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Junction conditions for odd-parity perturbations on most general spherically symmetric space-times

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Perturbations of a general background space-time with a hypersurface of discontinuity, such as the history of a collapsing star, are considered. The junction conditions that these perturbations obey are expressed in terms of the perturbed first and second fundamental forms (intrinsic metric and extrinsic curvature) of the (perturbed) set of hypersurfaces one of which is the discontinuous one. These junction conditions are applied to the odd-parity metric and hydrodynamical asymmetries of a slightly aspherical but otherwise general and realistic spherically collapsing star. The junction conditions are stated in terms of those metric and matter perturbational objects that are the most natural, economic, and versatile: gauge-invariant geometrical objects. For odd-parity perturbations these are scalars and vectors on the totally geodesic submanifold spanned by the time and radial coordinates. The end result is simple: The junction conditions amount to the continuity of the gradient of a master gauge-invariant scalar wave function from which all other perturbational quantities can be derived.

I. INTRODUCTION

The first detector of gravitational radiation was built nearly 17 years ago,[1] and the rate at which additional detectors have been put into operation has been more or less a monotonically increasing function of time ever since. The most likely sources of the radiation that these detectors are hoped to receive are supernovalike events associated with the births of neutron stars or black holes.[2,3] Such astrophysical events are in general nonspherical. Consequently, primarily during the collapse phase gravitational radiation is emitted which is expected to be received by these detectors. The catastrophic collapse is envisioned to proceed along various scenarios.[4,5] But at the present time no precise information about the detectable power, spectral flux, polarization, etc., is known for any one of them. Indeed, the present theoretical state of the art has not even permitted a detailed consideration of the simplest astrophysically relevant source of gravitational radiation: a slightly aspherical version of the spherical supernova-type collapse first considered by Colgate and White[6] and May and White.[7] Moreover, during such a collapse neutrinos are produced in large quantities and indications are that they can play a nontrivial dissipative role.[8] Consequently gravitational radiation has a viable competitor also capable of damping out perturbational asymmetries that a supernova-type event may have. This decreases the magnitude of the signal a gravitation radiation detector would receive otherwise.

It is clear that the formulation of the coupled hydrodynamics-radiation problem for the asymmetries of relativistic collapse must be preceded by several preliminary steps. They include, among others, (1) the availability of an economic and versatile formalism together with its equations that describe these asymmetries, on an arbitrarily given spherically symmetric background, (2) the junction conditions that join the asymmetries across a surface of discontinuity (e.g., the history of the surface of a star), and (3) a formulation of the initial-value problem governing the asymmetries. The above-mentioned formalism seems to exist only for asymmetries having odd-parity angular harmonic components.[9] This paper describes the first two steps for odd-parity asymmetries that evolve in accordance with the linearized Einstein field equations for an arbitrary spherically symmetric space-time.

Cunningham, Price, and Moncrief[10] have obtained results similar to ours, but only in the context of the interesting problem of radiation from a slightly aspherically collapsing homogeneous dust cloud. The results of this paper, by contrast, are applicable to perturbations away from any spherically symmetric background.

Notation: Greek indices run over $0,1,2,3$, capital Latin indices run over $0,1$ only, lower case Latin indices run over $2,3$ only, and they refer to the angular coordinates θ and φ.

II. SUMMARY

The results can be stated as follows: Consider a spherically symmetric space-time with metric

$$g_{\mu\nu}\,dx^\mu\,dx^\nu = g_{AB}\,dx^A dx^B + r^2(x^C)(d\theta^2 + \sin^2\theta\, d\phi^2)$$

and stress-energy tensor

$$t_{\mu\nu}dx^\mu\,dx^\nu = t_{AB}\,dx^A\,dx^B + \tfrac{1}{2}t^b{}_b r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\,.$$

Consider an odd-parity perturbation of these fields (suppress angular integers l and m) expressed in terms of the vector harmonic $S_a(\theta, \phi)$ which is transverse ($S_a{}^{:a} = 0$) on the unit two-sphere:

$$h_{\mu\nu}dx^\mu\,dx^\nu = h_A(x^C)S_a(dx^A dx^a + dx^a dx^A) + h(x^C)(S_{a:b} + S_{b:a})dx^a dx^b\,,$$

$$\Delta t_{\mu\nu}dx^\mu\,dx^\nu = \Delta t_A S_a(dx^A dx^a + dx^a dx^A) + \Delta t(S_{a:b} + S_{b:a})dx^a dx^b\,.$$

Construct the corresponding gauge-invariant metric and matter perturbations:

$$k_A = h_A - r^2(r^{-2}h)_{,A} \quad \text{(metric)}\,,$$

$$T_A = \Delta t_A - \tfrac{1}{2}t^b{}_b h_A \quad \text{(matter)}\,,$$

$$T = \Delta t - \tfrac{1}{2}t^b{}_b h \quad \text{(matter)}\,.$$

It is to be noticed that, given these gauge invariants, one can reconstruct all perturbations provided one imposes a suitable gauge condition on h. For $l \geq 2$ the relevant scalar equation governing these quantities is

$$[r^{-2}(r^4\Pi)^{|C}]_{|C} - (l-1)(l+2)\Pi = -\kappa S\,,$$

where

$$\Pi = -\tfrac{1}{2}[(r^{-2}k_A)_{|B} - (r^{-2}k_B)_{|A}]\epsilon^{AB}\,,$$

$$S = -\tfrac{1}{2}T_{A|B}\epsilon^{AB}\,,$$

$$k_A = [\kappa r^2 T_A + \epsilon_{AB}(r^4\Pi)^{|B}][(l-1)(l+2)]^{-1}\,,$$

$$\kappa = 16\pi G/c^4\,,$$

and

$$\epsilon_{AB} = [AB](-\det g_{CD})^{1/2}$$

is the antisymmetric unit tensor on the two-dimensional manifold spanned by the radial and time coordinates x^C $(C = 0, 1)$.

The junction conditions across a spherically symmetric hypersurface of discontinuity Σ with unit spacelike normal n_A consist of

$$\Pi,\ k_A n^A,\ k_A(g^{AB} - n^A n^B),\ \text{and}\ T_A n^A$$

being continuous across Σ.

For $l = 1$ the relevant scalar equation governing these quantities is

$$\psi^{|C}{}_{|C} = \frac{\kappa}{r^4}\int r^2 T^A \epsilon_{AB}\, dx^B$$

where

$$r^{-2}k_A = \psi^{|C}\epsilon_{CA} + \phi_{,A}$$

and

$$(r^2T^A)_{|A} = 0\,,$$

i.e., the integrand has zero curl. The exact vector field $\phi_{,A}$ is indeterminate. The junction conditions consist of ψ, Π, and $T_A n^A$ being continuous across the spherically symmetric hypersurface Σ.

III. CONTINUOUS TENSOR FIELDS

Consider a space-time M with a metric

$$ds^2 = g_{\mu\nu}dx^\mu\,dx^\nu\,,$$

whose signature is $-{+}{+}{+}$. Consider the one-parameter family of contours of a real-valued smooth function $f(x^\mu)$ defined on M. These contours are three-dimensional submanifolds. One of them, Σ, say, the history of the surface of a collapsing star, has the property that some property of space-time, e.g., the stress-energy tensor, is permitted to be discontinuous in such a way that the intrinsic geometry as well as the extrinsic curvature of these submanifolds are continuous across Σ. This condition is satisfied if the stress-energy tensor has no δ-function type of singularity across Σ.[11]

Consider the vector field of (spacelike) unit normals

$$n_\mu = (f_{,\alpha}\,f_{,\beta}\,g^{\alpha\beta})^{-1/2}f_{,\mu} \tag{3.1}$$

associated with the contours of the scalar f. Next consider a tensor field $t_{\mu\ldots\nu}$ on M. The restriction of this field to a given contour is said to *lie* on the contour, or, more briefly, $t_{\mu\ldots\nu}$ is *intrinsic* to the given contour of f if

$$t_{\mu\ldots\nu}n^\mu = \cdots = t_{\mu\ldots\nu}n^\nu = 0\,. \tag{3.2}$$

Thus one may consider the metric intrinsic to a given contour of f,

$$i_{\mu\nu} \equiv g_{\mu\nu} - n_\mu n_\nu, \quad i^{\mu\nu} \equiv g^{\mu\nu} - n^\mu n^\nu\,. \tag{3.3}$$

Indices of intrinsic tensors can be raised and lowered with $i_{\mu\nu}$ or with $g_{\mu\nu}$. Furthermore, $i_{\mu\nu}$ projects vectors and tensors onto the contours of f. Another tensor field which is intrinsic to the contours of f is the (symmetric) extrinsic curvature tensor field[12]

$$\begin{aligned} e_{\mu\nu} &\equiv n_{\alpha;\beta}\,i_\mu{}^\alpha i_\nu{}^\beta = n_{\mu;\nu} - n_{\mu;\sigma}n^\sigma n_\nu \\ &= f_{,\mu}g_{,\nu} + f_{,\nu}g_{,\mu} + g f_{,\mu;\nu} - g^2 f_{,\sigma}g_{,\rho}g^{\sigma\rho}f_{,\mu}f_{,\nu}\,, \end{aligned} \tag{3.4}$$

where

$$g = (f_{,\alpha}f_{,\beta}g^{\alpha\beta})^{-1/2}\,.$$

Thus, given the smooth function f, we assume the corresponding tensor fields $i_{\mu\nu}$ and $e_{\mu\nu}$ to be continuous tensor fields on M.

Similarly, consider space-time M with perturbed scalar $\tilde{f}$ and perturbed metric $\tilde{g}_{\mu\nu}$. The corresponding intrinsic metric $\tilde{i}_{\mu\nu}$ and extrinsic curvature tensor field $\tilde{e}_{\mu\nu}$ are also assumed to be continuous on M. The perturbations in the scalar

$$\Delta f = \tilde{f}(x^\sigma) - f(x^\sigma)$$

are continuous. The perturbations

$$\Delta i_{\mu\nu} = \tilde{i}_{\mu\nu}(x^\sigma) - i_{\mu\nu}(x^\sigma),$$

$$\Delta e_{\mu\nu} = \tilde{e}_{\mu\nu}(x^\sigma) - e_{\mu\nu}(x^\sigma)$$

are also continuous, but not intrinsic to the same contour f: The normals $\tilde{n}_\mu$ and n_μ do not coincide in general.

Recall, however, the specific feature that makes junction conditions across a particular contour of f, say Σ, so useful. This feature is the fact that the coordinate system used on one side of Σ need not be joined continuously onto the coordinate system used on the other side. This feature of relative arbitrariness in the coordinate systems on either side of Σ necessitates expressing the junction conditions in terms that are totally intrinsic to Σ. One thereby deals only with geometrical objects that lie strictly on the boundary common to two adjoining coordinate neighborhoods.

"Usefulness" is, however, not a sufficient criterion for using intrinsically defined data as material to be joined across Σ. The ultimate justification lies in the linearized Einstein field equations being expressed precisely in terms of the above-mentioned and the to-be-below-exhibited [Eqs. (3.5) and (3.6)] intrinsic data on some one-parameter family of 3-surfaces. Expressing these field equations in terms of intrinsic quantities amounts to what is known as a "three-plus-one" formulation.[13] This has been done very elegantly for the unperturbed field equations in terms of the first and second fundamental forms.[14] In this paper we shall not do this for the perturbed field equations.

The intrinsically defined junction conditions are expressed in terms of the equality of the projections of tensors from each side of Σ onto Σ itself. It follows that the perturbed tensor fields continuous across a given contour of f are the projections of $\Delta i_{\mu\nu}$ and $\Delta e_{\mu\nu}$, namely

$$i_\mu{}^\alpha i_\nu{}^\beta \Delta i_{\alpha\beta}, \tag{3.5}$$

$$i_\mu{}^\alpha i_\nu{}^\beta \Delta e_{\alpha\beta}. \tag{3.6}$$

IV. REDUCTION OF TENSOR FIELDS ON A SPHERICALLY SYMMETRIC SPACE-TIME

A general spherically symmetric background space-time has a metric of the form

$$g_{\mu\nu}dx^\mu dx^\nu = g_{AB}dx^A dx^B + r^2(x^C)(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{4.1}$$

Latin capitals A, B, C,... refer to some as-yet-unspecified radial and time coordinates. The functions $r(x^C)$ and $g_{AB}(x^C)$ are scalar and tensor fields on the totally geodesic two-dimensional space-time M^2 spanned by x^C $(C=0,1)$. The vector field

$$v_A = r_{,A}/r \tag{4.2}$$

is also on this submanifold. The scalar function f is independent of the angular variables

$$f(x^\mu) = f(x^C). \tag{4.3}$$

The intrinsic metric Eq. (3.3) and the extrinsic curvature Eq. (3.4) have the form

$$i_{AB} = g_{AB} - n_A n_B, \tag{4.4a}$$

$$i_{Aa} = 0,$$

$$i_{ab} = g_{ab} = r^2\gamma_{ab}, \tag{4.4b}$$

$$e_{AB} = (n_{A|B} - n_{A|C}n^C n_B) = n_C{}^{|C} i_{AB}, \tag{4.5a}$$

$$e_{Aa} = 0,$$

$$e_{ab} = \tfrac{1}{2}e^m{}_m g_{ab} = n_C v^C g_{ab}. \tag{4.5b}$$

Latin lower cases $a, b, \ldots, m$ refer to the angular coordinates θ and ϕ. The symbol γ_{ab} is the metric on the unit two-sphere. The vertical bar in Eq. (4.5a) refers to the covariant derivative with respect to g_{AB} on M^2. The second equality in Eq. (4.5a)

$$e_{AB} = e^C{}_C i_{AB}$$

is true for any symmetric tensor with the property $e_{AB}n^B = 0$ on M^2. The partial trace $e^m{}_m = e^2{}_2 + e^3{}_3 = 2v_C n^C$ in Eq. (4.5b) is a scalar on M^2.

It is clear that $f(x^C)$, i_{AB}, e_{AB}, $n_C{}^{|C}$, $n_C v^C$, and $r(x^C)$ are geometrical objects on M^2, and that their restrictions to the contours of $f(x^C)$ are intrinsic to and continuous across those contours.

Another geometrical object on M^2 is the antisymmetric unit tensor

$$\epsilon_{AB} = [AB](-\det g_{CD})^{1/2}.$$

It gives the integration differential

$$\tfrac{1}{2}\epsilon_{AB}dx^A dx^B = |g|^{1/2}dx^0 dx^1$$

for M^2. On the other hand, the vector field

$$\epsilon_{AC}n^C,$$

intrinsic to a given contour of $f(x^C)$, gives the integration differential $\epsilon_{AC}n^C dx^A$ for the contour, say Σ.

V. ODD-PARITY PERTURBATION JUNCTION CONDITIONS ON A SPHERICALLY SYMMETRIC SPACE-TIME

Perturbations away from the background metric Eq. (4.1) have "odd" harmonic components of parity $(-1)^{l+1}$. They have the form (suppress angular integers l and m)

$$h_{\mu\nu}dx^\mu dx^\nu = h_A(x^C)S_a(\theta,\phi)(dx^A dx^a + dx^a dx^A) + h(x^C)(S_{a:b}+S_{b:a})dx^a dx^b \,. \tag{5.1}$$

The covariant derivative of the transverse ($S_a{}^{:a}=0$) vector harmonic S_a on the unit two-sphere is indicated by a colon. Lower case Latin indices $a, b, \ldots$ refer to θ and ϕ. It is clear that the three expansion coefficients h_0, h_1, and h can be assembled into a covector and a scalar field

$$h_A dx^A \text{ and } h$$

on M^2. Similarly, odd-parity perturbations away from the intrinsic metric Eqs. (4.4) are given by

$$\Delta i_{\mu\nu}dx^\mu dx^\nu = h_A S_a(dx^A dx^a + dx^a dx^A) + h(S_{a:b}+S_{b:a})dx^a dx^b \,. \tag{5.2}$$

This follows from the fact that perturbation away from the unit normal $n_\mu dx^\mu$,

$$(\Delta n_\mu)dx^\mu = (\Delta g f_{,\mu} + g\Delta f_{,\mu})dx^\mu \,,$$

where $g=(f_{,\alpha}f_{,\beta}g^{\alpha\beta})^{-1/2}$, depend only on the scalar perturbations Δg and Δf. These are zero for odd-parity perturbations. For odd-parity perturbations one has therefore

$$(\Delta n_\mu)dx^\mu = 0\,, \tag{5.3}$$

and $\Delta i_{\mu\nu}$, Eq. (5.2), receives no contribution from Δn_μ.

Odd-parity perturbations away from the extrinsic curvature Eq. (4.5) have the form

$$(\Delta e_{\mu\nu})dx^\mu dx^\nu = \Delta e_A S_a(dx^A dx^a + dx^a dx^A) + \Delta e(S_{a:b}+S_{b:a})dx^a dx^b \,, \tag{5.4a}$$

where

$$\Delta e_A = \tfrac{1}{2}(h_{A|C} - h_{C|A} + 2h_C v_A)n^C \,, \tag{5.4b}$$

$$\Delta e = -\tfrac{1}{2}(h_C - h_{,C})n^C \,. \tag{5.4c}$$

This is because, with the help of Eq. (5.3),

$$\Delta e_{\mu\nu} = \Delta(n_{\mu;\nu}) - \Delta(n_{\mu;\sigma})n^\sigma n_\nu \,.$$

Its components can be evaluated with the aid of the expressions given in the Appendix.

The perturbed quantities that are continuous across, say, the surface of a collapsing star, are expressions (3.5) and (3.6). With the help of Eq. (5.2) these are

$$i_A{}^C h_C \text{ and } h\,, \tag{5.5}$$

and with the help of Eqs. (5.4b) and (5.4c)

$$i^C{}_A \Delta e_C \text{ and } \Delta e\,. \tag{5.6}$$

Although these expressions are continuous, they can be altered readily by performing arbitrary infinitesimal coordinate changes on the background space-time. The resultant "gauge" ambiguity in these quantities at best is an inconvenience and at worst obscures their important aspects. It is therefore desirable, if not mandatory, to reexpress all quantities in terms of gauge-invariant geometrical objects on M^2.

VI. ODD-PARITY GAUGE-INVARIANT JUNCTION CONDITIONS

We wish to construct from the perturbations Eqs. (5.2) and (5.4) geometrical objects on M^2 that are independent of gauge transformations. To this end consider any symmetric tensor field, say

$$e_{\mu\nu}dx^\mu dx^\nu = e_{AB}dx^A dx^B + \tfrac{1}{2}e^C{}_C g_{ab}dx^a dx^b$$

on a spherically symmetric space-time. An infinitesimal odd-parity coordinate perturbation

$$\xi_\mu dx^\mu = \xi(x^C)S_a dx^a$$

in the spherically symmetric background M induces a change in $e_{\mu\nu}dx^\mu dx^\nu$ and in $g_{\mu\nu}dx^\mu dx^\nu$ which is the Lie derivative with respect to ξ_μ:

$$\begin{aligned}(\pounds_\xi e_{\mu\nu})dx^\mu dx^\nu &= (e_{\mu\nu;\sigma}\xi^\sigma - e_{\sigma\nu}\xi^\sigma{}_{;\mu} - e_{\mu\sigma}\xi^\sigma{}_{;\nu})dx^\mu dx^\nu \\ &= -\tfrac{1}{2}e^C{}_C r^2(\xi/r^2)_{,A}S_a(dx^A dx^a + dx^a dx^A) \\ &\quad -\tfrac{1}{2}e^C{}_C\xi(S_{a;b}+S_{b:a})dx^a dx^b\,,\\ (\pounds_\xi g_{\mu\nu})dx^\mu dx^\nu &= -(\xi_{\mu;\nu}+\xi_{\nu;\mu})dx^\mu dx^\nu \\ &= -r^2(\xi/r^2)_{,A}S_a(dx^A dx^a + dx^a dx^A) \\ &\quad -\xi(S_{a:b}+S_{b:a})dx^a dx^b\,.\end{aligned}$$

It follows that the gauge-transformed odd-parity perturbations $\bar h_{\mu\nu} = h_{\mu\nu} + \pounds_\xi g_{\mu\nu}$ and $\Delta\bar e_{\mu\nu} = \Delta e_{\mu\nu} + \pounds_\xi e_{\mu\nu}$ are

$$\bar h_A = h_A - r^2(\xi/r^2)_{,A}\,,\quad \bar h = h - \xi \tag{6.1a}$$

$$\Delta\bar e_A = \Delta e_A - \tfrac{1}{2}e^C{}_C r^2(\xi/r^2)_{,A}\,,\ \Delta\bar e = \Delta e - \tfrac{1}{2}e^C{}_C\xi\,. \tag{6.1b}$$

The gauge-invariant geometrical objects on M^2 are constructed by taking those linear combinations of Eq. (6.1a) and (6.1b) which are independent of the gauge function ξ. The resultant gauge invariants evidently are

$$k_A \equiv h_A - r^2(h/r^2)_{,A}\,, \tag{6.2a}$$

$$E_A \equiv \Delta e_A - \tfrac{1}{2} e^a{}_a h_A$$
$$= \tfrac{1}{2}[(h_A/r^2)_{|C} - (h_C/r^2)_{|A}]n^C , \qquad (6.2b)$$
$$E \equiv \Delta e - \tfrac{1}{2} e^a{}_a h = -\tfrac{1}{2} k_C n^C . \qquad (6.2c)$$

Odd-parity geometrical objects that are continuous across the surface of a collapsing star are linear combinations of those in Eq. (5.5) and (5.6). The gauge-invariant Eq. (6.2c) has precisely this property already. Projecting Eqs. (6.2a) and (6.2b) with $i_A{}^C$ onto the history of the collapsing star's surface Σ, one obtains the remaining continuous gauge invariants,

$$i_A{}^C k_C \qquad (6.3a)$$

and

$$i_A{}^C E_C = -\tfrac{1}{2}[(r^{-2}k_A)_{|C} - (r^{-2}k_C)_{|A}]n^C = E_A . \qquad (6.3b)$$

Their continuity follows from Eqs. (5.5), (5.6), and the continuity of the intrinsic derivative $(h/r^2)_{,C} i^C{}_A$ of the continuous scalar h/r^2. Equations (6.3a), (6.3b), together with (6.2c), constitute the odd-parity gauge-invariant perturbation objects that are continuous across Σ. These continuity conditions can be summarized loosely by saying that k_A as well as the normal component of its curl and continuous across Σ.

VII. APPLICATION TO ODD-PARITY LINEARIZED FIELD EQUATIONS ON SPHERICAL SPACE-TIME

The odd-parity junction conditions which consist of the continuity of Eqs. (6.3a), (6.3b), and (6.2c) can be applied directly to solutions of the linearized field equations[9]

$$k^C{}_{|C} = \kappa T, \quad (2 \leq l) \qquad (7.1)$$
$$-[r^4(r^{-2}k_A)_{|C} - r^4(r^{-2}k_C)_{|A}]^{|C} + (l-1)(l+2)k_A = \kappa r^2 T_A \quad (1 \leq l) . \qquad (7.2)$$

Here $\kappa = 16\pi G/c^4$, and T and T_A are the two gauge invariants constructed from the odd-parity perturbation in the stress-energy tensor $t_{\mu\nu}dx^\mu dx^\nu$,

$$\Delta t_{\mu\nu}dx^\mu dx^\nu = \Delta t_A S_a (dx^A dx^a + dx^a dx^A) + \Delta t (S_{a:b} + S_{b:a}) dx^a dx^b ,$$

in a way identical to that given by Eqs. (6.2b) and (6.2c). The result is

$$T_A = \Delta t_A - \tfrac{1}{2} t^a{}_a h_A ,$$
$$T = \Delta t - \tfrac{1}{2} t^a{}_a h .$$

Solutions to the odd-parity equations are obtained from a master equation in a certain scalar Π. Indeed, any antisymmetric tensor F_{AB} in two dimension can be expressed in terms of a scalar and the antisymmetric unit tensor ϵ_{AB} ,

$$F_{AB} = \Pi \epsilon_{AB} , \qquad (7.3)$$

where

$$\Pi = -\frac{1}{2!} F_{CD} \epsilon^{CD} .$$

Let

$$F_{AB} = (r^{-2}k_A)_{|B} - (r^{-2}k_B)_{|A} ,$$

take the curl of Eq. (7.2), use Eq. (7.3), and obtain

$$2\{r^{-2}(r^4 \Pi \epsilon_{C[A})^{|C}\}_{|B]} + (l-1)(l+2)\Pi \epsilon_{AB} = \kappa S \epsilon_{AB} ,$$

where

$$S = -\frac{1}{2!} T_{A|C} \epsilon^{AC} .$$

Now use $\epsilon_{CA|B} = 0$, multiply by ϵ^{AB}, and use $\epsilon_{AB}\epsilon^{AB} = -2$. Thus obtain

$$-\frac{2}{2!}\{r^{-2}(r^4\Pi)^{|C}\}_{|B} \epsilon_{CA}\epsilon^{AB} + (l-1)(l+2)\Pi = \kappa S ,$$

which reduces to the master equation

$$-\{r^{-2}(r^4\Pi)^{|C}\}_{|C} + (l-1)(l+2)\Pi = \kappa S . \qquad (7.4)$$

The scalar function

$$\Pi = -\tfrac{1}{2}[(r^{-2}k_A)_{|B} - (r^{-2}k_B)_{|A}]\epsilon^{AB} \qquad (7.5)$$

is continuous across Σ. Indeed, with the help of Eq. (7.3), Eq. (6.3b) becomes

$$E_A = -\tfrac{1}{2}\Pi \epsilon_{AC} n^C .$$

The tensor $\epsilon_{AC}n^C$ is intrinsic to Σ and hence is continuous across Σ. Since E_A is continuous, the scalar Π is continuous.

It may be remarked that in an empty Schwarzschild background the scalar Π is not the same as the quantity Q originally introduced by Regge and Wheeler.[15] Instead, $Q = k_0$, the time component of k_A. It is the fact that the Schwarzschild time coordinate is a Killing orbit which cooperated in letting them write down their odd-parity wave equation for this quantity.

For $l \geq 2$ the gauge-invariant object k_A is given by Eq. (7.2),

$$k_A = [\kappa r^2 T_A + \epsilon_{AC}(r^4\Pi)^{|C}][(l-1)(l+2)]^{-1} ,$$

whose projections onto Σ, Eq. (6.3a), as well as perpendicular to Σ, Eq. (6.2c), and also continuous. Thus

$$(l-1)(l+2)k_A \epsilon^{AC} n_C = (r^4\Pi)^{|C} + \kappa r^2 T_A \epsilon^{AC} n_C , \qquad (7.6a)$$

as well as

$$\kappa r^2 T_A n^A \qquad (7.6b)$$

are continuous. The latter follows from the continuity of $k_A n^A$, Eq. (6.2c), and of

$$(r^4\Pi)_{|C} i^{CA}\epsilon_{AB} n^B = (r^4\Pi)^{|C}\epsilon_{CB} n^B .$$

For $l=1$ multiply Eq. (7.2) by the antisymmetric tensor ϵ^{AB}, use Eq. (7.5), and obtain

$$(r^4\Pi)^{|B} = \tfrac{1}{2}\kappa r^2 T_A \epsilon^{AB} .$$

Evidently the right-hand side to be exact must have zero curl [i.e., $(r^2 T_A)^{|A}=0$]. This fact is confirmed by the odd-parity linearized conservation equation[9]

$$(r^2 T_A)^{|A} = (l-1)(l+2)T .$$

Thus for $l=1$ the continuous scalar Π is

$$\tfrac{1}{2}[(r^{-2}k_A)_{|B} - (r^{-2}k_B)_{|A}]\epsilon^{AB} = \Pi = \frac{1}{2}\frac{\kappa}{r^4}\int r^2 T^A \epsilon_{AB}\, dx^B .$$

For this case use Helmholz's theorem on M^2 and let

$$r^{-2}k_A = \psi^{|C}\epsilon_{CA} + \phi_{,A} , \qquad (7.7)$$

and obtain the scalar equation

$$\psi^{|C}{}_{|C} = \frac{\kappa}{r^4}\int r^2 T^A \epsilon_{AB}\, dx^B . \qquad (7.8)$$

The complementary solution ($\psi_{\text{comp}}{}^{|C}{}_{|C}=0$) is not determined by the source, but instead is determined by boundary conditions. Loosely speaking, a particular solution of Eq. (7.8) is associated with rotating matter, whereas ψ_{comp} may be nonzero even if ostensible spinning matter is absent. The scalar ϕ is indeterminate. This is because for $l=1$ the scalar h, Eq. (5.1), is indeterminate. Thus k_A is not gauge invariant, although Π, the (dual of the) curl of $r^{-2}k_A$ still is.

VIII. CONCLUSION

The problem of odd-parity perturbation as described by the linearized Einstein field equations on an arbitrary spherically symmetric space-time has been reduced to the simpler problem of solving a scalar wave equation Eq. (7.4) for each angular mode, on the totally geodesic submanifold M^2. The continuity properties of the scalar Π are precisely those that one expects for such an equation; namely, Π and its gradient are continuous on M^2, even if M^2 has finite discontinuities in the background stress energy defined on it.

APPENDIX

Here we list the derivatives $h_{\mu\nu;\sigma}$ of a given perturbation mode l,m for odd parity, Eq. (5.1). The reduced form of the nonzero Christoffel symbols that is the basis of our perturbational formalism is

$$\Gamma_{\mu\nu}{}^{\sigma}:\ \Gamma_{AB}{}^{C},\quad \Gamma_{Aa}{}^{b} = v_A\delta^b{}_a,\quad \Gamma_{ab}{}^{A} = -v^A g_{ab} = -v^A r^2\gamma_{ab} .$$

Here $\Gamma_{AB}{}^{C}$ and $v_A = r_{,A/r}$ are defined on the totally geodesic submanifold M^2 whose metric is

$$g_{AB}dx^A dx^B ,\quad (A,B=0,1) .$$

The derivatives $h_{\mu\nu;\sigma}$ themselves have the following form for odd parity:

$$h_{AB;C} = 0 ,$$

$$h_{Aa;B} = (h_{A|B} - h_A v_B)S_a ,$$

$$h_{AB;a} = -(h_A v_B + h_B v_A)S_a ,$$

$$h_{Aa;b} = h_A S_{a:b} - v_A h(S_{a:b} + S_{b:a}) ,$$

$$h_{ab;A} = (h_{,A} - 2v_A h)(S_{a:b} + S_{b:a}) ,$$

$$h_{ab;c} = v^C h_C(S_a g_{bc} + S_b g_{ac}) + h(S_{a:b:c} + S_{b:a:c}) .$$

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[4] See, for example, R. Ruffini and J. A. Wheeler, in *Black Holes*, edited by C. DeWitt and B. S. DeWitt (Gordon and Breach, New York, 1973).
[5] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973), Chap. 35.
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[11] See, for example, Chap. 21 in Ref. 5.
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[13] See, for example, Chap. 21 of Ref. 5.
[14] R. P. Geroch, J. Math. Phys. 13, 956 (1972), Appendix A, can easily be adapted to hypersurfaces whose normals are spacelike.
[15] T. Regge and J. A. Wheeler, Phys. Rev. 108, 1063 (1957).