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Soluble gauge theory of a noncompact group

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It is shown that the gauge theory of the multiplicative group of complex numbers is the theory of two free vector mesons only one of which is massless.

It was recently shown to be possible to construct a gauge theory of a noncompact internal-symmetry group.[1] In this paper, one of the simplest of these quantum field theories, the pure-gauge theory of the multiplicative group of complex numbers, is solved exactly.

The Lagrange density of the gauge theory of the general linear group of all complex, nonsingular $n\times n$ matrices, $\mathrm{GL}(n,C)$, is

$$L=-(2e)^{-2}\ \mathrm{tr}(F^{\dagger}_{\mu\nu}gF^{\mu\nu}g^{-1}) + \tfrac{1}{2}(2f)^{-2}\ \mathrm{tr}(g_{;\mu}g^{-1}g^{;\mu}g^{-1})+\varphi^{\dagger}_{;\mu}g\varphi^{;\mu}-V(\varphi^{\dagger}g\varphi), \tag{1}$$

where the curvature tensor $F_{\mu\nu}$ is

$$F_{\mu\nu}=A_{\mu,\nu}-A_{\nu,\mu}+[A_\mu,A_\nu] \tag{2}$$

and the covariant derivatives of the matter field φ and of the internal metric tensor g are

$$\varphi_{;\mu}=\varphi_{,\mu}-A_\mu\varphi \tag{3}$$

and

$$g_{;\mu}=g_{,\mu}+gA_\mu+A^{\dagger}_{\mu}g\,. \tag{4}$$

The potential $V\geqslant 0$ represents a self-interaction of φ, the subscript, μ means $\partial/\partial x^\mu$, and e and f are coupling constants.

Under the gauge transformation associated with the matrix $a(x)$ in $\mathrm{GL}(n,C)$, the n complex scalar fields $\varphi_i(x)$ transform as

$$\varphi'_i(x)=a_{ij}(x)\varphi_j(x)\,. \tag{5}$$

The Lagrange density L is invariant under this transformation because the internal metric tensor $g(x)$ and the connection $A_\mu(x)$ transform as

$$g'(x)=a^{-1}(x)^{\dagger}g(x)a^{-1}(x) \tag{6}$$

and as

$$A'_\mu(x)=a(x)A_\mu(x)a^{-1}(x)+a(x)_{,\mu}a^{-1}(x)\,. \tag{7}$$

The problems[2] usually associated with gauge theories of noncompact groups, negative probability and negative energy, are avoided here through the use of the internal metric tensor g. The tensor g is Hermitian and non-negative and may be regarded as the product of a more fundamental Hermitian matrix h with its adjoint, $g=h^{\dagger}h$. Although the matrix $g(x)$ transforms (6) as a tensor in the internal space, it is composed of fields $g_{ij}(x)$ that are scalars under Lorentz transformations.

By linearizing the Lagrange density L, one may show that the n^2 complex vector mesons A_μ separate into n^2 real massless vector mesons C_μ, associated with the group $\mathrm{U}(n)\subset\mathrm{GL}(n,C)$, and n^2 real massive vector mesons W_μ of equal mass, $M_W=e/f$, associated with the noncompact part of $\mathrm{GL}(n,C)$. The longitudinal components of the massive vector mesons W_μ are supplied by the n^2 real scalar fields that make up the internal metric tensor g, with which no other physical particles are associated. These features of the linearized theory are explicitly exhibited by the exact solution for the group $\mathrm{GL}(1,C)$.

The simplest version of the theory described by the Lagrange density L occurs for $n=1$ when the matter fields φ are omitted. Then both the non-Abelian character and the matrix structure of the theory are absent, and L becomes

$$L=-(2e)^{-2}F^{\dagger}_{\mu\nu}F^{\mu\nu}+\tfrac{1}{2}(2f)^{-2}[g_{;\mu}g^{-1}g^{;\mu}g^{-1}]\,. \tag{8}$$

The square brackets in this equation represent an operator ordering[3] that allows one to write

$$[g_{;\mu}g^{-1}]=[(g_{,\mu}+gA_\mu+A^{\dagger}_{\mu}g)g^{-1}] =A_\mu+A^{\dagger}_\mu+(\ln g)_{,\mu}\,. \tag{9}$$

This interpretation of the classical Lagrange density (1) makes the theory exactly soluble.

If the vector-meson fields W_μ and C_μ are defined as

$$W_\mu=\tfrac{1}{2}(A_\mu+A^{\dagger}_\mu+(\ln g)_{,\mu}) \tag{10}$$

and

$$C_\mu=\tfrac{1}{2}(A_\mu-A^{\dagger}_\mu)\,, \tag{11}$$

then since the gradient of the logarithm of g does not contribute to a curl, the curl $F_{\mu\nu}=A_{\mu,\nu}-A_{\nu,\mu}$ may equally well be considered to be the sum

$$F_{\mu\nu}=W_{\mu,\nu}-W_{\nu,\mu}+C_{\mu,\nu}-C_{\nu,\mu}\,. \tag{12}$$

The Lagrange density (8) then becomes

$$L=-(2e)^{-2}F^{\dagger}_{\mu\nu}F^{\mu\nu}+\tfrac{1}{2}f^{-2}W_{\mu}W^{\mu}\,, \qquad (13)$$

which describes two free vector mesons, W_μ and C_μ. The meson W_μ has mass

$$M_W=e/f \qquad (14)$$

and is associated with gauge invariance under the multiplicative group of real numbers. The meson C_μ is massless and is associated with the gauge group U(1).

When written in the form (8), this theory is apparently nonrenormalizable but manifestly symmetric under GL(1, C). When written in the form (13), it is soluble and finite but only the U(1) part of the symmetry is obvious.

For $n>1$ the noncompact and non-Abelian aspects of the GL(n, C) gauge theory become intertwined and no exact solution is available. One may impose the ghost-free gauge condition, $g=1$, and fix the remaining compact gauge freedom, but it is not known whether the resulting perturbation theory would be renormalizable. However, if gauge theories of noncompact, non-Abelian groups are renormalizable or otherwise implementable, then they might be used to form unified theories of the strong and electro-weak interactions, for they possess an intrinsic geometrical mechanism that gives a mass to some of the vector mesons (namely those associated with the generators of the noncompact part of the group). Existing unified gauge theories use the Higgs mechanism, which is somewhat arbitrary and unmotivated, for that purpose.

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[1]K. Cahill, Phys. Rev. D 18, 2930 (1978).

[2]S. L. Glashow and M. Gell-Mann, Ann. Phys. (N.Y.) 15, 437 (1961).

[3]If quantization is carried out in the temporal gauge, $A_0=0$, then A_i and g commute and the brackets refer only to the operator ordering of g and $\dot{g}$ which can be arranged so that (9) follows.