Axially symmetric two-body problem in general relativity. III. Bondi mass loss and the failure of the quadrupole formula

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Suggested difficulties and criticism regarding earlier work is addressed. It is demonstrated that the rate of gravitational energy loss from the authors' model free-fall system employing the widely accepted Bondi method agrees precisely with the results described in prior works. Origins of the breakdown of the quadrupole formalism for free fall, previously indicated, are now delineated in detail. The role of source structure in the energy loss rate re-emerges, bringing into question much of the earlier work of others. The iterative technique with flat-space wave operators is justified. A new approach to quasiperiodic systems such as binary stars is described. Ideally modeled upon the actual birth of such systems, it evolves from an initially stationary configuration, again avoiding the problems and ambiguities regarding incoming radiation.

I. INTRODUCTION

Gravitational radiation has been and continues to be a subject of intense interest in general relativity. Its existence was predicted by Einstein[1] more than sixty years ago and since then hundreds of papers have been written about it. In spite of a world-wide effort, it has thus far eluded direct detection. However, new enthusiasm has been generated by the recent observations[2] of the binary pulsar PSR 1913+16 which, it is claimed, point to an indirect confirmation of the emission of gravitational radiation. Furthermore, it is suggested that basically the observations validate the Einstein quadrupole energy loss rate[1,3]

$$\dot{E} = -\frac{G}{45c^5}(\dddot{D}^{\alpha\beta})^2 \tag{1.1}$$

for such freely falling systems. This "quadrupole formula," which is a direct analog of the electromagnetic expression, has also had an interesting history. Based on the linearized version of the Einstein field equations, it correctly describes the lowest-order[4] emission from small weakly radiating slow-motion systems which are driven by nongravitational forces. However, for systems with motion deriving from gravitation itself, such as binary stars, the nonlinearities inherent in the field equations must be contended with and the quadrupole formula is case into doubt. Through the years, this has been emphasized by Eddington,[5] Bonnor,[6] Papapetrou,[7] and others. Numerous papers have been written, usually in conjunction with singular sources, in an effort to address the problem, and the authors have contradicted each other and at times themselves as well.[8] A major source of difficulty in these works stemmed from their uncertain avoidance of incoming radiation, the presence of which would clearly vitiate any definitive conclusion regarding the actual rate of energy loss from an insular system.

Five years ago, a new approach to the problem of gravitational radiation from freely falling bodies was initiated.[9] In this work (henceforth to be referred to as I) we considered a static axially symmetric Weyl–Levi-Civita type of system composed of two bodies held in equilibrium by a strut. The strut was severed and the induced dynamic perturbation of the metric was analyzed. By virtue of the static history, there was no ambiguity regarding the development of the field and no possibilities for incoming radiation.

The beautiful consistency of Einstein's theory manifested itself by yielding a continuous metric only when the breaking of the intervening stress was properly synthesized with the induced motion of the masses. Using the field expanded to second order in the gravitational constant G, linearized theory and the quadrupole formula were shown to be valid during the stress-breaking period. However, when the stress was completely broken and free fall began, it was demonstrated that the third-order field was required to deduce the radiation. Moreover, it was emphasized that one could no longer proceed under the artificial assumption of singular sources as a representation of the bodies. Indeed, an order-of-magnitude calculation led to the prediction of a potentially structure-dependent contribution to the free-fall energy loss rate which could be considerably more significant than that which would be deduced by a naive application of the quadrupole formula.

Two years later in a subsequent paper[10] (henceforth to be referred to as II), a nonsingular source was constructed, and the dominant contributions to the dynamic third-order metric were derived.

As anticipated in I, the nonlinearities and structure played a vital role, and the energy loss rate was found to exceed that of a straightforward application of the quadrupole formula by a factor of $(\alpha/\rho_0)^4$ where 2α is the distance between the sources and ρ_0 is their linear dimension. This calculation was carried through in all detail with every term accounted for because one could not exclude the possibility, however unlikely, that these large contributions would miraculously conspire to cancel, leaving the quadrupole formula in the end.

Since the publication of I, and particularly since II, several colleagues have offered an interesting variety of criticism and suggestions for which we are very grateful. The essential aim of this paper is to address in some detail the most frequently voiced criticisms of our work and clarify certain lesser problems.

The first criticism stems from our use of the much-maligned energy-momentum pseudotensor (or "complex" in the currently fashionable parlance) in deducing the energy loss rate. Ever since its introduction by Einstein, the pseudotensor has been roundly criticized, primarily because of its noncovariant character. Many researchers in the field have used it and Bonnor[11] has presented a justification of its use. However, a measure of energy loss which is of a considerably less controversial nature has been developed by Bondi.[12] Bondi's "news function" relates directly to the mass loss of the system and this is calculated in Sec. II for our problem. The answer for the energy loss rate is precisely the same as that which was found in II.

The second criticism derives from the confidence which many relativists have in the quadrupole formalism.[1,3] There is a tendency to trust its application to any weak-field, slow-motion situation. However, to condition the reader to the fact that this trust is ill-founded, we can easily point to interfering composite radiating systems spaced on the order of a characteristic wavelength. In spite of the fact that the individual radiating components are much smaller than the characteristic wavelength, the quadrupole formalism breaks down for such systems due to their interaction. Indeed such a model of two coaxial rotating rods was constructed,[13] and this was implemented by Braginskii[14] in his so-called heterodyne detector model for gravitational radiation. Further development and applications of this work followed,[15] and recently, this was nicely formalized in a somewhat different manner by Press.[16]

In spite of warnings dating back to the time of Eddington,[5] many have not recognized any limitations on the validity of the quadrupole formula to systems which are driven by nongravitational forces alone. Although we did state the reasons for its limitation in II, we did recognize then and now that a more detailed exposition is required to by fully convincing. This is the aim of Sec. III.

In Sec. IV, we address some other criticisms and recent work, and in Sec. V, we conclude with a summary and discussion of future avenues of research.

II. BONDI MASS LOSS

By a clever choice of coordinates, in particular by employing a retarded time coordinate u which is naturally matched to a system with outgoing radiation, Bondi[12] was able to relate the mass loss of an axially symmetric system to an angular integral of a squared "news function" which derives from his metric

$$ds^2 = \left(\frac{V}{R}e^{2\beta} - U^2R^2e^{2\gamma}\right)du^2 + 2e^{2\beta}du\,dR + 2UR^2e^{2\gamma}du\,d\theta - R^2(e^{2\gamma}d\theta^2 + e^{-2\gamma}\sin^2\theta\, d\phi^2)\,. \tag{2.1}$$

To ensure the outgoing-radiation condition, Bondi expanded his variables in the form of a power series in negative powers of R with coefficients which are functions of the retarded time. In particular,

$$\gamma = \frac{{}^1\gamma(u,\theta)}{R} + \frac{{}^3\gamma(u,\theta) - \frac{1}{6}[{}^1\gamma(u,\theta)]^3}{R^3} + \cdots\,, \tag{2.2}$$

where the R^{-2} coefficient had to vanish in order that the other functions satisfy the outward-radiation condition. By a transformation to the Weyl metric in the static case, Bondi identified the combination of functions which determine the mass and hence finally the mass loss during a radiative phase

$$m_0 = -\frac{1}{2G}\int_0^\pi ({}^1\gamma_0)^2 \sin\theta\, d\theta\,, \tag{2.3}$$

where the subscript 0 denotes $\partial/\partial u$ and ${}^1\gamma_0$ is called the news function.

To calculate the Bondi mass loss, in the context of the metric in I and II, in cylindrical polar coordinates

$$ds^2 = Ddt^2 - Adr^2 - Bdz^2 - Cd\phi^2\,, \tag{2.4}$$

we note that the transformation to the Bondi form [Eq. (2.1)] does not involve the ϕ coordinate and hence

$$C = r^2e^{-2\nu + c + c' + \cdots} = R^2\sin^2\theta e^{-2\gamma} \tag{2.5}$$

in the vacuum where c, the dynamic G^2 order perturbation, becomes static to order R^{-1} during freefall[9] and c' is the $O(G^3)$ perturbation which determines the lowest-order energy loss in freefall.

In the manner of Bondi,[12] the coordinate transformations are expanded in powers of R (Ref. 17):

$$r=R\sin\theta+{}^0f(u,\theta)+\frac{{}^1f(u,\theta)}{R}+\frac{{}^2f(u,\theta)}{R^2}+\cdots,$$
$$z=R\cos\theta+{}^0g(u,\theta)+\frac{{}^1g(u,\theta)}{R}+\frac{{}^2g(u,\theta)}{R^2}+\cdots. \tag{2.6}$$

The background function ν is given by[9]

$$\nu=-Gm\left(\frac{1}{r_1}+\frac{1}{r_2}\right), \tag{2.7}$$

where

$$(r_{1,2})^2=r^2+(z\pm\alpha)^2, \tag{2.8}$$

and hence

$$\nu=\frac{-2Gm}{R}+\frac{2Gm}{R^2}({}^0f\sin\theta+{}^0g\cos\theta)+O(R^{-3}). \tag{2.9}$$

Since the dipole moment of this system vanishes for all times,

$${}^0f={}^0g=0. \tag{2.10}$$

From Eqs. (2.5) and (2.2),

$$r[1-\nu+\tfrac{1}{2}(c+\nu^2)+\tfrac{1}{2}(-\nu c+c'-\tfrac{1}{3}\nu^3)+\cdots]=R\sin\theta\left(1-\frac{{}^1\gamma}{R}+\frac{{}^1\gamma^2}{2R^2}-\frac{{}^3\gamma}{R^3}+\cdots\right). \tag{2.11}$$

Equation (2.11) is differentiated with respect to u. From Eqs. (2.6)–(2.10), r_0 is $O(R^{-1})$ and ν_0 is $O(R^{-3})$. Moreover, since asymptotically[9,10]

$$c=\frac{{}^1c(\theta)}{R}+\frac{{}^2c(u,\theta)}{R^2}+\cdots,$$
$$c'=\frac{{}^1c'(u,\theta)}{R}+\cdots, \tag{2.12}$$

it follows from Eq. (2.11) that

$$\frac{c_0'}{2}R\sin\theta=-{}^1\gamma_0\sin\theta+\cdots. \tag{2.13}$$

From Eqs. (5.5) and (5.36) of II,

$$c_0'=\frac{8G^3m^3}{\alpha^2\rho_0{}^2}\frac{(t-R)}{R} \tag{2.14}$$

asymptotically during freefall and hence the news function is

$${}^1\gamma_0=\frac{-4G^3m^3u}{\alpha^2\rho_0{}^2}. \tag{2.15}$$

From Eqs. (2.3) and (2.15), the Bondi mass loss is

$$m_0=-\frac{16G^5m^6u^2}{\alpha^4\rho_0{}^4}, \tag{2.16}$$

which is precisely the result as obtained in II.

It is to be emphasized that we have found the complete news function from *nonlinear* as well as linear contributions to lowest contributing order in G and hence our approach is not subject to some of the early criticism of potentially incomplete applications of the Bondi method.[6]

III. FAILURE OF THE QUADRUPOLE FORMULA

In the usual quadrupole formalism,[3] one takes the metric form

$$g_{ik}=\eta_{ik}+h_{ik} \tag{3.1}$$

with coordinate conditions

$$\psi^k_{i,k}=0,\quad \psi^k_i\equiv h^k_i-\tfrac{1}{2}\delta^k_i h^j_j \tag{3.2}$$

to express the field equations as inhomogeneous wave equations[18]

$$\Box\psi^k_i=16\pi G\tau^k_i, \tag{3.3}$$

where the effective source τ^k_i is the usual energy-momentum tensor T^k_i plus nonlinear terms t^k_i. The retarded potentials are chosen,

$$\psi^k_i=-4G\int(\tau^k_i)_{t-R/c}\frac{dV}{R}, \tag{3.4}$$

and the assumption is made that for asymptotic field points, the $1/R$ can be set equal to $1/R_0$, the inverse of the origin to field point distance, and the retardation $t-R/c$ be replaced by $t-R_0/c$ for slow motions as in electrodynamics to yield

$$\psi^k_i=-\frac{4G}{R_0}\int(\tau^k_i)_{t-R_0/c}\,dV. \tag{3.5}$$

Rather than determine the field by Eq. (3.5), the coordinate conditions [Eqs. (3.2)] are employed in conjunction with Eq. (3.3) to give

$$\tau^k_{i,k}=0. \tag{3.6}$$

Repeated integrations with the Gauss theorem and the elimination of surface terms finally yields[3]

$$\psi_{\alpha\beta}=-\frac{2G}{R_0}(\ddot{d}_{\alpha\beta})_{\rm ret} \tag{3.7}$$

for the spatial components where

$$d_{\alpha\beta}=\int\epsilon x^\alpha x^\beta dV \tag{3.8}$$

is the untraced mass quadrupole tensor. Integrating the flux finally gives the quadrupole formula [Eq. (1.1)] where

$$D_{\alpha\beta}=3d_{\alpha\beta}-\delta_{\alpha\beta}d_{\gamma\gamma}. \tag{3.9}$$

There are several reasons for the breakdown of this formalism during freefall. First, the nonlinear contributions to τ^k_i are field sources which are noncompact and hence the R^{-1} cannot be casually replaced by R_0^{-1} and removed from the integral. Moreover, in a nonlinear theory, it would be illogical to relegate such contributions to propagation phenomena and pretend that they are somehow divorced from the wave generation itself. The logical approach is to allow the field equations to determine the nonlinear matter-field coupling and produce the net, asymptotic order-R^{-1} field in successive orders in G and hence deduce the energy loss as in Sec. II.

In our earlier work, it was demonstrated that there are numerous contributions which are more significant than those leading to the quadrupole formula. However, to demonstrate the incompleteness of the traditional method, we shall pursue a different route in this paper. Assuming the mantle of the advocate of the quadrupole formalism, we shall reach a stage where any claims that nonlinear contributions are negligible will be clearly untenable.

Let us even assume, for the sake of argument, that the step from Eq. (3.4) to Eq. (3.5) can be made. To arrive at Eq. (3.7), one makes repeated use of Eq. (3.6) to show very easily that

$$(\tau^{00}x^\alpha x^\beta)_{,00}=2\tau^{\alpha\beta}+(\tau^{\gamma\delta}x^\alpha x^\beta)_{,\gamma\delta} - 2(\tau^{\gamma\alpha}x^\beta+\tau^{\gamma\beta}x^\alpha)_{,\gamma}\,. \tag{3.10}$$

Integrating over all space and casually applying the Gauss theorem yields Eq. (3.7) where the mass density is taken as the approximation to τ^{00}. However, let us actually examine the disposal of the second and third terms on the right-hand side of Eq. (3.10). By Eq. (3.6) and the Gauss theorem, the integration of Eq. (3.10) can be expressed as[19]

$$\ddot{d}_{\alpha\beta}=2\int\tau^{\alpha\beta}dV-\int(\tau^{\gamma 0}{}_{,0}x^\alpha x^\beta+\tau^{\gamma\alpha}x^\beta+\tau^{\gamma\beta}x^\alpha)n_\gamma dS\,, \tag{3.11}$$

where the surface integral is to be evaluated over the bounding surface at infinity as well as over any surfaces of discontinuity which may exist in the domain of the original volume of integration. Consider a system of two bodies with sharp boundaries, in freefall. At the very least, there will be discontinuities in the second spatial derivatives of the order-G background potential ν as is totally familiar from classical theory. Moreover, since[3]

$$\tau^{ik}=\frac{1}{16\pi G}[(-g)(g^{ik}g^{lm}-g^{il}g^{km})]_{,lm}\,, \tag{3.12}$$

it is clear that discontinuities will exist in the integrand of the second integral of Eq. (3.11) over the surfaces of the bodies and hence invalidate the quadrupole formalism if these discontinuities induce a sizable correction.

The calculations to analyze these and other contributions to the net order G^3 field have been performed. As one would expect from II, there are numerous contributions which are indeed more important than those which one finds by the traditional quadrupole formalism. However, since the primary goal in this section is to demonstrate the inadequacy of the quadrupole formalism, we will focus upon a contribution from the discontinuities of the second derivatives of the background potential (see Appendix).

Because of the G^{-1} factor in Eq. (3.12), $O(G^2)$ contributions to τ^{ik} will arise from $O(G)$ background times $O(G^2)$ dynamic fields. These terms compete with terms arising from the quadrupole formalism to the same order in G [namely $O(G^3)$ terms by virtue of the extra factor of G in Eq. (3.5)]. Clearly, from Eq. (3.12), there will be such terms with the form ${}^1g^{ik}{}_{,lm}\,{}^2g^{lm}$ where the superscripts indicate orders in G. From Eqs. (2.7) and (2.8) of II,

$$\nu=\begin{cases}\nu_E(1)+\nu_E(2) \text{ outside the spheres}\,,\\ \nu_E(1)+\nu_I(2) \text{ inside sphere } 2\,,\\ \nu_I(1)+\nu_E(2) \text{ inside sphere } 1\,,\end{cases} \tag{3.13}$$

where

$$\nu_E(1,2)=-\frac{Gm}{r_{1,2}}\,,\qquad \nu_I(1,2)=-\frac{3Gm}{2\rho_0}\left(1-\frac{(r_{1,2})^2}{3\rho_0{}^2}\right)\,. \tag{3.14}$$

Thus the surface discontinuities, denoted by an asterisk, are

$$\nu^*_{,11}=-\frac{3Gmr^2}{\rho_0{}^5}\,,\quad \nu^*_{,22}=-\frac{3Gm(z\pm\alpha)^2}{\rho_0{}^5}\,,\qquad \nu^*_{,12}=-\frac{3Gmr(z\pm\alpha)}{\rho_0{}^5}\,, \tag{3.15}$$

where the $z\pm\alpha$ terms refer to the surfaces $r_{1,2}=\rho_0$, respectively.

A typical contribution to the surface integral of Eq. (3.11) will arise from a discontinuity in τ^{ik} of the form

$$\frac{1}{G}{}^1g^{ik}{}_{,lm}\,{}^2g^{lm}\sim\frac{1}{G}\nu^*_{,22}d\,. \tag{3.16}$$

From Eqs. (4.15) of II and Eq. (3.15), the dynamic part over the surface of body I is of the order

$$\frac{1}{G}\nu^*_{,22}d\sim\frac{1}{G}\left(\frac{Gm(z+\alpha)^2}{\rho_0{}^5}\right)\frac{G^2m^2}{\alpha^2}\,\frac{(\alpha+z)t^2}{\rho_0{}^3}\,, \tag{3.17}$$

and hence the contribution of such a term to the surface integral in Eq. (3.11) is of the order

$$\frac{1}{G}\oint (\nu^{*}_{,22}d)x^{\beta}dS \sim \int_0^{2\pi}\int_0^{\rho_0}\frac{G^2m^3t^2}{\rho_0{}^8\alpha^2}[(\rho_0{}^2-\rho^2)\rho^3\cos^2\phi]\,d\rho d\phi\ , \qquad (3.18)$$

where the spherical surface element $dS=\rho_0 r d\, r d\phi/(\alpha+z)$ for sphere I has been expressed in cylindrical polar coordinates. Thus in addition to the usual contribution of the quadrupole formula term

$$\ddot{d}_{\alpha\beta}\sim\frac{G^2m^3t^2}{\alpha^4} \qquad (3.19)$$

to Eq. (3.5), there is the additional contribution from Eq. (3.18) of the order

$$\frac{1}{G}\oint_1 (\nu^{*}_{,22}d)x^{\beta}\,dS\sim\frac{G^2m^3t^2}{\alpha^2\rho_0{}^2} \qquad (3.20)$$

and hence Eq. (3.7) is modified to

$$\psi_{\alpha\beta}\sim\frac{G}{R_0}\left(\frac{G^2m^3t^2}{\alpha^4}+\frac{G^2m^3t^2}{\alpha^2\rho_0{}^2}\right). \qquad (3.21)$$

Clearly, the second nonlinear contribution dominates the usual quadrupole term. It does so in precisely the same form as we found before in II by the direct method.

One might argue that the quadrupole formalism could be restored merely by smoothing out the boundaries. There is certainly no question that surface discontinuities will be avoided by smoothing the boundaries to the extent that matter pervades all of space. However, this would no longer constitute the freefall problem that we have in mind. Burke[20] has mentioned the possibility that this very assumption might have been the source of errors in some earlier works.

One might wish to smooth the boundaries more abruptly to retain the separateness of the bodies. However, on purely physical grounds, it would be inconceivable that such a cosmetic alteration could change the basic qualitative character of the radiation which was established for bodies with sharp boundaries. To establish this in full rigor would require that one enter into the mathematical niceties of smoothing functions where one must confront new boundaries with new continuity considerations. On physical grounds, one can be confident that the basic character must remain unaltered. Moreover, in the approach of our earlier work,[10] it was found by the direct method that the large contributions arise from volume integrals, the character of which would clearly be unaltered by minor variations of the boundary distributions. It is comforting to observe the appearance of the large contributions, indeed of the same orders in the physical parameters, by these different avenues of analysis.

IV. FURTHER CONSIDERATIONS

In our work, we have solved for the second- and third-order fields by combining the field equations to yield inhomogeneous wave equations for the successive orders with a flat-space wave operator. That Einstein's theory presents us with wave equations enhances our confidence in its inherent logic. This was shown by Bonnor,[11] building upon the work of Rosen and Shamir.[21] Some critics have argued that the waves propagate in a curved background and hence the flat-space wave operator is inappropriate. Indeed, there is no question about the fact that the background is curved. In fact the waves themselves add to the curvature. In strong-field situations, this could present considerable obstacles. However, we have built upon the premise that the field is everywhere weak and hence an iterative procedure in developing the field to successive orders of accuracy is justified. The curvature of the background *has* been taken into account. In fact, we have had to go even further and take into account the curvature contribution of the second-order dynamic field. The important point is this: In an iterative procedure, their contribution, which always involves elements other than the order of field which is being sought, belongs on the right-hand side of the d'Alembert equation, symbolically expressed as

$$\Box b'=\nu b+T \qquad (4.1)$$

[see II, Eq. (3.14)]. In order to account for the background curvature in the spirit with which many criticisms are directed, one would have to introduce contributions of the form $\nu b'$ and, in addition, bb' and hence change Eq. (4.1) to

$$\Box b'+\nu b'+bb'=\nu b+T \qquad (4.2)$$

or

$$\Box^{*} b'=(\text{source})\ , \qquad (4.3)$$

where $\Box^{*}$ denotes the curved-space wave operator. Clearly, however, the difference between the use of $\Box$ and $\Box^{*}$ involves higher-order corrections. Terms of the form $\nu b'$ and bb' are relevant only for the fourth- and fifth-order iterations respectively.

Recently, Rosenblum[22] has claimed that he has found an energy loss exceeding that given by the quadrupole formula by a factor of 2.3 for the small-angle scattering of freely gravitating point singularities. Although it is of course gratifying to have one's results in a certain sense corroborated, we must caution that Rosenblum's claims should be treated with considerable reservation. Firstly, the use of singularities for sources is fraught with pitfalls. They have certainly contributed to the history of conflicting claims in the radiation problem. Secondly, as was shown in I, it is precisely when one reaches the level of freefall gravitational radiation that one must have some knowledge of the structure of the source to proceed with a meaningful analysis of that radiation. Moreover, the effect of the background field intensity is manifest in the dependence of the dominant components of the radiation on the linear dimension of the sources. When those sources are allowed to shrink to point singularities, the extent of the radiation which one might wish to deduce could hardly be described as meaningful.

V. SUMMARY AND CONCLUDING REMARKS

In this paper, we have attempted to allay some of the misgivings of our critics. We have demonstrated that the energy loss of our freely falling two-body system as calculated by the Bondi method is precisely the same as that which was found earlier.[10] The commonly held conviction that the quadrapole formalism is universally applicable to all weak-field slow-motion sources has been shown to be untenable, since it has been pointed out that even in the non-freefall domain, one can construct such sources for which this is not the case.[13,15,16] That the formalism could break down in the freefall case should therefore come as a somewhat lesser surprise. We have pursued this formalism within the context of our problem and have shown its deficiencies. The unjustifiable removal of the R^{-1} from the integration over noncompact source terms was first discussed. This was followed by an analysis of the surface integrals arising from the application of the Gauss theorem which are never considered properly in the standard treatments. We then demonstrated that they harbor precisely the kind of large contributions which have been found before by a more direct route.[10] The role of source structure is evident in freefall, a result which clearly casts doubt upon the claims of proponents of point singularity sources.

The logic of the iteration method was emphasized and it was demonstrated that flat-space rather than curved-space wave operators were called for.

It must be stressed that neither our work nor that of anyone else to the present time has meaningfully predicted the gravitational energy loss from quasiperiodic sources such as binary stars. Our results pertain to problems of the collision type with freefall times τ limited to[10]

$$\tau^2 \ll \alpha^3/Gm\,.$$

Earlier work of others ignores source structure and is plagued with problems of avoidance of incoming radiation which we have surmounted by choosing a well-defined static history.

To shed light on problems of the binary-star type, we hope to develop a variant of the present work. The proposal is to consider an initially *stationary* configuration of fluid in two hemispherical envelopes which are initially joined together. At a certain time, a perturbation is induced to sever the bond and the constituents separate into freefall orbit. Indeed this could be regarded as a model reconstruction, albeit idealized, of the actual birth of a binary system. Because of the emission of gravitational radiation, no binary system can be envisaged to have existed as such in the arbitrarily distant past unless it was being supplied by energy from outside, such as by incoming radiation. Choosing the stationary history in our next phase of work would appear to be the truly natural path.

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APPENDIX

In deriving the quadrupole formula, the harmonic gauge is employed in conjunction with Cartesian coordinates. However, in our own work, cylindrical coordinates were used to exploit the symmetry and the coordinate conditions which diagnalized the metric rather than harmonic coordinates were chosen to simplify the metric. Thus, a transformation of coordinates is required to demonstrate the breakdown of the quadrupole formula. The metric describing the dynamics of the free-fall system is given by

$$\begin{aligned} ds^2 &= \exp(2\nu + d)dt^2 - \exp(2\gamma - 2\nu + a)dr^2 \\ &\quad - \exp(2\gamma - 2\nu + b)dz^2 \\ &\quad - \rho^2 \exp(-2\nu + c)d\phi^2\,, \end{aligned} \tag{A1}$$

where ν and γ are the static vacuum Weyl functions of $O(G^1)$ and $O(G^2)$, respectively, and a, b, c, d are the time-dependent dynamic functions.

A transformation from the metric (A1) to that of Eq. (3.1) using the fact that

$$r^2 = x^2 + y^2\,, \quad \phi = \tan^{-1}(y/x) \tag{A2}$$

[henceforth $(x^1, x^2, x^3) = (x, y, z)$, $x^0 = t$] yields

$$h_{ik} = {}^1h_{ik} + {}^2h_{ik} + \cdots ,$$

where the presuperscript n represents the value of h_{ik} to order G^n:

$$\begin{aligned}
&-{}^1h_{00} = -2\nu = {}^1h_{11} = {}^1h_{22} = {}^1h_{33} , \\
&{}^1h_{\alpha\beta} = 0, \quad \alpha \neq \beta \\
&{}^2h_{00} = d + 2\nu^2 , \\
&{}^2h_{11} = (x^2+y^2)^{-1}[(2\gamma + 2\nu^2 + a)x^2 + cy^2] , \\
&{}^2h_{33} = 2\gamma + 2\nu^2 + b , \\
&{}^2h_{22} = (x^2+y^2)^{-1}[(2\gamma + 2\nu + a)y^2 + cx^2] , \\
&{}^2h_{12} = -xy(x^2+y^2)[a + 2\gamma + 2\nu^2 - c] ,
\end{aligned} \tag{A3}$$

and all other metric components of $O(G^2)$ are zero.

The introduction of the harmonic condition can be performed[3] by making an infinitesimal coordinate transformation $\bar{x}^i = x^i + \xi^i$ which induces a transformation in the metric of the form $\bar{h}_{ik} = h_{ik} - \xi_{i,k} - \xi_{k,i}$. This freedom enables the harmonic condition to be satisfied. A complete calculation to account for all terms would require the ξ_i contributions as well.

Applying the harmonic condition on Eq. (3.12) yields contributions to $\tau^{\gamma\alpha}$ of the form

$$\tau^{\gamma\alpha} \sim \frac{1}{G}(g^{\gamma\alpha}{}_{,lm}\, g^{lm} - g^{\gamma m}{}_{,l}\, g^{\alpha l}{}_{,m}) . \tag{A4}$$

Henceforth we shall restrict our attention to the discontinuities resulting from the second spatial derivatives of the $O(G^1)$ Newtonian potential, i.e., those resulting from Eqs. (3.15). Therefore only contributions of the form $(1/G)({}^1g^{\gamma\alpha}{}_{,lm}\, {}^2g^{lm})$ will be analyzed here.

We now wish to evaluate those contributions to $\ddot{d}_{\alpha\beta}$ which are neglected in the usual derivation of the quadrupole formula, namely

$$\ddot{I}_{\alpha\beta} \equiv \int_{S^*} \tau^{*\gamma\alpha} x^\beta n_\gamma dS , \tag{A5}$$

where $\tau^{*\gamma\alpha}$ is a value of the discontinuity of $\tau^{\gamma\alpha}$ on the surface S^* over which the integral is evaluated.

Since the discontinuity of $\nu_{,\alpha\beta}$ exists on the surface of the spherical bodies, the surface integral will be evaluated on those surfaces by splitting each sphere into two hemispheres. The following analysis will be carried out on sphere 1 but a similar analysis follows for sphere 2. The figure shows how the surface of sphere 1 is split into two hemispherical surfaces S_1 and S_2 which have explicit representations

$$z = \pm(\rho_0{}^2 - x^2 - y^2)^{1/2} - \alpha , \tag{A6}$$

respectively.

The unit normals to the two hemispherical surfaces are taken to be in the outward direction; thus $\hat{n}(1)$ and $\hat{n}(2)$ have non-negative and nonpositive z components, respectively. The surface integral (A5) may now be evaluated as the sum of surface integrals over the hemispheres:

$$\begin{aligned}
\ddot{I}_{\alpha\beta} &= \ddot{I}_{\alpha\beta}(1) + \ddot{I}_{\alpha\beta}(2) \\
&= \int_{S_1} \tau^{*\gamma\alpha}(1) x^\beta n_\gamma(1) dS_1 \\
&\quad + \int_{S_2} \tau^{*\gamma\alpha}(2) x^\beta n_\gamma(2) dS_2 .
\end{aligned} \tag{A7}$$

The unit normals may be easily determined from the explicit representation $f(x, y) = z$,

$$\hat{n}(1,2) = \pm \frac{-(\partial f/\partial x)\hat{i} - (\partial f/\partial y)\hat{j} + \hat{k}}{[1 + (\partial f/\partial x)^2 + (\partial f/\partial y)^2]^{1/2}} \tag{A8}$$

which from (A6) yields

$$\begin{aligned}
\hat{n}(1) &= \rho_0{}^{-1}[x\hat{i} + y\hat{j} + (\rho_0{}^2 - x^2 - y^2)^{1/2}\hat{k}] , \\
\hat{n}(2) &= \rho_0{}^{-1}[x\hat{i} + y\hat{j} - (\rho_0{}^2 - x^2 - y^2)^{1/2}\hat{k}] .
\end{aligned} \tag{A9}$$

The evaluation of the surface element may be carried out using the x and y variables for a description in the parametric representation of the two hemispheres, i.e., each hemisphere S_1, S_2 is mapped in a one-to-one fashion onto the circular disc $D = \{(x, y) \mid x^2 + y^2 \le \rho_0{}^2\}$. If $dT = dx\,dy$ is the surface element on the x-y plane resulting from the projection of dS_1 (or dS_2) onto that plane, than $dS_i = dT \sec\gamma_i$ where γ_i is the angle measured between the unit normal $\hat{n}(i)$ and the unit coordinate vector $\hat{k}$. (See Fig. 1.)

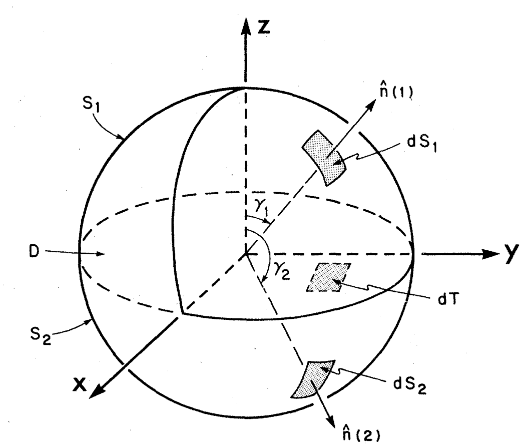

FIG. 1. Separation of sphere 1 into two hemispheres [radius ρ_0, center $(x, y, z) = (0, 0, -\alpha)$].

Therefore,

$$dS_1 = \sec\gamma_1 dT = \frac{\rho_0 dx\,dy}{(\rho_0^{\,2} - x^2 - y^2)^{1/2}}\,,$$
$$dS_2 = \sec\gamma_2 dT = -\sec\gamma_1 dT = \frac{-\rho_0 dx\,dy}{(\rho_0^{\,2} - x^2 - y^2)^{1/2}}\,. \tag{A10}$$

Therefore, using (A9) and (A10) in (A7),

$$\ddot{I}^{(1)}_{\alpha\beta} = \int\int_D x^\beta[(\rho_0^{\,2} - x^2 - y^2)^{-1/2} \times (\tau^{1\alpha}_{(1)}x + \tau^{2\alpha}_{(1)}y) + \tau^{3\alpha}_{(1)}]dx\,dy\,,$$
$$\ddot{I}^{(2)}_{\alpha\beta} = \int\int x^\beta[-(\rho_0^{\,2} - x^2 - y^2)^{-1/2} \times (\tau^{1\alpha}_{(2)}x + \tau^{2\alpha}_{(2)}y) + \tau^{3\alpha}_{(2)}]dx\,dy\,. \tag{A11}$$

We are now in a position to determine the discontinuity $\tau^{*\gamma\alpha}$. Choosing the term $(1/G)({}^1g^{\gamma\alpha}{}_{,lm}\,{}^2g^{lm})$ while making use of (A3) yields

$$\tau^{*\gamma\alpha} \sim \frac{1}{G}\left[\nu^*_{,11}\left(\psi\frac{x^2}{x^2+y^2} + c\frac{x^2-y^2}{x^2+y^2}\right) + \nu^*_{,22}\left(\psi\frac{y^2}{x^2+y^2} + c\frac{y^2-x^2}{x^2+y^2}\right) + \nu^*_{,33}b + 2\nu^*_{,12}(2c-\psi)\frac{xy}{x^2+y^2}\right], \tag{A12}$$

where $a + c = \psi$. Since we are interested only in the energy flux, the expression (A12) retains only the time-dependent $O(G^2)$ terms of (A3).

We now evaluate a typical expression (e.g., $\nu^*_{,33}b$) noting that $b = d$ since to lowest order in ρ_0/α, $\lambda \equiv b - d = 0$.

From Eqs. (4.15) of II and Eq. (3.15)

$$\nu^*_{,33} = \frac{Gm(z+\alpha)^2}{\rho_0^{\,5}}\,, \quad d = \frac{G^2m^2(z+\alpha)}{\alpha^2\rho_0^{\,3}} \tag{A13}$$

[note that $\nu^*_{,33}$ above, in Cartesian coordinates, is $\nu^*_{,22}$ of Eq. (3.15) in cylindrical coordinates], and the value of $\tau^{\gamma\alpha}$ on the hemispheres is

$$\tau^{*\gamma\alpha}_{(1)} \sim \frac{G^2m^3(\rho_0^{\,2} - x^2 - y^2)^{3/2}t^2}{\alpha^2\rho_0^{\,8}}\,,$$
$$\tau^{*\gamma\alpha}_{(2)} \sim \frac{-G^2m^3(\rho_0^{\,2} - x^2 - y^2)^{3/2}t^2}{\alpha^2\rho_0^{\,8}}\,. \tag{A14}$$

Choosing for example $x^\beta = x^1 = x$, Eqs. (A11) become

$$\ddot{I}^{(1)}_{\alpha\beta} \simeq \frac{G^2m^3t^2}{\alpha^2\rho_0^{\,8}}\int\int[(\rho_0^{\,2} - x^2 - y^2)(x^2 + yx) + x(\rho_0^{\,2} - x^2 - y^2)^{3/2}]dx\,dy\,,$$
$$\ddot{I}^{(2)}_{\alpha\beta} \sim \frac{G^2m^3t^2}{\alpha^2\rho_0^{\,8}}\int\int[(\rho_0^{\,2} - x^2 - y^2)(x^2 + yx) - x(\rho_0^{\,2} - x^2 - y^2)^{3/2}]dx\,dy\,. \tag{A15}$$

Rather than performing the integration over the Cartesian coordinates x, y, we use polar coordinates ρ, ϕ in the plane

$$x = \rho\cos\phi\,, \quad y = \rho\sin\phi\,,$$

and the integrals over the disc D become

$$\ddot{I}^{*(1)}_{\alpha\beta} \sim \frac{G^2m^3t^2}{\alpha^2\rho_0^{\,8}}\int_0^{2\pi}\int_0^{\rho_0}[(\rho_0^{\,2} - \rho^2)\rho^3(\cos^2\phi + \cos\phi\sin\phi) + \rho^2\cos\phi(\rho_0^{\,2} - \rho^2)^{3/2}]d\rho\,d\phi$$
$$\sim \frac{G^2m^3t^2}{\alpha^2\rho_0^{\,8}}(\rho_0^{\,6}) = \frac{G^2m^3t^2}{\alpha^2\rho_0^{\,2}}\,,$$
$$\ddot{I}^{*(2)}_{\alpha\beta} \sim \frac{G^2m^3t^2}{\alpha^2\rho_0^{\,8}}\int_0^{2\pi}\int_0^{\rho_0}[(\rho_0^{\,2} - \rho^2)\rho^3(\cos^2\phi + \cos\phi\sin\phi) - \rho^2\cos\phi(\rho_0^{\,2} - \rho^2)^{3/2}]d\rho\,d\phi$$
$$\sim \frac{G^2m^3t^2}{\alpha^2\rho_0^{\,8}}(\rho_0^{\,6}) = \frac{G^2m^3t^2}{\alpha^2\rho_0^{\,2}}\,.$$

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