

## Lattice models of quark confinement at high temperature

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We consider the behavior of lattice models of quark confinement at high temperature. We find that confinement is strictly a low-temperature phenomenon. At high temperatures a transition to a plasma-like phase occurs. In this phase free gluons form a plasma which Debye screens the quarks.

### I. INTRODUCTION

The properties of matter under extreme conditions of temperature and pressure have always appealed to the curiosity of chemists and physicists. Currently we know enough about the basic constituents of matter to understand temperatures corresponding to electron-positron pair production, i.e.,  $10^{11}$  K. At about  $10^{12}$  K hadrons begin to be produced. As long as not too many are produced, it is reasonable to use the ordinary hadrons as fundamental constituents. Eventually the energy density will become comparable to that within a hadron. Then we must cease describing matter in terms of protons, neutrons, pions, etc. and substitute the degrees of freedom of a more fundamental theory. We assume that this means quantum chromodynamics (QCD).

Quantum chromodynamics is a theory of strongly interacting colored quarks and gluons. It is believed to have the property of quark confinement. That is to say, only color-neutral systems have finite energy. The ionization energy for separating a colored subsystem from an overall neutral system is infinite.

The quark-confining property is known to be true in strongly coupled lattice approximations to QCD but only at zero temperature.<sup>1,2</sup> We shall see that at sufficiently high temperatures lattice QCD undergoes a transition to an unconfined phase. Roughly speaking, the colored gluons form a plasma which Debye screens the color of the quark and therefore destroys the long-range confining potential.

In Sec. II we review the principles of Hamiltonian lattice gauge theory.<sup>2</sup> In Sec. III the finite-temperature behavior of strongly coupled Abelian lattice gauge theory is studied. We find that for low temperature the confining force is the usual linear potential. However, a critical point separates this confined phase from a phase in which the force law is the ordinary Coulomb law. In Sec. IV we continue the study of the Abelian theory including a charge-carrying field. Non-Abelian

lattice chromodynamics is the subject of Sec. V. As in the Abelian theory a transition between confined and unconfined phases exists. This time the unconfined phase is characterized by a short-range force because the Coulomb force is screened by a plasma of gluons. The last section discusses the validity of the conclusions for continuum chromodynamics.

### II. LATTICE GAUGE THEORY

Consider a simple cubic lattice in 3 space dimensions whose sites are labeled by triplets of integers

$$r = x, y, z = \text{site}.$$

Directed links of the lattice are indicated by a site and one of 6 unit lattice vectors called  $\hat{n}_x, \hat{n}_y, \hat{n}_z, \hat{n}_{-x}, \hat{n}_{-y}, \hat{n}_{-z}$ :

$$(r, \hat{n}) = \text{link}.$$

Each link of the lattice has a degree of freedom  $V(r, \hat{n})$  which for the Abelian theory is a phase

$$V = e^{i\phi} \text{ (Abelian)}.$$

For the  $SU_N$  theory  $V$  is a special unitary  $N$ -dimensional matrix. Each link  $(r, \hat{n})$  has a mate which is just the same link but oppositely oriented. It is the link

$$(r + \hat{n}, -\hat{n}).$$

The two degrees of freedom  $V(r, \hat{n})$  and  $V(r + \hat{n}, -\hat{n})$  are related:

$$V(r + \hat{n}, -\hat{n}) = V^*(r, \hat{n}). \quad (1)$$

For the Abelian theory

$$\phi(r, \hat{n}) = -\phi(r + \hat{n}, -\hat{n}). \quad (2)$$

In the Abelian theory each link carries an electric flux  $E(r, \hat{n})$  which satisfies

$$E(r, \hat{n}) = -E(r + \hat{n}, -\hat{n}), \quad (3)$$

$$[\phi(r, \hat{n}), E(r, \hat{n})] = i. \quad (4)$$

$E(r, \hat{n})$  is the conjugate momentum to  $\phi$  and since

$\phi$  is an angle  $E$  has the integers for its spectrum,

$$E(\mathbf{r}, \hat{n}) = \text{integer}. \quad (5)$$

In the non-Abelian theory  $E$  is a member of the adjoint representation of  $SU_N$ . For our example we will work with  $SU_2$  so that  $E$  is a 3-vector. It satisfies

$$\begin{aligned} [E^\alpha(\mathbf{r}, \hat{n}), E^\beta(\mathbf{r}, \hat{n})] &= i\epsilon^{\alpha\beta\gamma} E^\gamma(\mathbf{r}, \hat{n}), \\ [E^\alpha(\mathbf{r}, \hat{n}), E^\beta(\mathbf{r}', \hat{n}')] &= 0, \quad (\mathbf{r}, \hat{n}) \neq (\mathbf{r}', \hat{n}'). \end{aligned} \quad (6)$$

Thus the  $E$ 's are angular momenta and have the spectrum of integer and  $\frac{1}{2}$  integer angular momenta. The  $E$ 's generate left and right group transformations on the  $V$ 's,

$$[E^\alpha(\mathbf{r}, \hat{n}), V(\mathbf{r}, \hat{n})] = \tau^\alpha V(\mathbf{r}, \hat{n}), \quad (7)$$

where  $\tau^\alpha$  are the Pauli matrices.

The  $E$ 's do not satisfy Eq. (3) but rather

$$E(\mathbf{r} + \hat{n}, -\hat{n}) = -V(\mathbf{r}, \hat{n}) E(\mathbf{r}, \hat{n}) V^\dagger(\mathbf{r}, \hat{n}), \quad (8)$$

where

$$E = E^\alpha \tau^\alpha. \quad (9)$$

However, the  $E$ 's do satisfy

$$E(\mathbf{r}, \hat{n})^2 = E(\mathbf{r} + \hat{n}, -\hat{n})^2. \quad (10)$$

In order to express the Hamiltonian we will need a set of operators to identify with the magnetic field energy. Thus consider an elementary square of the lattice bounded by the directed links 1, 2, 3, 4 as in Fig. 1. We label such a square  $\Gamma$  and define (Abelian theory)

$$\begin{aligned} V(\Gamma) &= V(1)V(2)V(3)V(4) \\ &= e^{i[\phi(1)+\phi(2)+\phi(3)+\phi(4)]} \end{aligned} \quad (11)$$

and (non-Abelian theory)

$$\text{Tr } V(\Gamma) = \text{Tr } V(1)V(2)V(3)V(4). \quad (12)$$

The Hamiltonian for Abelian lattice gauge theory (LGT) is

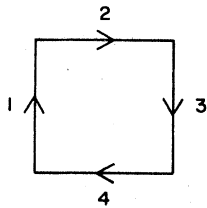


FIG. 1. An elementary lattice square.

$$\begin{aligned} H &= \sum_{\text{links}} \frac{g^2 E^2}{2a} \\ &\quad - \sum_{\text{boxes}} \frac{1}{2ag^2} [V(\Gamma) + V^\dagger(\Gamma)], \end{aligned} \quad (13)$$

where  $\sum_{\text{links}}$  and  $\sum_{\text{boxes}}$  indicate sums over undirected and unoriented links and boxes. For non-Abelian LGT

$$\begin{aligned} H &= \sum_{\text{links}} \frac{g^2 E^2}{2a} \\ &\quad - \sum_{\text{boxes}} \frac{1}{ag^2} \text{Tr} [V(\Gamma) + V^\dagger(\Gamma)]. \end{aligned} \quad (14)$$

The space of states includes unphysical states which are purged by applying a subsidiary condition. The physical subspace consists of vectors  $|\psi\rangle$  satisfying

$$\nabla_r \cdot E = \sum_n E(\mathbf{r}, \hat{n}) |\psi\rangle = 0 \quad \text{all } \mathbf{r}, \quad (15)$$

where  $\sum_n$  indicates a sum over the 6 lattice directions. Equation (15) is the lattice version of Gauss's law which says that the total flux leaving a site must equal zero. It is modified when sources are included.

### III. ABELIAN GAUGE THEORY AT FINITE TEMPERATURE

We shall argue later that if a transition to an unconfined phase occurs the transition temperature is bounded from above by ignoring the magnetic term in  $H$ . In other words, if we find a transition to the unconfined phase when  $H_{\text{magnetic}}$  is ignored then the full Hamiltonian surely has one at a lower  $T_c$ . For now we shall simply work in the strong-coupling approximation in which

$$H = \sum_{\text{links}} \frac{g^2}{2a} E(\mathbf{r}, \hat{n})^2. \quad (16)$$

In this approximation confinement is rigorous at zero temperature.<sup>2</sup>

The calculation of the partition function for the Hamiltonian (16) would be trivial if we did not have to impose Gauss's law on the allowable states. We define

$$\begin{aligned} Z(\beta) &= \sum' \exp(-\beta H) \\ &= \sum' \exp\left(-\beta \sum \frac{g^2}{2a} E^2\right), \end{aligned} \quad (17)$$

where  $\sum'$  means that the sum over states is restricted to those states satisfying (15).

We may impose Gauss's law by introducing  $\delta$  functions of total flux at each site. Thus introduce a factor for each site

$$\delta(\nabla_r \cdot E) = \delta\left(\sum_{\hat{n}} E(r, \hat{n})\right) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp\left[i\alpha \sum E(r, \hat{n})\right]. \quad (18)$$

Accordingly

$$Z(\beta) = \sum_E \prod_{\text{links}} \exp\left(-\frac{\beta g^2}{2a} E^2\right) \prod_{\text{sites}} \int_{-\pi}^{\pi} d\alpha(r) \exp\left[i\alpha(r) \sum E(r, \hat{n})\right]. \quad (19)$$

In (19) we have dropped an insignificant factor of  $1/2\pi$  for each site.

By an obvious manipulation we may rewrite  $Z$  in the form

$$Z(\beta) = \int_{-\pi}^{\pi} \prod_r d\alpha(r) \prod_{\text{links}} \left[ \sum_E \exp\left(-\frac{\beta g^2}{2a} E(r, \hat{n})^2 + i[\alpha(r) - \alpha(r + \hat{n})] E(r, \hat{n})\right) \right]. \quad (20)$$

To understand Eq. (20) better we introduce the so-called periodic Gaussian function

$$\sum_m e^{-\gamma(\phi + 2m\pi)^2} \equiv \tilde{e}^{-\gamma\phi^2}. \quad (21)$$

It consists of a periodic superposition of Gaussians and closely resembles the function

$$e^{-2\gamma} \exp(2\gamma \cos \phi) \quad (22)$$

for large  $\gamma$ .

To express (20) in terms of periodic Gaussians we use the identity

$$\sum_E e^{cE^2 + i\alpha E} = \left(\frac{\pi}{c}\right)^{1/2} \tilde{e}^{-(1/4c)\alpha^2}, \quad (23)$$

giving

$$Z(\beta) = \int_{-\pi}^{\pi} d\alpha(r) \prod_{\text{links}} \left\{ \tilde{e}^{-(a/2\beta g^2)[\alpha(r) - \alpha(r + \hat{n})]^2} \times \left(\frac{2\pi a}{\beta g^2}\right)^{1/2} \right\}. \quad (24)$$

The numerical factor  $(2\pi a/\beta g^2)^{1/2}$  for each link may be ignored for our purposes. The remaining structure defines the well-known Villain approximation to the planar Heisenberg model<sup>6</sup> defined by

$$Z = \int_{-\pi}^{\pi} d\alpha(r) \prod_{\text{links}} \exp\left\{-\frac{a}{\beta g^2} \cos[\alpha(r) - \alpha(r + \hat{n})]\right\}. \quad (25)$$

Note that the usual *inverse temperature* of the Heisenberg magnet is  $-a/\beta g^2$ , which is proportional to the *temperature* of the original problem. In fact as the temperature of the lattice gauge world

increases the effective temperature of the magnetic model decreases leading to a transition to a magnetized or ordered state.

The Villain model is known to have the following properties: (1) For small  $a/\beta g^2$  the system is disordered. This means

$$\langle e^{i\alpha(0)} e^{-i\alpha(r)} \rangle \underset{r \rightarrow \infty}{\sim} e^{-\mu|r|}. \quad (26)$$

(2) For large  $a/\beta g^2$  the system is magnetized or ordered:

$$\langle e^{i\alpha(0)} e^{-i\alpha(r)} \rangle \rightarrow \text{const.} \quad (27)$$

In fact for very large  $a/\beta g^2$  spin-wave analysis shows the correlation to behave like

$$\langle e^{i\alpha(0)} e^{-i\alpha(r)} \rangle \rightarrow c \exp\left(\frac{\beta g^2}{r}\right). \quad (28)$$

Next let us consider the Abelian theory with a pair of static charges of magnitude  $\pm g$  at locations  $r=0$  and  $r=R$ . These sources are introduced by changing Gauss's law to read

$$\begin{aligned} \sum_{\hat{n}} E(r, \hat{n}) &= 0, \quad r \neq 0 \text{ or } R \\ \sum_{\hat{n}} E(0, \hat{n}) &= 1, \\ \sum_{\hat{n}} E(R, \hat{n}) &= -1. \end{aligned} \quad (29)$$

This is implemented by changing the  $\delta$  functions in (18) at  $r=0$  and  $r=R$ . The result is to introduce an extra factor of

$$\exp[i\alpha(0)] \exp[-i\alpha(R)]$$

into the integrands of (19), (20), and (24). Thus in the presence of sources we define

$$Z(\beta, R) = \int_{-\pi}^{\pi} d\alpha(r) \prod_{\text{links}} \tilde{e}^{-(a/2\beta g^2)[\alpha(r) - \alpha(r+n)]^2} \times e^{i\alpha(0)} e^{-i\alpha(R)} \\ = Z(\beta) \langle e^{i\alpha(0)} e^{-i\alpha(R)} \rangle. \quad (30)$$

Now consider the free energy of the system with sources (subtracting the free energy of the sourceless system):

$$W(\beta R) = -[\ln Z(\beta R) - \ln Z(\beta)]/\beta \\ = \frac{1}{\beta} \ln \langle e^{i\alpha(0)} e^{-i\alpha(R)} \rangle. \quad (31)$$

From Eq. (26) we see that for small temperature ( $1/\beta < g^2/a$ ) the potential is confining and linear:

$$W(\beta R) \rightarrow \frac{\mu}{\beta} |R|. \quad (32)$$

For high temperature ( $\beta > g^2/a$ ) Eq. (28) gives

$$W(\beta, R) \rightarrow \frac{g^2}{R}, \quad (33)$$

i.e., a normal Coulomb force.

#### IV. ABELIAN CHARGE-CARRYING FIELDS

The existence of charge-carrying fields modifies the results of Sec. III. The charge-carrying field we will use is particularly simple. At each site introduce a phase variable  $\phi(r)$ . The complex charge-carrying field at site  $r$  is  $\exp[i\phi(r)]$ . The

charge in units of  $g$  is the canonical conjugate to  $\phi$  which we call  $Q(r)$ .  $Q(r)$  has integer spectrum.

The energy stored in the field  $\phi$  will be taken to be

$$mQ^2,$$

where  $m$  can be thought of as the mass of a singly charged site. In a more refined model, gauge-invariant hopping terms can be added to  $H$  to allow the charges to move through the lattice.

The new model Hamiltonian is

$$H = \sum_{\text{links}} \frac{g^2}{2a} E(r, n)^2 + m \sum_{\text{sites}} Q(r)^2 \quad (34)$$

and Gauss's law must be modified to account for the sources on the sites:

$$\nabla_r \cdot E = Q(r). \quad (35)$$

The partition function in Eq. (19) is modified in two ways. First the  $\delta$  function is modified to

$$\delta \sum_n E(r, n) - Q(r) \quad (36)$$

and second we must include the factor

$$\prod_{\text{sites}} e^{-\beta m Q^2(r)}. \quad (37)$$

Thus

$$Z(\beta) = \sum_{E, Q} \prod_{\text{links}} \exp\left(-\frac{\beta g^2}{2a} E^2\right) \prod_{\text{sites}} e^{-\beta m Q^2} \int_{-\pi}^{\pi} d\alpha(r) \exp\left[i\alpha(r) \sum_n E(r, n)\right] \exp[-i\alpha(r) Q(r)]. \quad (38)$$

Using Eq. (23) gives

$$Z(\beta) = \int d\alpha(r) \prod_{\text{links}} \tilde{e}^{-(a/2\beta g^2)[\alpha(r) - \alpha(r+n)]^2} \prod_{\text{sites}} \tilde{e}^{-(1/4\beta m)[\alpha(r)]^2}. \quad (39)$$

This is the Villain approximation to the planar Heisenberg model with an external field:

$$Z_{\text{Heisenberg}} = \int d\alpha \prod_{\text{links}} e^{(a/\beta g^2) \cos \alpha(r) - \alpha(r+n)} \prod_{\text{sites}} e^{(1/2\beta m) \cos \alpha(r)}. \quad (40)$$

As before, the temperature for the magnetic model in Eq. (40) is  $\beta g^2 a$  which is *inversely* proportional to the temperature of the lattice gauge theory. The external field is given by  $1/2m$  so that in the limit  $m \rightarrow \infty$  the charges have no

effect.

The properties of the Villain model in the presence of an external symmetry-breaking field are summarized by the phase diagram in Fig. 2.

In Fig. 2 the horizontal axis separates states

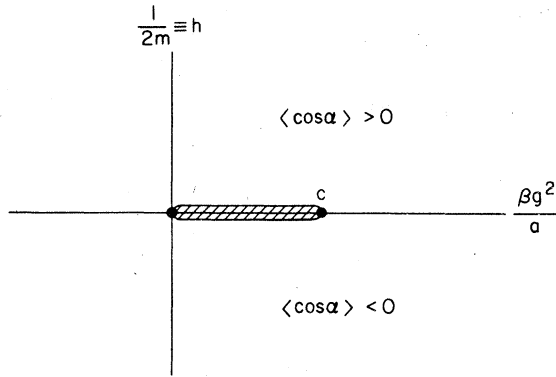


FIG. 2. Phase diagram for the Villain model in an external field.

in which the magnetization  $M (= \cos \alpha)$  is positive and negative. Along the line between the origin and the critical point labeled  $c$  there is spontaneous magnetization. The magnetization vanishes on the remainder of the horizontal axis. For various values of  $h$  the magnetization behaves like Fig. 3.

For  $h \neq 0$  the magnetization is a smooth function of  $a/\beta g^2$  and is always nonzero. The implications of this for the force between external sources is interesting. Equation (31) still applies. Since  $\langle e^{i\alpha(0)} e^{-i\alpha(R)} \rangle$  must tend to the square of the magnetization as  $R \rightarrow \infty$  and  $M$  does not equal zero for any  $a/\beta g^2$  (unless  $h = 0$ ) we find that the potential  $V(R)$  always tends to a finite limit as  $R \rightarrow \infty$ . Thus it might seem that there is no confinement of charge even at zero temperature. This is not the right interpretation as we shall see. But first let us consider the behavior of  $V(R)$  for large  $R$ .

In the presence of an external field the correlations in a magnet always behave like

$$[1 + c(R) e^{-mR}] M^2 \tag{41}$$

for large  $R$  where  $c(R)$  falls at most like a power. This gives

$$W(R) = C(R) e^{-mR} + \ln M^2, \tag{42}$$

i.e., a short-range Yukawa-type force law.

The result of a short-range force between sources even at zero temperature would seem to conflict with the known confining properties of our model. However, this is false. Consider a pair of sources separated by distance  $R$  on a lattice. When no quantized sources exist the force law at zero temperature is

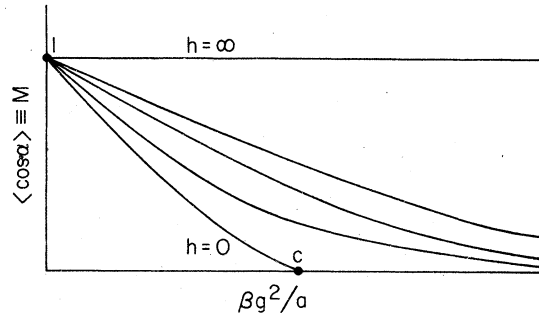


FIG. 3. Magnetization temperature at fixed external field.

$$W(R) \sim \frac{g^2}{2a^2} R \tag{43}$$

corresponding to a string of flux between the sources. However, if quantized sources exist a configuration with much less energy can exist. It is constructed by simply creating an opposite charge from the quantum source right on the external sources thus neutralizing them. This gives an energy  $2m$  which is consistent with the magnetic picture. Thus it seems that for  $m \neq \infty$  a sharp transition between confined and unconfined phases *does not exist*. However, we do believe that there is a very large qualitative difference between low and high temperatures. At high temperatures we think the long-range Coulomb force found previously is screened as in a plasma. This is very different from the pairing which takes place in the confined phase. For example, if a fractional charge is immersed in a plasma it is screened just as effectively as an integer charge. Therefore it would be very interesting to embed external charges of fractional magnitude in the high-temperature quark configuration and see if the long-range forces are screened.

Unfortunately the formalism we are using does not permit fractional charges. This is because the  $E$  fields are integers and therefore

$$\nabla_r \cdot E = \text{integer}.$$

However, we can replace the quantized charged described by the field  $\phi$  by sources with two units of charge. It then becomes interesting to ask if the high-temperature configuration of doubly charged objects can screen a singly charged external source.

The model Hamiltonian is identical with Eq. (34) but now Gauss's law becomes

$$\nabla_r \cdot E = 2Q(r) \tag{44}$$

and Eqs. (39) and (40) are replaced by

$$Z(\beta) = \int d\alpha \prod_{\text{links}} \tilde{e}^{(a/2\beta g^2)[\alpha(r) - \alpha(r+n)]^2} \times \prod_{\text{sites}} \tilde{e}^{-(1/16\beta m)[2\alpha(r)]^2}, \quad (45)$$

$$Z(\beta) = \int d\alpha \prod_{\text{links}} e^{(a/\beta g^2)\cos[\alpha(r) - \alpha(r+n)]} \times \prod_{\text{sites}} e^{(1/2\beta m)\cos 2\alpha(r)}. \quad (46)$$

Evidently the expression in Eq. (45) is periodic with period  $\pi$  instead of  $2\pi$ .

The symmetry-breaking field now has a discrete symmetry

$$\alpha(r) \rightarrow \alpha(r) + \pi; \quad (47)$$

at low temperature [high temperature for the Villain-Heisenberg (VH) model] this symmetry is not broken. Thus

$$\langle \cos \alpha(r) \rangle = 0, \quad (48)$$

$$\langle e^{i\alpha(r)} e^{-i\alpha(0)} \rangle \rightarrow \exp(-\mu r). \quad (49)$$

The external field does cause

$$\mathfrak{M} = \langle \cos 2\alpha(r) \rangle \neq 0$$

so that

$$\langle e^{2i\alpha(r)} e^{-2i\alpha(0)} \rangle \rightarrow \mathfrak{M}^2(1 + c e^{-\mu r}). \quad (50)$$

Equations (49) and (50) mean that at low temperature singly charged sources experience linearly forces but doubly charged sources do not. This is consistent with the usual confinement picture.

At high temperatures (low in the VH model) the system becomes magnetized,

$$M = \langle \cos \alpha \rangle \neq 0, \quad (51)$$

$$\langle e^{i\alpha(0)} e^{-i\alpha(r)} \rangle \rightarrow M^2(1 + c e^{-\mu r}), \quad (52)$$

thus resulting in the screened short-range force between singly charged sources. This type of charge neutralization is entirely different from the pairing of opposite charges to form bound neutral systems. It occurs in plasmas and is called Debye screening.

## V. NON-ABELIAN THEORY

For the non-Abelian theory we consider the model strong-coupling Hamiltonian

$$H = \frac{g^2}{2a} \sum_{\text{links}} E^\alpha(r, n) E^\alpha(r, n) \quad (53)$$

supplemented by the constraint

$$\nabla_r \cdot E^\alpha = \sum_n E^\alpha(r, n) |\psi\rangle = 0. \quad (54)$$

The operators  $\sum_n E^\alpha$  are algebraically angular momenta. To implement Eq. (54) we introduce projection operators at every site which project onto the zero-angular-momentum state. These operators are constructed as follows:

Parametrize the three-dimensional rotations by vectors  $\vec{\mathbb{I}}$ . The direction of  $\vec{\mathbb{I}}$  is the axis of rotation and the magnitude is the angle of rotation. If we form the operator

$$\mathcal{O}(r) = \int \exp\left(i \vec{\mathbb{I}}^\alpha \sum_n E^\alpha(r, n)\right) d\vec{\mathbb{I}}, \quad (55)$$

where  $d\vec{\mathbb{I}}$  is the group-invariant volume element, then Eq. (55) is the operator which projects out vectors satisfying Eq. (54).

The partition function is

$$Z(\beta) = \text{Tr} \prod_{\text{links}} \exp\left(-\beta \frac{g^2}{2a} E^2\right) \prod_{\text{sites}} \mathcal{O}(r) \\ = \text{Tr} \int d\vec{\mathbb{I}}(r) \prod_{\text{links}} \exp\left[-\frac{\beta g^2}{2a} E^2 - \vec{\mathbb{I}}(r) \cdot E(r, \hat{n}) - \vec{\mathbb{I}}(r + \hat{n}) \cdot E(r + \hat{n}, -\hat{n})\right]. \quad (56)$$

Let us now focus attention on a single link  $(r, \hat{n})$ . Define  $\vec{\mathbb{I}}(r) = \vec{\mathbb{I}}$ ,  $\vec{\mathbb{I}}(r + \hat{n}) = \vec{\mathbb{I}}'$ ,  $E(r, \hat{n}) = E$ ,  $E(r + \hat{n}, -\hat{n}) = E'$ . The link  $(r, \hat{n})$  contributes the factor

$$\text{Tr} \exp\left(-\beta \frac{g^2}{2a} E^2 - \vec{\mathbb{I}} \cdot E - \vec{\mathbb{I}}' \cdot E'\right). \quad (57)$$

The states of a single link are spanned by a basis

$$|j; a, b\rangle,$$

where  $j$  is an integer or half integer and  $a$  and  $b$  are "magnetic" quantum numbers. The magnetic quantum numbers each run over  $2j + 1$  values

$$a, b = j, j - 1, j - 2, \dots, -j.$$

The states  $|j, a, b\rangle$  are eigenvectors of  $E^2$ ,  $(E')^2$ ,  $E_3$ , and  $E'_3$ :

$$E^2 |j, a, b\rangle = (E')^2 |j, a, b\rangle = j(j + 1) |j, a, b\rangle, \tag{58}$$

$$E_3 |j, ab\rangle = a |j, ab\rangle, \quad E'_3 |j, ab\rangle = b |j, ab\rangle. \tag{59}$$

Thus the trace in Eq. (57) is a sum over all integer and  $\frac{1}{2}$  integer  $j$ ,

$$\sum_{j, a, b} \langle j, ab | \exp\left(-\beta \frac{g^2}{2a} E^2 - \vec{1}\vec{1} \cdot E - \vec{1}\vec{1}' \cdot E'\right) | j, ab \rangle = \sum_j \exp\left(-\beta \frac{g^2}{2a} j(j + 1)\right) x_j(l) x_j(l'), \tag{60}$$

where  $x_j$  is the character of the group element  $l$ .

$$x_j(\vec{1}) = \text{Tr} \exp(-i\vec{1} \cdot T_j), \tag{61}$$

where  $T_j$  are the angular momentum matrices for spin  $j$ . They are easily evaluated,

$$x_j(\vec{1}) = \frac{\sin(j + \frac{1}{2})l}{\sin \frac{1}{2}l}, \tag{62}$$

where  $l$  is the angle of rotation (magnitude of  $\vec{1}$ ). Thus Eq. (60) becomes

$$\frac{1}{\sin \frac{1}{2}l \sin \frac{1}{2}l'} \sum_j \exp[-\beta g^2 / 2aj(j + 1)] \sin(j + \frac{1}{2})l \sin(j + \frac{1}{2})l'. \tag{63}$$

The sum can be evaluated in terms of periodic Gaussians giving

$$\frac{e^{\beta/g^2/8a}}{\sin \frac{1}{2}l \sin \frac{1}{2}l'} \left\{ -\tilde{e}^{-(a/2\beta g^2)(l+l')/2} + \tilde{e}^{-(a/2\beta g^2)(l-l')/2} \right\}. \tag{64}$$

The functions

$$\tilde{e}^{(a/2\beta g^2)[(l \pm l')/2]}$$

are periodic in  $l \pm l'$  with period  $4\pi$ . The factor  $\exp(\beta g^2/8a)$  can be ignored as it will contribute a numerical factor to  $Z$  which will cancel when averages are computed. The factor in curly brackets will be denoted

$$F(l + l', l - l') = -\tilde{e}^{-(a/2\beta g^2)[(l+l')/2]} + \tilde{e}^{-(a/2\beta g^2)[(l-l')/2]}. \tag{65}$$

It is periodic in  $l \pm l'$  with period  $4\pi$  and is antisymmetric under interchange of  $l + l'$  and  $l - l'$ . It also has the property of being vanishingly small for large  $\beta$  unless

$$l = l' \text{ or } l = -l'. \tag{66}$$

The entire partition function can now be written

$$\int \prod_r dl(r) \sin^{-2}[\frac{1}{2}l(r)] \prod_{\text{links}} \frac{F(l(r) + l(r+n), l(r) - l(r+n))}{\sin \frac{1}{2}l(r) \sin \frac{1}{2}l(r+n)}. \tag{67}$$

The integrations over  $l$  must go from 0 to  $4\pi$  in order to cover the whole  $SU_2$  group space. The factor  $dl/sm^2l/2$  is the group-invariant volume element.

Before considering the possible phase transitions which can occur, let us consider the partition function in the presence of static sources. The sources are a pair of  $SU_2$  quarks of color  $\frac{1}{2}$ . They are described by  $\tau$  matrices and have color operators  $Q_1$  and  $Q_2$ . They are located at  $r = 0$  and  $r = R$ . Thus Gauss's law is modified to

$$\sum_n E(r, \hat{n}) = 0, \quad r \neq 0, R$$

$$\sum E(0, \hat{n}) = Q_1,$$

$$\sum E(R, \hat{n}) = Q_2,$$

and the partition function integrand has the extra factors

$$\begin{aligned} \text{Tr} e^{\mathbb{I}(0) \cdot Q_1} e^{\mathbb{I}(R) Q_2} &= x_{1/2} [l(0)] x_{1/2} [l(R)] \\ &= \cos \frac{1}{2} l(0) \cos \frac{1}{2} l(R). \end{aligned} \quad (68)$$

Following the logic of the Abelian model we find the potential between quarks to be

$$V(R) = \frac{-1}{\beta} \ln \langle \cos \frac{1}{2} l(0) \cos \frac{1}{2} l(R) \rangle. \quad (69)$$

The integrand of Eq. (67) has the symmetry

$$l \rightarrow l + 2\pi \quad (\text{all } r). \quad (70)$$

Unless this symmetry is broken

$$\langle \cos \frac{1}{2} l \rangle = 0 \quad (71)$$

since  $\cos \frac{1}{2} l$  changes sign under Eq. (70). In this case

$$\langle \cos \frac{1}{2} l(0) \cos \frac{1}{2} l(R) \rangle \rightarrow e^{-\mu R}$$

and the quark potential is linear. This is certainly the case for large  $\beta$ .

For small  $\beta$  the symmetry (70) is almost certainly broken. Let us see why. For small  $\beta$   $|F(l+l', l-l')|$  is vanishingly small except for  $l=l'$  or  $l=-l'+4\pi$ . Thus suppose  $l(0)=\phi$ . Then the neighboring sites must have

$$l = \phi \quad \text{or} \quad l = 4\pi - \phi.$$

Repeating the argument we find that to avoid vanishing small probabilities all the  $l$ 's must equal  $\phi$  or  $4\pi - \phi$ .

Now consider the factors  $1/[\sin \frac{1}{2} l(r)]^4$  for each site. Since  $\frac{1}{2} \sin \phi = \sin \frac{1}{2} (4\pi - \phi)$  we will have a factor

$$\left( \frac{1}{(\sin \frac{1}{2} \phi)^4} \right)^N,$$

where  $N$  is the total number of sites. This means that the values

$$\phi \approx 0 \quad (\text{same as } \phi = 4\pi)$$

or

$$\phi \approx 2\pi$$

are very strongly favored and will dominate the configurations.

The values  $l \approx 0$  and  $l \approx 2\pi$  will not both occur at different sites. The factors  $F$  enforce all the  $l$ 's to be almost the same (mod  $4\pi$ ). We have therefore a conventional spontaneous breaking of symmetry; the broken symmetry is Eq. (70). We see that in the spontaneously broken states

$$\langle \cos \frac{1}{2} l \rangle = \pm 1$$

for infinite temperature. Accordingly

$$\langle \cos \frac{1}{2} l(0) \cos \frac{1}{2} l(R) \rangle \rightarrow M^2 (1 + ce^{-\mu R})$$

and quarks are unconfined.

The mechanism for the short-range force undoubtedly involves the fact that the non-Abelian gauge field can carry color. Thermal fluctuations eventually cause the colored gluons to form a plasma which Debye screens the quark.<sup>4</sup>

## VI. CONCLUSIONS

We have considered the influence of thermal fluctuations on a particularly simple model of confinement. The model consists of dropping the magnetic part of the lattice-gauge-theory Hamiltonian and retaining only the electric terms. Two questions naturally arise. The first is: To what extent are the conclusions modified when the magnetic terms are included? The second and far more difficult is whether the phase transition occurs in the continuum limit.

The first question we feel is relatively easy. When the magnetic terms are turned on the critical temperature decreases. We have not proved this in general, but it seems obvious that eliminating the magnetic parts of  $H$  maximizes the confining tendencies of LGT. We have been working on various methods of incorporating  $H_{\text{magnetic}}$  and the message is always the same. If a phase transition to an unconfined phase is possible without  $H_{\text{mag}}$ , it is even easier with it. Thus we feel the following statement is correct:

Without  $H_{\text{mag}}$  if a phase transition occurs it is pure LGT at a temperature  $\beta_c^{-1}$  satisfying

$$\frac{\beta_c g^2}{2a} = \lambda^{-1} \quad \text{or} \quad T_c = \lambda \frac{g^2}{2a},$$

where  $\lambda$  is a number of order of magnitude 1.

When  $H_{\text{mag}}$  is turned on the critical temperature decreases,

$$T_c = \frac{g^2}{2a} \lambda f(g^2),$$

where  $f(g^2)$  is a fraction less than 1.

Now what about question number two? This unfortunately is much more difficult. The continuum limit of LGT involves letting  $g \rightarrow 0$  with  $a$  according to the rule

$$g^2(a) \rightarrow \frac{c}{-\ln a}$$

or

$$a \rightarrow e^{-c/g^2}.$$



The critical temperature behaves like

$$T_c = \lim_{g \rightarrow 0} \frac{\lambda g^2}{2e^{-c/g^2}} f(g^2)$$
$$= \lim_{g \rightarrow 0} \frac{\lambda g^2 e^{c/g^2}}{2} f(g^2).$$

Thus it is not possible to say with certainty whether

the transition occurs in the limiting theory without knowing the behavior of  $f(g^2)$  for  $g^2 \rightarrow 0$ .

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<sup>2</sup>J. Kogut and L. Susskind, Phys. Rev. D 11, 395 (1975).

<sup>3</sup>J. Villain, J. Phys. 36, 581 (1975).

<sup>4</sup>See M. B. Kislinger and P. D. Morley, Phys. Rev. D 13,

2765 (1976); 13, 2771 (1976) for a discussion of the plasma effect in perturbation theory for Yang-Mills theory.