Cross-section calculations for $\bar{\nu}_e + d \rightarrow n + p + \bar{\nu}_e$

Tino Ahrens and Lawrence Gallaher
Georgia Institute of Technology, Atlanta, Georgia 30332
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The cross section of the reaction $\bar{\nu}_e + d \rightarrow n + p + \bar{\nu}_e$ has been calculated for an equilibrium fission spectrum as a function of the proton energy chosen to serve as a variable detection threshold. The results are intended to permit as accurate a determination of the neutral-current isovector axial-vector coupling constant as the experimental conditions permit. With the theoretically plausible constant $G_A = -1.49 \times 10^{-11}$ MeV^{-2} and a most recent semiempirical equilibrium fission spectrum the total cross section is calculated to be 7.27×10^{-45} cm^2; half of this cross section leads to proton energies of 150 keV and above.

OBJECTIVE

Since feasibility has been established of an experiment for the determination of the axial-vector coupling constant associated with the semileptonic interaction of the neutral components of the isovector current,[1] it is appropriate to provide theoretical values in terms of said constant of the actually-to-be-measured quantity and, of course, with an accuracy at least as high as the experimentally expected one. The quantity to be measured is the cross section of the reaction

$$\bar{\nu}_e + d \rightarrow n + p + \bar{\nu}_e$$

for incident equilibrium fission neutrinos, as a function of the detection-threshold proton energy. Calculation of the theoretical values required for comparison will thus necessitate (1) calculation of the cross section differential with respect to proton energy for monoenergetic incident antineutrinos $d\sigma/dE$, followed by (2) its integral over the fission spectrum, and (3) the integral over the desired proton energy range.

An experimental accuracy of 25% is expected in the short term, eventually to be followed, it is hoped, by one close to 10%. Any ingredient of the theoretical cross section readily lending itself to calculation with greater accuracy than the mentioned experimental ones is treated accordingly.

As is rather well established, the fission neutrino spectrum decreases by one order of magnitude within the first 3 MeV beyond the binding energy of the deuteron and again within the next 2 MeV. At 10 MeV it is more than 3000 times smaller.[2] The cross-section calculation may thus be performed with sufficient accuracy in the allowed approximation.

THE MATRIX ELEMENT

With the modest but realistic accuracy aspired to here, the basic theoretical expression is, except for G_A, the model-independent

$$\frac{d\sigma}{dE} = 2\pi G_A{}^2 \int d\Omega\, d\Omega' \rho |\mathfrak{M}|^2 ,$$
$$\mathfrak{M} = \phi'^* \vec{\sigma}_L \phi \cdot \int dv \tfrac{1}{3} \sum_{m_D} \Psi_{\rm if}^* \sum_N \vec{\sigma}_N \tau_{3N} \Psi_{m_D} . \quad (1)$$

$\Psi_{\rm if}$ denotes the interacting final-state wave function, the subscripts N and L stand for nucleon and lepton, respectively, and ρ is used for the density of final states; the solid angle element $d\Omega$ is associated with the proton of energy E, and $d\Omega'$ with the emitted antineutrino whose two spinor is ϕ'. The other symbols should be self-explanatory.

With the deuteron as a pure S-state configuration, the matrix element has been treated elsewhere,[3] evaluation of the space part of the heavy-particle factor generally proceeding according to Bethe and Longmire. Keeping the angular correlation, one has

$$|\mathfrak{M}|^2 = \frac{8\pi\gamma}{1-\gamma r_t} \frac{\sin^2\delta_s}{k^2} \times \left(\frac{\gamma + k\cot\delta_s}{\gamma^2+k^2} - \frac{r_t + r_s}{4}\right)^2 \left(1 - \frac{\hat{q}\cdot\hat{q}'}{3}\right). \quad (2)$$

Here, $\gamma = \sqrt{MB}$, with M the nucleon mass and B the magnitude of the deuteron binding energy, and $k^2 = ME_r$, with E_r the exit energy of the two nucleons in their c.m. frame; $\hat{q}$ and $\hat{q}'$ are the directions of the antineutrino momenta. In the well-known manner, $\cot\delta_s$ may be usefully expressed as follows:

$$\cot\delta_s = -\frac{1}{a_s} + \frac{r_s}{2} k^2 . \quad (3)$$

At this point, the "effective ranges" appearing in Eqs. (2) and (3) are still functions of energy,

$$r_s = 2\int (v_{\rm if}{}^2 - u_{\rm if}{}^2)dr ,$$
$$r_t = 2\int (v_D{}^2 - u_D{}^2)dr , \quad (4)$$

$u_{\rm if}$ and u_D being the radial parts of $\Psi_{\rm if}$ and Ψ_{m_D}, respectively, and v_{it} and v_D their respective asymptotic forms, normalized to one at the origin. Equation (2) is then accurate to within better than 1%.[4] Substituting the standard effective ranges at zero energy has little effect: Firstly, the energies replaced by zero, B, and E_r are very small compared to the nuclear potentials; secondly, Eq. (2) depends on the effective ranges in a way which makes it rather insensitive to changes in them. In fact, eliminating r_s and r_t altogether brings about a change in the right side of Eq. (2) of only a few percent. For a straightforward demonstration consider Eq. (2), using Eq. (3) and letting $k^2 \ll \gamma^2$,

$$|\mathfrak{M}|^2 = \frac{8\pi\gamma}{1-\gamma r_t}\,\frac{1}{k^2+1/a_s^2}\left(\frac{\gamma-1/a_s}{\gamma^2}-\frac{r_s+r_t}{4}\right)^2\left(1-\frac{\hat{q}\cdot\hat{q}'}{3}\right)$$
$$\simeq \frac{8\pi\gamma}{1-\gamma r_t}\,\frac{1}{k^2+1/a_s^2}\,\frac{1}{\gamma^2}\left(1-\gamma\,\frac{r_s+r_t}{2}\right)\left(1-\frac{\hat{q}\cdot\hat{q}'}{3}\right). \quad (5)$$

The last approximation is fairly good since $1/a_s \ll \gamma$ and $\gamma(r_s+r_t)\lesssim 1$. Noting finally that $r_s \approx r_t$ allows one now to infer quite physically the relative unimportance of the ranges. Without the subsequent simplifying approximation, Eq. (5) contains a "range correction" of 5%.

THE PHASE-SPACE FACTOR

This brings us to the phase-space factor, the central point of this paper:

$$\frac{dN}{dp_1 d\Omega_1 d\Omega_3} = \frac{1}{(2\pi)^6}\,\frac{E_2 E_3 p_1^2 p_3^3}{(E_2+E_3)p_3^2 - E_3\vec{p}_3\cdot(\vec{p}-\vec{p}_1)} \quad (6)$$

is an exact expression for the density of states available to three particles of total momentum $\vec{p}$. With a stationary target of mass M_D, the total incident energy shall be called $Q+M_D$ and the incident momentum $\vec{q}$. The equation

$$M_D + m_q = \sum_{k=1}^{3} m_k - B \quad (7)$$

defines a reaction energy B. The conservation laws are

$$Q + M_D = \sum_{k=1}^{3} E_k \text{ and } \vec{q} = \vec{p}\,.$$

For the sake of approximating, below we note the relation

$$E_2 - m_2 + E_3 \leq T - (B - m_3)\,,$$

where T is the incident kinetic energy.

For incoming neutrinos $m_q = 0$, with energies within the fission spectrum and outgoing neutrinos or electrons as particle 3, we have

$$q - (B - m_3) \lesssim M/100\,,$$

with M the "nucleon mass," and thus to within 1%

$$\frac{dN}{dp_1 d\Omega_1 d\Omega_2} = \frac{1}{(2\pi)^6}\,\frac{M E_3 p_1^2 p_3^3}{M p_3^2 - E_3\vec{p}_3\cdot(\vec{q}-\vec{p}_1)}\,.$$

$B - m_3$ is the magnitude of the binding energy of the target. Noting that also within 1% (henceforth E_1 means $E_{1\,\mathrm{kin}}$)

$$p_1^2 dp_1 = (2M^3E_1)^{1/2} dE_1\,,$$

we obtain for incoming neutrino and outgoing lepton

$$\rho = \frac{1}{(2\pi)^6}\,\frac{E_3 p_3^3 (2M^3E_1)^{1/2}}{p_3^2 - E_3\vec{p}_3\cdot(\vec{q}-\vec{p}_1)/M}\,. \quad (8)$$

In particular, for $m_3 = 0$ [$\vec{p}_3 \equiv \vec{q}'$, see Eq. (5)]

$$\rho = \frac{1}{(2\pi)^6}\,\frac{q'^2 (2M^3E_1)^{1/2}}{1 - \hat{q}'\cdot(\mathrm{q}-\mathrm{p}_1)/M}\,. \quad (9)$$

Since our independent variables are an outgoing nucleon energy, the associated direction, and the outgoing lepton direction, we need a suitable expression for the outgoing lepton energy. We use

$$E_3 = q - B - \frac{p_1^2 + p_2^2}{2M}\,,$$

where (10)

$$\vec{p}_2 = \vec{q} - \vec{p}_1 - \vec{p}_3\,.$$

With $m_3 = 0$ we obtain, again to within 1%,

$$q' = \left(q - B - 2E_1 + \frac{p_1 q}{M}\,\mu_1 - \frac{q^2}{2M}\right)\Big/\left(1 + \frac{p_1}{M}\,\mu_{13}\right)\,, \quad (11)$$

where μ_1 and μ_{13} stand for the cosines of the angles between the incoming direction and that of particle 1, and between the directions of particles 1 and 3, respectively. From Eqs. (10) or (11) we find the condition on μ_1,

$$\mu_1 \geq \frac{M(q - B - 2E_1) - q^2/2}{q(2M_1)^{1/2}}\,. \quad (12)$$

For a chosen q the maximum nucleon energy is

$$E_{1\,\max} = \frac{q-B}{2} + \frac{q}{2}\left(\frac{q-B}{M} - \frac{q^2}{4M^2}\right)^{1/2}, \quad (13)$$

corresponding to the momentum

$$p_{1\,\max} = \frac{q}{2} + \left[M(q-B) - \frac{q^2}{4}\right]^{1/2},$$

the minimum being zero as long as $q \gtrsim B(1+B/2M)$, an extremely mild condition.

Instead of as in Eq. (10), the final neutrino energy may also be expressed by

$$q' = q - B - E_r - E_c , \tag{14}$$

where E_r is the previously mentioned relative energy of the two nucleons [see Eqs. (2), (3), and (5)], and E_c the energy of their c.m. motion, $E_c = p_c^2/4M$. On the relation (14) one can build the phase-space factor

$$\frac{1}{(4\pi)^2}\frac{dN}{dE_r} = \frac{1}{(2\pi)^6}\frac{1}{2}\frac{q'^2(M^3E_r)^{1/2}}{1-\hat{q}'\cdot(\vec{q}-\vec{q}')/2M} . \tag{15}$$

The distributions in E_r and E_1 [see Eq. (9)] are similar. Let us look at the differences: (1) The anisotropy of the denominator on the right of Eq. (9) is relatively weak, though it may reach several percent; that in Eq. (15) is negligible even by our present standards [Eq. (15) is based on a "reduced nucleon," which, of course, is not involved in the momentum balance]. (2) q' is effectively a much simpler function of E_r than of E_1 [see Eq. (11)], since, again to within 1%,

$$q' = q - B - E_r - q^2/4M . \tag{16}$$

However, when E_r and $2E_1$ differ appreciably, which they can at their smaller values, q' is hardly affected by either one. On the other hand, large nucleon energies, necessitating dominating antiparallel components of the nucleon momenta, make the percentage difference of E_r and $2E_1$ small. Besides, for very large nucleon energies, q'^2 and with it the phase-space factor become quite small. For these reasons the reduced nucleon spectrum has previously been regarded as representative of the nucleon, specifically the proton spectrum (see Ahrens and Lang, Ref. 3). But, in that procedure, clearly no meaningful error bars can be given, Not only are we dealing with the complex comparison of the two phase-space factors, the matrix element also enters the picture, and it contains, due to the final state interaction, the low-energy sensitive term $(E_r+E_s)^{-1}$. What can be said is that for very low proton energies the rigorous differential energy cross section will be considerably smaller than that based on dN/dE_r [see Eq. (15)] and the identification $E_r = 2E_1$. This is made explicit by considering the following equation:

$$2E_1 - E_r = \frac{\vec{p}_1}{M}\cdot\vec{p}_c - \frac{p_c^2}{4M} . \tag{17}$$

For small E_1 the asymmetry of E_r about E_1 is appreciable. Thus for $q = 4$ MeV and at $E_1 = 25$ keV we obtain 17 keV $\leq E_r \leq$ 99 keV. By comparison, we get at $E_1 = 500$ keV, 850 keV $\lesssim E_r \lesssim$ 1150 keV.

Thus, *to obtain a reliable cross-section differential with respect to the proton energy is to evaluate the integral on the right of Eq.* (1) *with the energy E equated to the energy E_1 of the phase-space considerations.* The energy E_r occurring in the matrix element must therefore be expressed in terms of the integration variables,

$$E_r = \frac{q-B}{2} + E_1 - \frac{qp_1\mu_1}{2M} - \frac{q'}{2}\left(1 - \frac{p_1\mu_{13}}{M}\right) . \tag{18}$$

Because E_r may be (and most likely is)[3] the small difference of large quantities

$$E_r = q - B - q' - E_c ,$$

it becomes necessary to draw on Eq. (10) rather than Eq. (11):

$$q' = \frac{q - B - 2E_1 + q(p_1\mu_1 - q/2)/M}{1 + (p_1\mu_{13} - q\mu_3 + q'/2)/M} . \tag{19}$$

The desired accuracy is obtained by iterating once.

The differential cross section as given in Eq. (1) refers to monoenergetic incident antineutrinos. If one wishes to integrate over an incident spectrum, then the lower limit on q for a particular nucleon (in particular proton) energy is

$$q = M + p_1 - [(M+p_1)^2 - 2(MB + p_1^2)]^{1/2}$$
$$\simeq (B+E_1)(1 - p_1/M) + E_1 . \tag{20}$$

The numerical integrations here were carried out twice, that is, by two different methods. One program code was a PASCAL adaption of a Monte Carlo multidimensional quadrature with adaptive stratified sampling, originally published in Algol 60.[5] The second method (in FORTRAN) used adaptive quadrature, that, for multiple dimensions, hybridizes Gauss, Tchebychef, Newton-Cotes, and Simpson quadratures. The calculations were each done twice, in the belief that this redundancy is more likely to turn up errors. The two methods seldom differed by as much as 1%, and the consistency across methods was as good as the consistency within. The computations were carried out on a Cyber 74 (CDC 6400/6600).

CROSS SECTIONS

A previous evaluation of the differential cross section $d\sigma/dE_r$ with and without effective ranges [see Eq. (2)] demonstrates the relative unimportance of the ranges: For energies between 50 keV and 1.5 MeV the values $r_s = 1.22 \times 10^{-2}$/MeV and $r_t = 8.67 \times 10^{-3}$/MeV result in a range correction everywhere less than 2%; for comparison see Eq. (5) and the subsequent remarks. The total cross sections from $q = 2.5$ MeV on up are never affected by more than 1.5%.[6] Therefore, in this work, the calculations of $d\sigma/dE$ are based on the matrix ele-

TABLE I. Differential cross section as a function of proton (or neutron) energy E for incident antineutrinos of energy q and for equilibrium fission antineutrinos. Energies in MeV, cross sections in 10^{-45} cm^2/MeV.

E \ q	2.5	3.0	4.5	7.0	10	Fission
0.01	3.06	26.9	231	920	2030	26.4
0.02	3.00	29.5	266	1110	2560	30.5
0.04	2.11	26.9	265	1170	2890	30.7
0.08	0.653	18.4	218	1030	2730	25.2
0.16		7.50	138	722	2040	16.2
0.32		0.453	60.0	387	1170	7.10
0.64			11.2	139	490	1.85
1.28				22.1	124	0.166
2.56				0.171 (−2)	10.1	0.175 (−2)

ment with zero ranges,

$$|\mathfrak{M}|^2 = \frac{8\pi}{M^{3/2}} \frac{B^{1/2}(B^{1/2}+E_s^{1/2})^2}{(E_r+E_s)(E_r+B)^2}, \qquad (21)$$
$$E_s = 1/Ma_s^2.$$

It may be of interest that the final-state interaction introduces the factor $(B^{1/2}+E_s^{1/2})^2/(E_r+E_s)$. For representative values of the differential cross section see Table I. For integral cross sections see Table II. In Table III is listed the quantity of specific experimental interest, the integral cross section with the detection-threshold energy of the proton as the lower limit.

The coupling constant used in the computations is

$$G_A = -1.49 \times 10^{-11}\ \text{MeV}^{-2}. \qquad (22)$$

This value is based on the nuclear-β-decay vector coupling constant $G_V' = 1.42 \times 10^{-49}$ erg cm^3, the ratio of β-decay coupling constants $G_A'/G_V' = -1.25$, and the Cabibbo angle θ whose sine is 0.238.[7] Using the relation

$$G_A = G_A'/\cos\theta$$

conforms with the theory of Glashow *et al.*[8] Recent observations on the branching ratios of D-meson decays[9] have given crucial support to this theory. The other constants used are

$$E_s = 1/Ma_s^2 = 0.0738\ \text{MeV}$$
and $\qquad (23)$
$$B = 2.23\ \text{MeV}.$$

The antineutrino spectrum due to equilibrium fission is more recent than that in Ref. 2.[10] Some representative values are recorded in Table IV.

Comparison of the values in Table I and further results not exhibited here with values of $d\sigma/dE_r$ interpreted with the identification $E_r = 2E$ shows the following: For the incident neutrino energies considered here, the two kinds of differential cross section intersect near $E = 100$ keV. This energy increases slowly with increasing q. For $q = 6$ MeV, e.g., the overestimate due to use of the approximate cross section amounts to 12% at $E = 12.5$ keV, the underestimate 21% at 1.60 MeV.

The cross sections appearing in Tables I and III have not been calculated previously. Those of Table II have, since the complete integration does

TABLE II. Integral cross section for monoenergetic and for fission antineutrinos.

σ (10^{-45} cm^2)	q (MeV)
0.172	2.5
3.41	3.0
59.3	4.5
362	7.0
1120	10.0
7.27	Fission
7.36	3.25

TABLE III. The differential cross section for fission antineutrinos $\langle d\sigma\rangle/dE$ integrated from the lower proton (neutron) energy limit E_{th} to $E_{\max}$.

$\int_{E_{\text{th}}} \langle d\sigma\rangle$ (10^{-45} cm^2)	E_{th} (MeV)
0.0204	1.5
0.123	1.0
0.259	0.8
0.551	0.6
0.812	0.5
1.21	0.4
1.84	0.3
2.28	0.25
2.86	0.2
3.60	0.15
4.05	0.125
4.57	0.1
5.18	0.075
5.87	0.05
6.64	0.025

TABLE IV. Number of antineutrinos per MeV due to the equilibrium fission of ^{235}U, normalized to 6.1 per fission (Ref. 10).

No./MeV	q (MeV)
1.338	2.0
0.9875	2.5
0.7301	3.0
0.3356	4.0
0.1341	5.0
0.4752 (−1)	6.0
0.3797 (−2)	8.0
0.3450 (−3)	10.0
0.4544 (−5)	12.0

not require the *complex* intermediate step of obtaining cross sections differential with respect to a nucleon (e.g., the proton) energy. Thus we are able to compare our σ (fission spectrum) with previously computed values. Limiting ourselves to recent computations, one of us, in collaboration with Lang, obtained 6.6×10^{-45} cm^2,[3] Lee obtained 7.1×10^{-45} cm^2,[11] Avignone and Greenwood obtained 7.4×10^{-45} cm^2,[2] and Avignone, using his latest $\bar{\nu}$ spectrum, obtained 6.5×10^{-45} cm^2.[10] The difference between our 7.3×10^{-45} cm^2 and Avignone's value can be completely accounted for by the difference in coupling constants used.

To summarize, within the approximation of an S-state deuteron, the results obtained here and exhibited in the various tables and figures should, conservatively speaking, be accurate to within at least 5%. This statement implies, of course, that the fission spectrum, whenever used as input, is taken at face value. Judging by its history, this spectrum may still lead to errors of between 10 to 15%; once one has confidence in an accuracy of better than 10%, the publication of deuteron disintegration calculations involving finer details such as the D-state admixture and weak magnetism becomes indicated.

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