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Finite-time corrections to the chromoelectric-flux-tube model

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The probability for pair production in a uniform electric field which has been suddenly switched on is calculated as a function of time. The result is then applied to the problem of quark pair production in confining chromoelectric-flux-tube configurations.

I. INTRODUCTION

A rather widely accepted conjecture considers that quarks interact at a large distance through linearly rising potentials. This assumption leads to reasonable spectroscopic predictions and is very appealing as a means of quark confinement. In the framework of quantum chromodynamics (QCD) linear potentials may be achieved as a result of peculiar nonperturbative behavior of chromodynamics at large distances: The force-field lines are assumed to get collimated in tubes of constant width. Thus free quarks must carry infinitely long tubes which cost an infinite amount of energy. This picture is supported by lattice formulations of QCD.

These tubes, if they exist, can in fact be the major nonperturbative factor which establishes the gross features of hadronic reactions. The constant field within the tubes can create new pairs of quarks, thus changing the structure of the interacting hadronic matter.

In a recent paper[1] approximate formulas for the production of quarks by tunneling were derived and various observable implications were studied. The work dealt mainly with e^+e^- annihilation into hadrons. In our opinion the results that were obtained justify a more thorough analysis of the model. A small step in this direction is contained in the present article.

This paper is concerned with one specific aspect of the previous treatment, namely, the implicit assumption of adiabatic switching on of the field within the tubes. It will be shown that the fact that we are dealing with finite times leads to some sizable changes in the numerical results of Ref. 1.

The program of the paper is as follows: Section II briefly reviews the flux-tube model and presents the motivation for studying the influence of short times. In Sec. III the basic formula of this paper is derived. This formula gives the rate of pair production in an Abelian uniform electric field as a function of the time that elapsed since the field was suddenly switched on. Section IV considers the implications of this result on some items that were tackled in Ref. 1. A short concluding discussion is presented in Sec. V.

II. THE FLUX-TUBE MODEL

This section presents a very brief outline of the chromoelectric-flux-tube model set up in Ref. 1. The basic assumptions and approximations of the model are listed below:

(1) The relevant scale of the processes under consideration is such that quarks may be treated as massive Dirac particles. The relevant masses are the "constituent" masses ($m_u = m_d = 350$ MeV, $m_s = 500$ MeV).

(2) In a $q\bar{q}$ system confinement is implemented through the generation of a chromoelectric flux tube of universal thickness for which the quark and antiquark act as source and sink. If g is the strong coupling constant and Λ the radius of the tube one finds with the aid of Gauss's law that the field E is given by $E = g/2\pi\Lambda^2$. Constant forces imply linear Regge trajectories. If the Regge slope is α' then

$$\tfrac{1}{2} gE = \frac{1}{\pi\alpha'} . \tag{1}$$

(3) The unique process that is treated quantum mechanically is the creation of a pair by the chromoelectric field within the tube. This field is considered as a c-number external source in Dirac's equation. Interactions between particles created in this way are neglected until a point is reached where the members of the pair are subjected on their turn to the confinement hypothesis. Then they may screen the field which created them.

Clearly the model is well suited to deal with e^+e^- annihilation. The basic information needed in order to be able to make quantitative predictions is the probability of pair creation at given transverse momentum. For a uniform field which fills all space it is known that [1,2]

$$\lim_{T\to\infty}\langle \text{vac}|e^{-iHT}|\text{vac}\rangle \sim \exp\left[-VT\frac{gE}{8\pi^3}(-1)\int d^2p_T \ln(1-e^{-2\pi(m^2+p_T^2)/gE})\right]. \quad (2)$$

V is the three-dimensional volume of the world and m is the quark mass. The probability for pair creation may be read off as

$$P(\vec{p}_T)d^2p_T = -\frac{gE}{8\pi^3}\ln[1-e^{-2\pi(m^2+p_T^2)/gE}]. \quad (3)$$

The integrated probability (per unit time and per unit volume) to create a pair of quarks of mass m is

$$P_m = -\frac{g^2E^2}{16\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^2}e^{-(2\pi m^2/gE)n}. \quad (4)$$

This model has been used to predict the following experimentally observable features:

(a) Jet structure in $e^+e^-\to$ hadrons. Equation (3) above states that the transverse momentum of the quarks created within the tube will be dynamically cut off by a factor of barrier penetration. It is a rather complicated matter to relate this to the transverse-momentum distribution observed in the laboratory, but rough estimates turn out to be in good agreement with the traditional value of the average transverse momentum (~350 MeV).[3]

(b) Suppression of strangeness. Because of their larger mass it is harder to produce strange quarks than up or down quarks. Therefore, in addition to the known kinematical suppression[4] of K's versus π's in e^+e^-, there also exists a calculable dynamical suppression factor. With the help of Eq. (4) one obtains a range of values which is in reasonable agreement with experiment.

(c) Baryon production. A classical treatment of the color degree of freedom shows that within the tube there exists a competing mechanism which tries, via production of quarks of perverse color, to give birth to configurations of baryonic type. The effective field strength is only half the value of the field which leads to usual meson production and, therefore, baryon production is dynamically suppressed. Equation (4) may be employed and the factor thus obtained seems reasonable.

(d) Hadronic lifetimes of mesons. If the width Λ of the tube is known, one may use the model to estimate the width/mass ratio for mesons.

In Ref. 1 Eq. (2) was used for finite times (T). Such an approximation is rather crude. It turns out that the tube spanned by an energetic pair of electromagnetically produced quarks will stay empty for 5 GeV^{-1} on the average.

Other relevant time scales are of the same order of magnitude and thus some doubts are thrown on the applicability of the asymptotic equation. The single phenomenological result which may be insensitive to this problem is the predicted width/mass ratio for mesonic resonances. Indeed the measured width is extracted from the supposed pole structure of some scattering amplitude which reflects the behavior at asymptotic times.

The fact that we have an approximate expression for the decay amplitude of an unstable system as a function of time, for all times, could be turned into some information on the scattering amplitude for a process in which the respective system makes a resonant contribution. One may be able to improve on the standard various Breit-Wigner fits used in such cases. It is our feeling that research in this direction could be of some interest.

III. DERIVATION OF THE FORMULA

In this section the following exercise is solved: Calculate the vacuum persistence probability as a function of time for a world governed by Dirac's equation given that a uniform external electric field has been suddenly switched on.

The most straightforward approach will be used. We will explicitly diagonalize the Dirac Hamiltonian in the presence of an external electric field. The formally unitary transformation which connects the creation and annihilation operators of the free Dirac Hamiltonian will be explicitly written down. If U denotes the "unitary" operator then the new vacuum $|0\rangle$ may be expressed in terms of the old vacuum $|v\rangle$ by

$$|v\rangle = U|0\rangle. \quad (5)$$

With the aid of U one may calculate the object of interest, namely, $|\langle v|\exp(-iHT)|v\rangle|$.

First we have to solve Dirac's equation in the presence of an external field. This allows us to identify the creation and annihilation operators which separate the problem. The equations to be solved are

$$\begin{aligned}&[\not\partial - e\not A - m]\psi = 0,\\ &A_0 = -Ez, \quad \vec{A}=0, \\ &\{\psi^\dagger_\alpha(\vec{x},t),\psi_\beta(\vec{y},t)\} = \delta^3(\vec{x}-\vec{y}).\end{aligned} \quad (6)$$

Here and throughout this section we use $e=g/2$. The solution to (6) is

$$\psi(\vec{x}_T,z,t) = \int\frac{d^2p_T dk d\omega}{(2\pi)^2(eE)^{1/2}}e^{i\omega(k/eE-t)}[V^\dagger(k,\vec{p}_T)]_{i,j} \times \chi_i c_j(\omega,\vec{p}_T)\exp[i(kz+\vec{p}_T\cdot\vec{x}_T)], \quad (7)$$

$$\{c_i(\omega,\vec{p}_T),c_j^\dagger(\omega',\vec{p}_T')\}=\delta(\omega-\omega')\delta^2(\vec{p}_T-\vec{p}_T')\delta_{ij}\,,$$
$$x_i^T x_j=\delta_{ij}\,. \tag{8}$$

V is a 4×4 unitary matrix given in the Appendix. There one may also find the definition of the constant spinors χ_i.

Equation (7) tells us that the Hamiltonian is given by

$$H=\int d^2p_T d\omega\, \omega c_i^\dagger(\omega,\vec{p}_T)c_i(\omega,\vec{p}_T)\,. \tag{9}$$

Clearly, one should consider $c_i(\omega,\vec{p}_T)$ as annihilation operators for $\omega>0$ and as creation operators for $\omega<0$.

In terms of the spinors χ_i a free Dirac field has the following representation

$$\psi_f(\vec{x}_T,z,0)=\int\frac{d^2p_Tdk}{(2\pi)^{3/2}}\chi_i[W^\dagger(k,\vec{p}_T)]_{ij}\phi_j(k,\vec{p}_T)\times\exp[i(\vec{x}_T\cdot\vec{p}_T+zk)]\,, \tag{10}$$

where

$$\{\phi_i^\dagger(k,\vec{p}_T),\phi_j(k',p_T')\}=\delta_{ij}\delta(k-k')\delta^2(\vec{p}_T-\vec{p}_T')\,. \tag{11}$$

The 4×4 unitary matrix W is given in the Appendix. ϕ_1 and ϕ_2 are destruction operators and ϕ_3 and ϕ_4 are creation operators.

Let $a_i(\omega,\vec{p}_T)$ and $b_i(k,\vec{p}_T)$ denote the annihilation operators of the interaction and free theories, respectively. Up to phases we have the following relations:

$$a_i(\omega,\vec{p}_T)=\begin{cases}c_i(\omega,\vec{p}_T), & \omega<0\,,\\ c_i(\omega,\vec{p}_T), & \omega>0,\end{cases}$$
$$b_i(k,\vec{p}_T)=\begin{cases}\phi_i(k,\vec{p}_T), & i=1,2,\\ \phi_i(k,\vec{p}_T), & i=3,4\,.\end{cases} \tag{12}$$

The "unitary" transformation which connects the a's and b's may be split into four distinct factors: (1) a four-dimensional unitary rotation defined by the matrices V and W, (2) a fourier transform, and (3) two particle-hole transformations given in Eq. (12). We now define the generators of these transformations (arguments are suppressed):

$$\begin{aligned}
&s_1=f_1(\phi_i):\ e^{-is_1}\phi_i e^{is_1}=(VW^\dagger)_{ij}\phi_j\,,\\
&s_2=f_2(c_i):\ e^{is_2}c_i e^{-is_2}=(VW^\dagger)_{ij}\phi_j\,,\\
&s_3=f_3(\phi_i):\ e^{i(\pi/4)s_3}\phi_i e^{-i(\pi/4)s_3}=b_i\,,\\
&s_4=f_4(c_i):\ e^{i(\pi/4)s_4}c_i e^{-i(\pi/4)s_4}=a_i\,.
\end{aligned} \tag{13}$$

Thus the operator U [Eq. (5)] is given by

$$U=e^{i(\pi/4)s_3}e^{is_1}e^{is_2}e^{-i(\pi/4)s_4}. \tag{14}$$

The explicit form of the functions $f_{1,2,3,4}$ is given in the Appendix.

We turn now to our project, that is, the evaluation of

$$|\langle v|e^{-iHt}|v\rangle|=|\langle 0|U^\dagger e^{-iHt}U|0\rangle|\,. \tag{15}$$

It is easy to see that

$$|v\rangle=e^{-i(\pi/4)f_4(a_i)}e^{if_2(a_i)}\times e^{i(\pi/4)f_3((WV^\dagger a)_i)}|0\rangle\,. \tag{16}$$

Now one observes that

$$e^{-if_2(\vec{a})}e^{i(\pi/4)f_4(\vec{a})}He^{-i(\pi/4)f_4(\vec{a})}e^{if_2(\vec{a})}=ieE\int d^2\vec{p}_Td\omega\frac{\partial a_i}{\partial\omega}a_i\equiv H'\,, \tag{17}$$

and, therefore,

$$e^{-iH't}a_i(\omega,\vec{p}_T)e^{iH't}=a_i(\omega+eEt,\vec{p}_T)\,. \tag{18}$$

After a few manipulations one arrives at

$$\langle v|e^{-iHt}|v\rangle=\langle 0|e^{-i(\pi/4)f_3(a_i)}e^{i(\pi/4)f_3(R_{ij}(\omega,\vec{p}_T;t)a_j(\omega,\vec{p}_T))}|0\rangle\,,$$
$$R(\omega\vec{p}_T;t)=[WV^\dagger]\left(\omega-\frac{eEt}{2},\vec{p}_T\right)[VW^\dagger]\left(\omega+\frac{eEt}{2},\vec{p}_T\right). \tag{19}$$

A straightforward computation leads one to the following expression in which discrete notation has been used:

$$|\langle v|e^{-iHt}|v\rangle|^2=\prod_{\vec{p}_T,\omega}\det[\rho_t(\omega,\vec{p}_T)\rho_t^*(\omega,\vec{p}_T)]\,. \tag{20}$$

$\rho_t(\omega,\vec{p}_T)$ is a 2×2 submatrix of R:

$$R=\begin{pmatrix}\rho' & \rho''\\ \rho''' & \rho_t\end{pmatrix}. \tag{21}$$

The infinite product has the obvious interpretation

$$\prod_{\vec{p}_T, \omega} f_t(\omega, \vec{p}_T) = \exp\left\{\frac{V}{(2\pi)^3}\left[\int d^2\vec{p}_T d\omega \ln(f_t(\omega, \vec{p}_T))\right]\right\}. \tag{22}$$

where V denotes the (infinite space volume of the system. Thus our final result is

$$|\langle v|e^{-iHt}|v\rangle|^2 = \exp\left\{\frac{V}{(2\pi)^3}\int d^2\vec{p}_T \int d\omega[\ln(|\det\rho_t|^2)]\right\}. \tag{23}$$

The cylindrical symmetry of the system is reflected by the fact that $\rho_t(\omega, \vec{p}_T)$ turns out to depend only on $E_T = (m^2 + p_T^2)^{1/2}$. We define

$$P(E_T, t) = -\frac{1}{(2\pi)^3}\int d\omega \ln(|\det\rho_t|^2). \tag{24}$$

$(\partial P/\partial t)(E_T, t)$ is the rate per unit volume for creation of pairs at given transverse energy. The rate for creation of pairs of a given mass is

$$\pi\frac{\partial}{\partial t}\int_{m^2}^{\infty} dE_T^{\,2} P(E_T, t) \equiv \frac{\partial}{\partial t}P_m(t). \tag{25}$$

The explicit form of ρ_t is rather complicated and is given in the Appendix. Asymptotically we know that we should have [see Eq. (3)].

$$\lim_{t\to\infty}\frac{\partial P}{\partial t}(E_T, t) = \frac{eE}{4\pi^3}\ln(1 - e^{-\pi E_T^2/eE}). \tag{26}$$

We have no analytic closed expressions for $P(E_T, t)$ and $P_m(t)$. The integrals on ω and E_T were evaluated numerically. The function $P(E_T, t)$ is shown in Figs. 1 to 4. The precise definitions of the curves drawn are given in the Appendix. The function $P_m(t)$ as defined in Eq. (25) is not drawn. The reason for this is, as will be explained in the next section, that we are in fact interested in the expression

$$P_m(t, p_T^c) = \pi\int_{m^2}^{m^2+(p_T^c)^2} dE_T^{\,2} P(E_T, t), \tag{27}$$

where p_T^c is a cutoff imposed by the physical aspects of our problem. The relevant curves are shown in Fig. 5. For notations see the Appendix.

The qualitative behavior of the function $P(E_T, t)$ is easily understood. For very short times the first order in perturbation theory gives a good approximation. On dimensional grounds one may write

$$|\langle 0|e^{-iHt}|0\rangle|^2 \underset{t\to 0}{\simeq} \exp\left[-\text{const}\times(eE)^2 V t^2 \times\int_{m^2}^{m^2+(p_T^c)^2} dE_T^{\,2}\frac{1}{E_T}\right]. \tag{28}$$

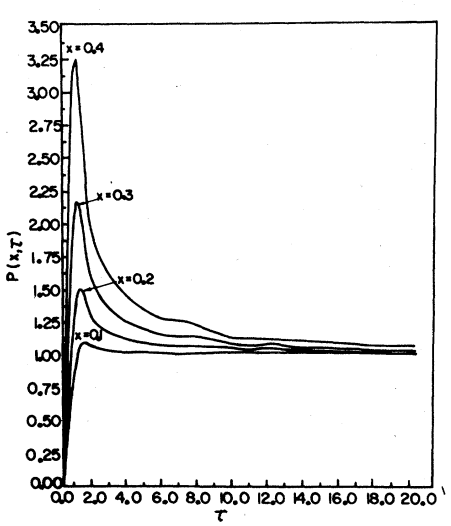

FIG. 1. Relative probability for pair creation at given transverse energy E_T, $E_T^{\,2} = 2x/(\pi\alpha')$ (α' is the Regge slope). The ordinate τ is proportional to the time t, $\tau = t/(2\pi\alpha')^{1/2}$. The four curves are, in ascending order, drawn with x held fixed at 0.1, 0.2, 0.3, and 0.4. The values of the function are rescaled by the asymptotic expressions of Eq. (A17). Therefore as $\tau\to\infty$ the curves should level off at the value of 1. It is seen that the asymptotic behavior is achieved only after very long times when the transverse energy is large.

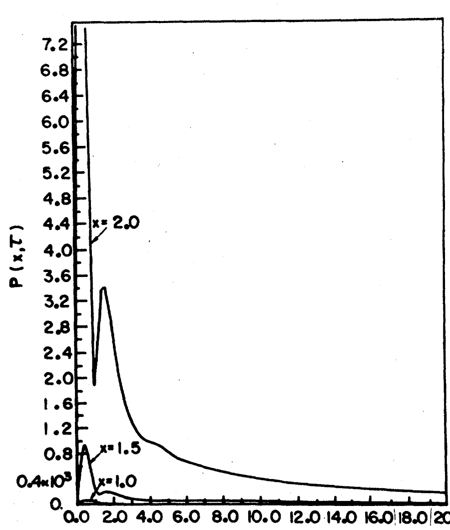

FIG. 2. Same as Fig. 1 only for different values of x: 0.5, 1.0, 1.5, and 2.0. It is impossible to distinguish between the τ-axis and the first curve ($x = 0.5$). This shows the strong dependence of the approach to the asymptotic value on E_T.

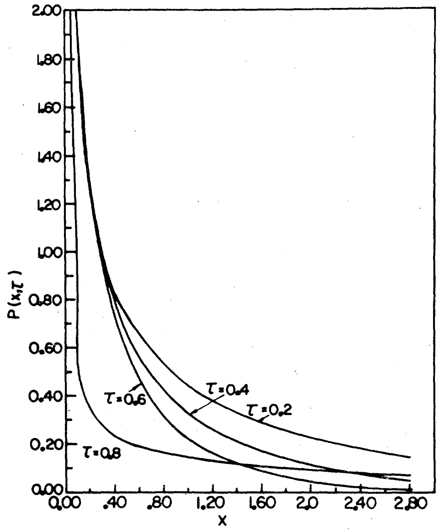

FIG. 3. The probability of pair creation is shown as a function of transverse energy for different values of time. The variables are the same as in Fig. 1 and 2. The highest value of time corresponds to the lowest curve drawn. The values of τ are $\tau=0.2$, 0.4, 0.6, and 0.8. No rescalings are done. It is seen that the decrease at high transverse energies gets steeper as time increases. As expected, when the time increases the dependence of the curves on time is weakened.

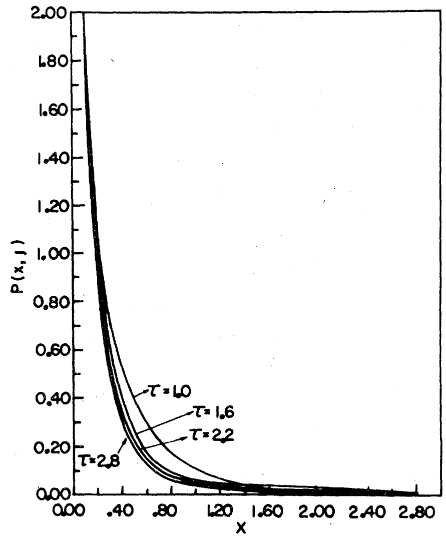

FIG. 4. Same as Fig. 3 only for different values of τ: $\tau=1.0$, 1.6, 2.2, and 2.8. The curves are not very different from one another signaling the approach to a limit.

Therefore the approximation gets better numerically as the energy E_T increases. The asymptotic value is overshot and approached slowly from above (for E_T larger than some value). This reflects the physical fact that for finite times, energy need not be conserved and thus the volume of states the system may decay into is larger than for the asymptotic case.

We should remark that our numerical analysis has been checked by observing the $t\to 0$ and $t\to\infty$ limits. In these regions the exact results are known [the missing constant in Eq. (28) is easy to calculate].

We are now prepared to proceed to investigate the possible influence of these results on the items discussed in the previous section.

IV. QUANTITATIVE IMPLICATIONS

The results of the previous section will now be used to calculate the average transverse momentum of the quarks produced within the tube, the ratio between strange and nonstrange quarks and the relative probability for production of baryonic configurations. We will not deal with the width to mass ratio of meson resonances because the effects of finite times on it should be rather small, as we already explained.

It should be clear that what we are going to do is not the correct treatment of the problem even with respect to time dependence. In reality the field is not switched on suddenly. What the application of the results of the previous section will do for us is that it will give us an upper bound on the errors which are to be expected on account of the time dependence of the field in the tube. With

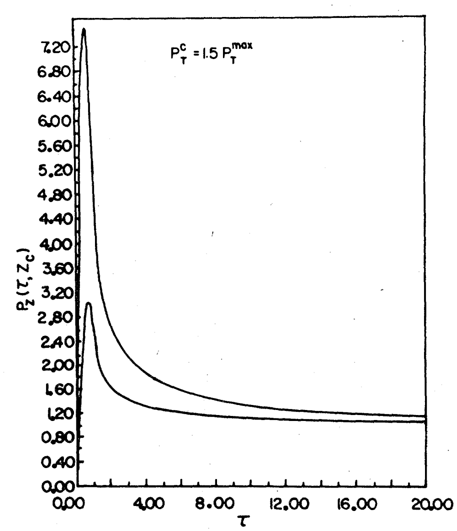

FIG. 5. The probability for creating quarks of a given mass divided by its aymptotic value. The lower curve describes nonstrange quarks and the upper curve is for strange quarks. All other parameters were taken at some reasonable value which is irrelevant for our illustrative purposes. The ordinate is proportional to time. The curves should level off at 1 for $\tau\to\infty$.

this "worst case analysis" policy in mind we decide to calculate the quantities we are interested in for the first pair that is produced. This pair is created in the tube that connects the two quarks which resulted from the initial photon (we are dealing with $e^+e^- \to$ hadrons). Since it is the first to be created it will be most sensitive to the finiteness of the time that elapsed since the field and the tube were produced.

The tube mentioned above will stay empty (except for the chromoelectric field it contains) for a time t if nowhere along its extension was a pair produced. Let the index f denote flavor and m_f the respective constituent mass. For high enough energies the ends of the tube move with the velocity of light. Then the probability that nothing will happen in the expanding tube for a time t at least is

$$\prod_{x=-t}^{t} \exp\left[-\pi\Lambda^2 \sum_f P_{m_f}(t-|x|, p_T^c)\Delta x\right] = \exp\left[-2\pi\Lambda^2 \sum_f \int_0^t dt' P_{m_f}(t', p_T^c)\right] \equiv G(t). \quad (29)$$

We should now comment on the cutoff p_T^c. This reflects the fact that the width of the tube is finite. It ensures that one excludes the possibility of creating quarks with such a large transverse momentum that they would escape from the tube even before materializing.[1] This qualitative argument is easily turned into a semiclassical estimate for an upper bound on p_T, $p_T^{\max}$:

$$p_T^{\max} = \frac{gE}{\pi}\Lambda. \quad (30)$$

In numerical evaluations we used $p_T^c = \eta p_T^{\max}$ where η was varied around the value $\eta = 1$. Thus we are able to get a feeling about the sensitivity of our results to the transverse shape of the tube.

For $t \to \infty$ we have [see Eq. (4) and (A17)]

$$G(t) \underset{t\to\infty}{\simeq} \exp\left(-\pi\Lambda^2 t^2 \sum_f P_{m_f}\right). \quad (31)$$

On numerical grounds the transverse-momentum cutoff has a negligible influence on the asymptotic limit. For finite times, however, the situation is different.

Using Eq. (29) the average "lifetime" of the "empty" tube may be calculated:

$$\langle t\rangle = -\int_0^\infty t\,\frac{dG}{dt}\,dt = \int_0^\infty G(t)\,dt. \quad (32)$$

If we replace $G(t)$ by its asymptotic form we find

$$\langle t\rangle_{\text{asym}} = \frac{1}{2\Lambda}\frac{1}{\left(\sum_f P_{m_f}\right)^{1/2}}. \quad (33)$$

Table I compares the average times given by

TABLE I. The average time that elapsed until the creation of the first quark pair in the tube ($\langle t\rangle$). All units are in powers of GeV.

Quark mass m_f	Regge slope α'	Tube radius Λ	$p_T^c/p_T^{\max}$ η	Asymptotic approximation	Sudden approximation
$m_u = m_d = 0.35$ $m_s = 0.50$	0.9	2.5	0.75	7.15	5.91
			1.00	6.64	5.05
			1.25	6.50	4.60
		3.0	0.75	5.65	4.33
			1.00	5.43	3.76
			1.25	5.39	3.44
	1.0	2.5	0.75	8.96	7.37
			1.00	8.19	6.22
			1.25	7.96	5.62
		3.0	0.75	7.00	5.33
			1.00	6.65	4.58
			1.25	6.58	4.17
$m_u = m_d = 0.30$ $m_s = 0.45$	0.9	2.5	0.75	6.04	5.21
			1.00	5.62	4.49
			1.25	5.50	4.11
		3.0	0.75	4.77	3.87
			1.00	4.59	3.39
			1.25	4.56	3.13
	1.0	2.5	0.75	7.23	6.26
			1.00	6.64	5.35
			1.25	6.46	4.87
		3.0	0.75	5.67	4.61
			1.00	5.40	4.01
			1.25	5.34	3.69

Eqs. (32) and (33) at different values of the parameters of the model.

The pair which has been created after a time $\langle t\rangle$ will have its location along the tube distributed according to $\rho(x)\,dx$:

$$\rho(x)=C\sum_f P_{m_f}(\langle t\rangle-|x|,p_T^c),\quad |x|<\langle t\rangle\,. \tag{34}$$

For the asymptotic case we have

$$\rho(x)=c'(\langle t\rangle-|x|),\quad |x|<\langle t\rangle\,. \tag{35}$$

These two distributions are compared in Fig. 6. Some typical values of the parameters have been assumed, and the range of x has been rescaled. Both distributions have been normalized. We see that the contribution of the finite time corrections is to produce a flatter distribution, almost uniform. It is an intriguing question whether this effect could yield also an asymmetrical distribution (in the sense that the tube would prefer to split, after screening takes place, into two parts of unequal length). We cannot resist the temptation to raise the extremely far-fetched conjecture that some similar effect may be responsible for the observed asymmetry in nuclear fission.

At this point it should be remarked that, in principle at least, it seems that the $e^+e^-\to$ hadrons data could be analyzed in such a way as to yield some information on the distribution of the location of the first pair along the tube. In our opinion this point deserves further investigation.

The averages over transverse momentum are done taking into account the joint probability for the first pair to be produced at time t and at location x, $|x|<t$:

$$\langle p_T^{\,2}\rangle=2\pi\Lambda^2\sum_f\int_0^\infty dt\,G(t)\times\int_0^{(p_T^c)^2}P((m_f^{\,2}+p_T^{\,2})^{1/2};t)\,p_T^{\,2}d^2p_T\,. \tag{36}$$

A similar equation gives $\langle p_T\rangle$. The results are listed for comparison with the asymptotic values in Table II. It is observed that the numbers obtained within the sudden approximation are strongly dependent on the semiclassical cutoff which resulted from the finite width of the tube. Thus the dynamical suppression of the high p_T quarks results not only from the barrier penetration factor (as was the case in the asymptotic approximation) but also from the finite width of the tube. In order to avoid confusion let us stress that the influence of the width of the tube is in the opposite direction to what one would expect on the basis of the uncertainty relation: An increase of the radius of the tube causes an increase in the average p_T and not

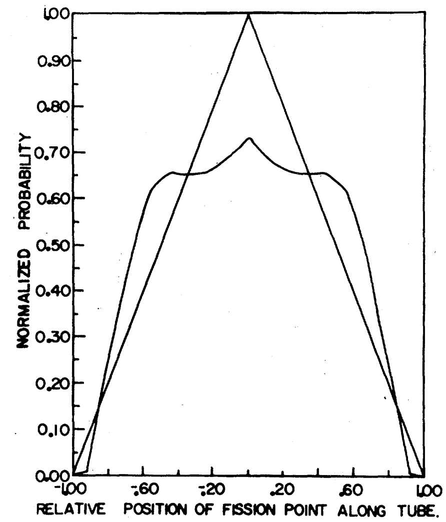

FIG. 6. The distribution of the relative location of the new pair along the tube. The triangular curve is drawn for asymptotic time and the second curve is drawn for a time equal to the average time of the tube. The area under both curves is equal to unity. All other parameters are irrelevant for our illustrative purpose.

a decrease. Our relation between $p_T^{\max}$ and Λ [Eq. (30)] is a semiclassical result.

The relative probability for the creation of strange quarks versus nonstrange quarks is given by $N_s/N_{\rm ns}=r/(1-r)$ where

$$r=2\pi\Lambda^2\int_0^\infty dt\,G(t)P_{m_s}(t,p_T^c)\,. \tag{37}$$

Table III compares the present results with the values obtained within the asymptotic approximation. The differences are serious but not alarming.

If the quarks created within the tube have a different color from the original pair one should expect to find baryons emerging from the reaction.[1] Quarks of a perverse color are produced by a field weaker by a factor of 2 than the field which produces mesonic configurations. Therefore perverse color is relatively rare and this is the *a posteriori* justification for the neglect of the color degrees of freedom until now. Using this fact we may infer that the production of baryons is dynamically suppressed:

$$\frac{N_B}{N_M}=2\times 2\pi\Lambda^2\int_0^\infty dt\,G(t)\sum_f P_{m_f}^B(t,p_T^c)\,. \tag{38}$$

The extra factor of 2 in Eq. (38) counts the number of perverse colors. By scaling arguments one observes that

$$P_{m_f}^B(t,p_T^c)=2^{-3/2}P_{2^{1/2}m_f}(2^{-1/2}t,2^{1/2}p_T^c)\,. \tag{39}$$

Notice that $p_T^{\max}$ for baryons is now smaller by a factor of 2 than the value given in Eq. (30). The values of N_B/N_M are exhibited in Table IV.

TABLE II. The r.m.s. transverse momentum of the first quark pair($\langle p_T^2 \rangle^{1/2}$). All units are in powers of GeV.

Quark mass m_f	Regge slope α'	Tube radius Λ	p_T^c/p_T^{max} η	Asymptotic approximation	Sudden approximation
$m_u = m_d = 0.35$ $m_s = 0.50$	0.9	2.5	0.75	0.209	0.238
			1.00	0.261	0.314
			1.25	0.284	0.375
		3.0	0.75	0.244	0.291
			1.00	0.281	0.373
			1.25	0.292	0.442
	1.0	2.5	0.75	0.204	0.224
			1.00	0.253	0.291
			1.25	0.278	0.347
		3.0	0.75	0.237	0.270
			1.00	0.275	0.344
			1.25	0.288	0.406
$m_u = m_d = 0.30$ $m_s = 0.45$	0.9	2.5	0.75	0.215	0.242
			1.00	0.264	0.315
			1.25	0.287	0.375
		3.0	0.75	0.248	0.293
			1.00	0.283	0.373
			1.25	0.294	0.440
	1.0	2.5	0.75	0.189	0.214
			1.00	0.239	0.282
			1.25	0.265	0.338
		3.0	0.75	0.223	0.261
			1.00	0.262	0.336
			1.25	0.275	0.398

TABLE III. The dynamical strangeness-suppression factor (N_s/N_{ns}). All units are in powers of GeV.

Quark mass m_f	Regge slope α'	Tube radius Λ	p_T^c/p_T^{max} η	Asymptotic approximation	Sudden approximation
$m_u = m_d = 0.35$ $m_s = 0.50$	0.9	2.5	0.75	0.29	0.44
			1.00	0.29	0.48
			1.25	0.29	0.51
		3.0	0.75	0.29	0.49
			1.00	0.29	0.53
			1.25	0.29	0.57
	1.0	2.5	0.75	0.26	0.40
			1.00	0.26	0.44
			1.25	0.26	0.47
		3.0	0.75	0.26	0.45
			1.00	0.26	0.49
			1.25	0.26	0.52
$m_u = m_d = 0.30$ $m_s = 0.45$	0.9	2.5	0.75	0.32	0.46
			1.00	0.32	0.50
			1.25	0.32	0.53
		3.0	0.75	0.32	0.51
			1.00	0.32	0.56
			1.25	0.32	0.59
	1.0	2.5	0.75	0.28	0.41
			1.00	0.28	0.46
			1.25	0.29	0.49
		3.0	0.75	0.28	0.46
			1.00	0.29	0.51
			1.25	0.29	0.54

V. DISCUSSION

The direct result of this paper is a quantitative estimate of the numerical errors one might expect on account of nontrivial time development of the tube. It is hoped that the results in Sec. III might be useful in future investigations of the flux-tube model. The approximate time development of the system has to be known in order to set up a complete semiclassical picture of quark production in $e^+e^- \to$ hadrons. Viewed from a different angle Sec. III gives an explicit and complete solution of an exactly soluble unstable system.

Even with this work taken into account the flux-tube model is far from being exhausted. The complete scenario of the cascade has to be designed, and it is to be expected that the extraction of observable predictions by detailed calculations will be a complicated theoretical and numerical matter. On different lines there is an impressive number of approximations which have been made, the treatment of which does not present any difficulties in principle. For example, one might calculate QCD perturbative corrections to pair production within the tube, or finite width effects. On the other hand, in the framework of the model there exist not less numerous approximations the treatment of which is an open problem in QCD (e.g. chiral-symmetry breaking and dynamical quark-mass generation). The model rests also on some assumptions which may be classified somewhere in between the two extremes discussed above. Progress is being constantly made in various works in this direction. For example, although the detailed mechanism of confinement is not yet known, it is believed that definite statements about the large-distance quark interaction may be made. Such statemements may be derived on the basis of the well-known magnetic superconductor analogy.[5]

To summarize, there is a lot of work left to be done before the flux-tube model may be either accepted or rejected.

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APPENDIX

1. The spinors χ_i

The spinors satisfy the equations

$$\begin{aligned} &\chi_i^T \chi_j = \delta_{ij}\,, \\ &\chi_3 = \gamma^0\chi_1, \quad \chi_4 = \gamma^0\chi_2, \quad \gamma_5\chi_{1,2} = \chi_{1,2}\,, \\ &\gamma^3\chi_1 = -\chi_3, \quad \gamma^3\chi_2 = \chi_4\,, \end{aligned} \tag{A1}$$

TABLE IV. The dynamical factor of suppression of baryons (N_B/N_M). All units are in powers of GeV.

Quark mass m_f	Regge slope α'	Tube radius Λ	$p_T^c/p_T^{\max}$ η	Asymptotic approximation	Sudden approximation
$m_u = m_d = 0.35$ $m_s = 0.50$	0.9	2.5	0.75	0.14	0.25
			1.00	0.14	0.28
			1.25	0.14	0.31
		3.0	0.75	0.14	0.28
			1.00	0.14	0.32
			1.25	0.14	0.34
	1.0	2.5	0.75	0.13	0.23
			1.00	0.13	0.26
			1.25	0.13	0.29
		3.0	0.75	0.13	0.27
			1.00	0.13	0.30
			1.25	0.13	0.32
$m_u = m_d = 0.30$ $m_s = 0.45$	0.9	2.5	0.75	0.19	0.29
			1.00	0.19	0.31
			1.25	0.19	0.34
		3.0	0.75	0.19	0.31
			1.00	0.19	0.34
			1.25	0.19	0.37
	1.0	2.5	0.75	0.17	0.26
			1.00	0.17	0.29
			1.25	0.17	0.31
		3.0	0.75	0.17	0.29
			1.00	0.17	0.32
			1.25	0.17	0.35

where the γ^μ are Dirac matrices.

2. The matrix V

The notation for the functions of the parabolic cylinder is that used in Bateman.[6] The following notations will be used:

$$\begin{aligned} &h = e^{i\pi/4},\\ &\nu = -i\,\frac{E_T^{\,2}}{2eE},\\ &E_T \pm p_T = m e^{\pm 2\theta_T},\\ &p_\pm = p_T \exp[\pm i(\alpha_T + \pi/4)] = p^1 \pm i p^2 . \end{aligned} \tag{A2}$$

The arguments of the parabolic cylinder functions will be suppressed according to the following conventions:

$$\begin{aligned} &D_{\nu-1}\left(h\left(\frac{2}{eE}\right)^{1/2} k\right) \equiv D_{\nu-1} = D^*_{-\nu-1},\\ &D_{-\nu}\left(h^*\left(\frac{2}{eE}\right)^{1/2} k\right) \equiv D_{-\nu} = D^*_{\nu},\\ &D_{-\nu-1}\left(h^*\left(\frac{2}{eE}\right)^{1/2} k\right) \equiv D^*_{\nu-1} = D_{-\nu-1},\\ &D_{\nu}\left(h\left(\frac{2}{eE}\right)^{1/2} k\right) \equiv D_{\nu} = D^*_{-\nu}. \end{aligned} \tag{A3}$$

With these conventions the matrix V is given by

$$V = (m/2E_T)^{1/2} e^{-\pi/4|\nu|}\, V',$$

$$V' = \begin{pmatrix} |\nu|^{1/2} e^{-\theta_T} D^*_{\nu-1} & e^{-\theta_T - i\alpha_T} D^*_\nu & h e^{\theta_T} D^*_\nu & |\nu|^{1/2} h^* e^{\theta_T - i\alpha_T} D^*_{\nu-1} \\ |\nu|^{1/2} e^{\theta_T} D^*_{\nu-1} & -e^{\theta_T - i\alpha_T} D^*_\nu & h e^{-\theta_T} D^*_\nu & -|\nu|^{1/2} h^* e^{-\theta_T - i\alpha_T} D^*_{\nu-1} \\ -e^{-\theta_T + i\alpha_T} D_\nu & |\nu|^{1/2} e^{-\theta_T} D_{\nu-1} & |\nu|^{1/2} h e^{\theta_T + i\alpha_T} D_{\nu-1} & -h^* e^{\theta_T} D_\nu \\ e^{\theta_T + i\alpha_T} D_\nu & |\nu|^{1/2} e^{\theta_T} D_{\nu-1} & -|\nu|^{1/2} h e^{-\theta_T + i\alpha_T} D_{\nu-1} & -h^* e^{-\theta_T} D_\nu \end{pmatrix}. \tag{A4}$$

3. The matrix W

We denote

$$E' \pm k = E_T e^{\pm 2y}. \tag{A5}$$

The matrix W is given by

$$W = \tfrac{1}{2}(m/E')^{1/2} W',$$

$$W' = \begin{pmatrix} e^{-\theta_T - y} & e^{\theta_T - y} & e^{-\theta_T + y} & e^{\theta_T + y} \\ h e^{-\theta_T + y + i\alpha_T} & -h e^{\theta_T + y + i\alpha_T} & -h e^{-\theta_T - y + i\alpha_T} & h e^{\theta_T - y + i\alpha_T} \\ e^{\theta_T + y} & e^{-\theta_T + y} & -e^{\theta_T - y} & -e^{-\theta_T - y} \\ h e^{\theta_T - y + i\alpha_T} & h e^{-\theta_T - y + i\alpha_T} & h e^{\theta_T + y + i\alpha_T} & -h e^{-\theta_T + y + i\alpha_T} \end{pmatrix}. \tag{A6}$$

4. The generators of the unitary transformations

$VW^\dagger$ is a unitary matrix. Hence there exists a Hermitian matrix Q such that

$$VW^\dagger = e^{-iQ}. \tag{A7}$$

Then f_1 is given by

$$f_1(\phi_i) = \int d^2p_T\, dk\, \phi_i^\dagger(k, \vec{p}_T) Q_{ij}(k, \vec{p}_T) \phi_j(k, \vec{p}_T). \tag{A8}$$

The generator of the Fourier transform is given by f_2[7]:

$$f_2(c_i) = -\frac{\pi}{4}\int d^2p_T\, dk\, c_i^\dagger(k, \vec{p}_T) \times \left(eE\frac{\partial^2}{\partial k^2} - \frac{k^2}{eE} + 1\right) c_i(k, \vec{p}_T). \tag{A9}$$

The generators for the particle-hole transformations f_3 and f_4 are

$$\begin{aligned} f_3(\phi_i) &= \sum_{j=3}^{4}\int d^2p_T\, dk[\phi_j(k, \vec{p}_T)\phi_j(-k, -\vec{p}_T) + \phi_j^\dagger(-k, -\vec{p}_T)\phi_j^\dagger(k, \vec{p}_T)]\epsilon(p_x),\\ f_4(c_i) &= \int d^2p_T \int_{-\infty}^{0} d\omega[c_i(\omega, \vec{p}_T)c_i(\omega, -\vec{p}_T) + c_i^\dagger(\omega, -\vec{p}_T)c_i^\dagger(\omega, \vec{p}_T)]\epsilon(p_x). \end{aligned} \tag{A10}$$

5. The matrix ρ_t

We first introduce some notation:

$$k_\pm = \omega \pm \frac{eEt}{2},$$

$$\mathfrak{z} = \left(\frac{2}{eE}\right)^{1/2}\omega,$$

$$\mathfrak{z}_{\pm} = \left(\frac{2}{eE}\right)^{1/2} k_{\pm}\,,$$
$$x = i\nu; \quad \mathfrak{z} = \frac{m^2}{2eE}, \qquad \text{(A11)}$$
$$\tau = \left(\frac{eE}{2}\right)^{1/2} t\,,$$
$$E_{\pm} \pm k_{\pm} = E_T e^{\pm 2y_{\pm}}, \quad \xi = y_+ + y_-, \quad \eta = y_- - y_+\,.$$

It is convenient to define the following functions:

$$F_x(k_{\pm}) = D_{ix}(h^*\mathfrak{z}_-)D_{-ix}(h\mathfrak{z}_+) + xD_{-ix-1}(h\mathfrak{z}_-)D_{ix-1}(h^*\mathfrak{z}_+)\,,$$
$$G_x(k_{\pm}) = h^*\sqrt{x}\,[D_{ix}(h^*\mathfrak{z}_-)D_{-ix-1}(h\mathfrak{z}_+) - D_{ix}(h^*\mathfrak{z}_+)D_{-ix-1}(h\mathfrak{z}_-)]\,. \qquad \text{(A12)}$$

The 2×2 matrix ρ_t is proportional to the unit matrix 1_2:

$$\rho = \frac{x^{1/2}e^{-(\pi/2)x}}{2[x^2 + \frac{1}{2}x(z^2+\tau^2) + \frac{1}{16}(z^2-\tau^2)^2]^{1/4}} \times [e^{\xi}F_x + e^{-\xi}F_x^* + i(e^{\eta}G_x + e^{-\eta}G_\eta^*)]1_2 \equiv \hat{\rho}1_2\,. \qquad \text{(A13)}$$

Using the dimensionless variables previously defined one can write the following expression for the vacuum-persistence probability:

$$|\langle v|e^{-iHt}|v\rangle|^2 = \exp\left\{Vt\left(\frac{eE}{2\pi}\right)^2\left[\frac{1}{\tau}\int_z^{\infty} dx\int_{-\infty}^{\infty} d\mathfrak{z}\,\ln(|\hat{\rho}|^2)\right]\right\}. \qquad \text{(A14)}$$

As explained in the text (Sec. IV) we are actually interested in the case where the range of transverse momenta is cut off by some number depending on the field strength and the tube radius [see Eq. (30)]. Let us denote by z_c the quantity $(p_T^c)^2/(2eE)$. The functions defined in the text can be expressed in terms of dimensionless quantities:

$$P(E_T, t) = \frac{(2eE)^{1/2}}{8\pi^3}\tau\mathcal{P}(x,\tau)\,,$$
$$P_m(t, p_T^c) = \frac{[2(eE)^3]^{1/2}}{4\pi^2}\tau\mathcal{P}_z(\tau; z_c)\,, \qquad \text{(A15)}$$

where

$$\mathcal{P}(x,\tau) = -\frac{1}{\tau}\int_{-\infty}^{\infty} d\mathfrak{z}\,\ln(|\hat{\rho}|^2)\,,$$
$$\mathcal{P}_z(\tau; z_c) = \int_z^{z+z_c}\mathcal{P}(x,\tau)dx\,. \qquad \text{(A16)}$$

For $\tau\to\infty$ one should find [see Eqs. (3) and (4) in the text]

$$\mathcal{P}(x,\tau) \sim -2\ln(1-e^{-2\pi x})\,,$$
$$\mathcal{P}_z(\tau; z_c) \sim \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}e^{-2\pi nz}(1-e^{-2\pi nz_c})\,. \qquad \text{(A17)}$$

Applying perturbation theory one finds for $\tau\to 0$,

$$\mathcal{P}(x,\tau) \sim \frac{\pi}{4}\frac{\tau}{x^{1/2}}\,. \qquad \text{(A18)}$$

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[1] A. Casher, H. Neuberger, and S. Nussinov, Phys. Rev. D 20, 179 (1979).
[2] W. Heisenberg and H. Euler, Z. Phys. 98, 714 (1936); J. Schwinger, Phys. Rev. 82, 664 (1951); E. Brezin and C. Itzykson, Phys. Rev. D 2, 1191 (1970).
[3] G. Feldman and M. Perl, Phys. Rep. 19C, 234 (1975); 33C, 285 (1977); Gail G. Hanson, SLAC Report No. SLAC-PUB-2118, 1978 (unpublished).
[4] J. Bjorken and G. Farrar, Phys. Rev. D 9, 1449 (1974).
[5] E.g. T. Banks and M. Spiegelglas, Nucl. Phys. B152, 478 (1979).
[6] Bateman Manuscript Project, edited by A. Erdelyi (McGraw-Hill, New York, 1955).

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