} \le m \le 0 \\ +\infty\,, & \text{if } m > 0 \end{cases}$$

$$\lim_{t\to 0^+} \Delta l(t) = \begin{cases} +\infty\,, & \text{if } -\frac{3}{16} \le m \le 0 \\ 0\,, & \text{if } m > 0\,. \end{cases}$$

This behavior is evidently related to the rate of expansion in the $e_{(1)}$ direction, $\theta_{11}$, which is given by Eq. (4.4). It follows from this equation that if $m < 0$, the background spacetime ($w' = 0$) is contracting in the $e_{(1)}$ direction (i.e., $\theta_{11} < 0$) as $t$ increases, while if $m > 0$, there is expansion in this direction.

## VII. THE SPATIAL GEOMETRY

We finally consider the effect of the wave pulse on the geometry of the spacelike hypersurfaces which are orthogonal to the fluid flow. The primary tensor of interest is the Ricci tensor since for a metric tensor in three dimensions, the Riemann-Chrisoffel tensor is determined algebraically by the Ricci tensor. Another tensor that is associated with a three-metric is the so-called Cotton-York[17] tensor (also known as the York[18] tensor). This tensor depends on the third derivatives of the metric tensor, but nevertheless plays a role analogous to the Weyl conformal curvature tensor in four dimensions, in that its vanishing is necessary and sufficient for the three-metric to be conformally flat. We refer to Ref. 17 for definitions and more details. It has also been suggested[19] that the Cotton-York tensor may be related in some way to the presence of gravitational waves, and it is for this reason that we study its behavior in the present situation. The components of the spatial Ricci and Cotton-York tensors relative to an orthonormal frame are denoted by $R^*_{\alpha\beta}$ and $C^*_{\alpha\beta}$, with $\alpha, \beta = 1, 2, 3$. Relative to the frame (4.2) we obtain

$$R^*_{11} = -R^*_{22} = R^*_{33} = -\tfrac{1}{2} t^{-1} (w')^2 e^{-2k}\,, \qquad (7.1)$$
$$R^*_{23} = -\tfrac{1}{2} t^{-1/2} [w'' - \tfrac{1}{2}(w')^3] e^{-2k}\,,$$

$$\begin{aligned} C^*_{11} &= t^{-3/2} (w')^3 e^{-3k}\,, \\ C^*_{22} &= -2C^*_{11} + t^{-1/2}[w''' - \tfrac{5}{2}(w')^2 w'' + \tfrac{1}{2}(w')^5] e^{-3k}\,, \\ C^*_{33} &= -C^*_{11} - C^*_{22}\,, \\ C^*_{23} &= 3t^{-1} w' [w'' - \tfrac{1}{2}(w')^3] e^{-3k}\,. \end{aligned} \qquad (7.2)$$

We note that in order to calculate the $C^*_{\alpha\beta}$, we have to assume that the function $A(v)$ in Eq. (2.6) is of class $C^3$, so that $w$ is piecewise $C^3$.

It follows that in regions I and III ($w' = 0$), the spacelike hypersurfaces are flat. However, within the wave pulse, the slices have nonzero Cotton-York tensor, and hence are not conformally flat.

Since $w$ is only of class $C^1$ globally, both $R^*_{\alpha\beta}$ and $C^*_{\alpha\beta}$ are discontinuous across the null hypersurfaces which bound the wave. It follows from Eqs. (7.1), (7.2), (2.1), (2.6), and (2.7) that

$$[R^*_{23}] = -\tfrac{1}{2}[w''] t^{-2m-1/2} e^{-n}$$
$$[C^*_{22}] = -[C^*_{33}] = -[w'''] t^{-3m-1/2} e^{-3n}\,.$$

We note that the discontinuities occur only in components which are transverse to the direction of propagation of the wave pulse (i.e., the $e_{(1)}$ direction). It can be shown, using the results of Ref. 17, that this type of behavior will occur in a large class of solutions which admit a two-parameter Abelian group of isometries.

## VIII. CONCLUSION

We have established that the spacetime of Sec. II can be interpreted as a gravitational wave pulse which propagates in a spatially homogeneous spacetime. We have not, however, been able to define the amplitude and polarization of the wave pulse. In the case of pp gravitational waves, Ehlers and Kundt[4] were able to give a coordinate-independent definition of the amplitude and polarization, by using the complex Weyl tensor and the equation of geodesic deviation. Their approach, however, depended essentially on the Weyl tensor being of Petrov type $N$, and hence cannot be applied in the present case. The standard line element[4] for pp gravitational waves contains two arbitrary functions which are constant on null hypersurfaces, and the approach of Ehlers and Kundt relates these functions to the amplitude and polarization of the waves. The fact that the line element for the present solutions contains only *one* arbitrary function suggests that the amplitude and polarization must be restricted in some sense. The fact that the wave pulse *rotates* the expansion eigenframe in the two-space orthogonal to the direction of propagation implies that we cannot regard the wave as being linearly polarized. In contrast, it should be noted that other recent papers which deal with gravitational radiation in spatially inhomogeneous (but plane or cylindrically symmetric) cosmologies, e.g., Liang[20] and Centrella and Matzner,[21] assume a diagonal metric which permits only gravitational waves of one linear polarization.

To conclude, we mention some possible extensions of this work. Firstly, the present solution describes a wave pulse in a spatially homogeneous spacetime of Bianchi type I. It is also possible to construct solutions in which the wave pulse propagates in a spatially homogeneous spacetime of Bianchi type V and type $VII_h$. These solutions will be presented elsewhere.[22] It would be of interest to determine for which of the Bianchi types[1] of spatially homogeneous spacetimes this type of construction can be performed.

Secondly, Szekeres[23] and Kahn and Penrose[24] have given exact vacuum solutions which represent the collision of two pp wave pulses. Szekeres has shown that curvature singularities develop in the interaction zone, and has suggested that this behavior is due to the very high symmetry of the pp wave metrics. In view of this it would be of interest to construct a spacetime which represents a collision of two wave pulses of the type considered in this paper, so as to be able to study the geometry in the interaction zone.


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## APPENDIX

In order to calculate the Weyl tensor, we introduce a null tetrad as follows

$$l_a dx^a = \frac{1}{\sqrt{2}}(w^{(0)} - w^{(1)}),$$

$$n_a dx^a = \frac{1}{\sqrt{2}}(w^{(0)} + w^{(1)}),$$

$$m_a dx^a = -\frac{1}{\sqrt{2}}(w^{(2)} + iw^{(3)}),$$

where the $w^{(a)}$ are given following Eq. (4.2). The nonzero complex components of the Weyl tensor, relative to this null tetrad are given by

$$\begin{aligned} \psi_0 &= -\tfrac{1}{4} m t^{-2} e^{-2k}, \\ \psi_2 &= \tfrac{1}{8}[\tfrac{4}{3}(m - \tfrac{3}{8})t^{-2} + iw' t^{-3/2}]e^{-2k}, \\ \psi_4 &= [-\tfrac{1}{4} m t^{-2} + \tfrac{3}{4}(w')^2 t^{-1} \\ &\quad + i(w'' - (w')^3 - m w' t^{-1}) t^{-1/2}] e^{-2k}. \end{aligned} \tag{A1}$$

On account of Eqs. (2.6) and (2.7), the only component which is discontinuous across the null hypersurface boundaries of region II is $\psi_4$, and

$$[\psi_4] = i[w''] t^{-1/2} e^{-2k},$$

i.e., the jump in the Weyl tensor is Petrov type $N$, with the normal to the null hypersurfaces defining the repeated principal null direction.

Relative to this tetrad, the spin coefficients $\rho$, $\sigma$, which describe the expansion and shear of the wave hypersurfaces (i.e., the hypersurfaces whose null geodesic generators are defined by $l_a$), are given by

$$\rho = -\frac{1}{2\sqrt{2}} t^{-1} e^{-k},$$

$$\sigma = -\tfrac{1}{2}\rho.$$

The spin coefficients $\mu$ and $\lambda$, which describe the expansion and shear of the null congruence defined by $n_a$, are given by

$$\mu = \frac{1}{2\sqrt{2}} t^{-1} e^{-k},$$

$$\lambda = -\frac{1}{4\sqrt{2}} t^{-1}(1 + 4it^{1/2} w') e^{-k}.$$

These quantities describe the distortion of light signals which pass through the wave region in the negative $x$ direction.

In order to calculate the curvature scalar (3.1), we also need the null tetrad components of the Ricci tensor. The only nonzero components are:

$$\phi_{00} = \phi_{22} = 2\phi_{11} = \tfrac{1}{4}\alpha^2 t^{-2} e^{-2k}.$$

The formulas given in the Appendix of Ref. 25 can be used to show that the invariant $I$, as defined by Eq. (3.1), is given by

$$I = \tfrac{2}{3}[\psi_0\bar\psi_0 + \psi_4\bar\psi_4 + \psi_0\psi_4 + \bar\psi_0\bar\psi_4 + 3(\psi_2{}^2 + 2\psi_2\bar\psi_2 + \bar\psi_2{}^2)].$$

Equation (3.2) is an immediate consequence of this equation and the expressions (A1).


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