Analogs of merons and meron pairs in the XY model

Daniel R. Stump
Physics Department, Indiana University, Bloomington, Indiana 47405
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A path-integral formula for the vacuum wave functional is used to show that merons in quantum chromodynamics produce long-range vacuum fluctuations of the form of Wu-Yang monopoles, whereas meron pairs or instantons produce only short-range fluctuations. The existence of long-range vacuum fluctuations may be related to the problem of quark confinement. The XY model, which describes classical two-component spins on a two-dimensional lattice, is reviewed. Vortex spin-field configurations, which are responsible for the existence of a disordered phase in the XY model, are shown to be analogous to merons in that they produce long-range fluctuations of the spins along a given axis. The analogy is emphasized by considering the analog of the vacuum wave functional in the XY model. It is argued that the vortex-antivortex pairs that occur in the ordered phase of the XY model are analogous to meron pairs or instantons in quantum chromodynamics.

I. MERONS IN QUANTUM CHROMODYNAMICS

Merons are gauge-field configurations in non-Abelian gauge theories in four-dimensional Euclidean space.[1] They satisfy the Euclidean field equations but are singular at isolated points and have infinite action. It has been suggested[1,2] that quark confinement in quantum chromodynamics (QCD) is a consequence of the contribution of merons to path integrals.

In this paper it will be shown that merons produce long-range vacuum fluctuations of the gauge fields. These vacuum fluctuations are precisely of the form of the Wu-Yang monopole configuration.[3] Therefore in an ionized-meron phase, i.e., a phase in which path integrals are dominated by meron configurations, the vacuum would contain arbitrary superpositions of monopole fields and would resemble the monopole vacuum described by Mandelstam.[4] It has been suggested that the existence of long-range vacuum fluctuations might account for quark confinement in QCD.[4,5]

The main purpose of this paper is to point out in some detail an analogy between the meron gauge-field configuration in QCD and the vortex spin-field configuration in the classical XY model. The XY model is a well understood model in statistical mechanics that describes classical two-component spins on a two-dimensional square lattice.[6-8] In this model vortex spin-field configurations play a special role in disordering the spins and lead to a phase transition between a disordered phase at high temperatures and a relatively more ordered phase at low temperatures.[7] The analogy between the vortex configuration and the meron solution in QCD is based on the observation that each of these configurations produces long-range fluctuations of the corresponding fields. In this paper the analogy will be drawn by examining in each theory the vacuum wave functional, defined below. The meron solution will be considered not in QCD but in the pure Yang-Mills theory with gauge group SU(2).

In addition it will be shown that the XY model is an example of a model in which instanton effects are non-negligible at suitable values of the temperature, in contrast to some other two-dimensional spin models considered recently by Witten.[9,10] The field configuration in the XY model that is analogous to an instanton in QCD is a vortex-antivortex pair.

The outline of the paper is as follows. In the remainder of Sec. I the vacuum wave functional will be defined and a path-integral formula for it derived. Also it will be shown that contributions to the path integral from merons produce long-range vacuum fluctuations of the gauge fields that are of the form of the Wu-Yang monopole field. In Sec. II well-known properties of the XY model will be reviewed. In order to make the analogy with QCD as precise as possible, the role of vortex configurations in producing long-range fluctuations of the spins will be discussed in terms of the analog of the vacuum wave functional in the XY model. Finally in Sec. III the parallels between merons in QCD and spin vortices in the XY model will be summarized. The analogy leads to speculations about the existence and nature of an ionized-meron phase in QCD that might explain quark confinement.[1]

The vacuum wave functional of a field theory is the Schrödinger wave function of the vacuum state of the theory. It is a functional of time-independent configurations of the fields. For example, consider a field theory of a self-interacting scalar field $\varphi(x)$ with action $S(\varphi)$. Let $|\Omega\rangle$ be the vacuum state of the theory and $|\eta(\vec{x})\rangle$ the eigenstate of the field operator at time $x_0=0$ with eigenvalue $\eta(\vec{x})$.

The vacuum wave functional $\Psi(\eta)$ is defined as

$$\Psi(\eta)=\langle\eta|\Omega\rangle\,. \tag{1.1}$$

$\Psi(\eta)$ is the probability amplitude that the field $\varphi(x)$ at time $x_0=0$ is in the configuration $\eta(\vec{x})$ in the vacuum state. In this paper the argument $\eta(\vec{x})$ will be called a vacuum fluctuation of the field $\varphi(x)$.

It is useful to derive a Feynman path-integral formula for $\Psi^2(\eta)$. Consider the Euclidean path integral

$$F(T;\eta)=\int_{P(T;\eta)}D\varphi' e^{-S(\varphi')}\,, \tag{1.2}$$

where $\varphi'(x)$ is a path in field-configuration space, and the set of paths $P(T;\eta)$ is the set of paths $\varphi'(x)$ that begin and end at φ_0, the vacuum expectation value of $\varphi(x)$, at time $x_4=\pm T$ and pass through the configuration $\eta(\vec{x})$ at $x_4=0$:

$$\varphi'(x)=\begin{cases}\varphi_0 \text{ at } x_4=\pm T\\ \eta(\vec{x}) \text{ at } x_4=0\,.\end{cases} \tag{1.3}$$

The limit $T\to\infty$ will be taken eventually. Since any path in $P(T;\eta)$ can be split into a path from φ_0 to $\eta(\vec{x})$ followed by another from $\eta(\vec{x})$ to φ_0, the Feynman path-integral formula[11] implies that $F(T;\eta)$ is the product of two Green's functions

$$F(T;\eta)=\langle\varphi_0|e^{-HT}|\eta\rangle\langle\eta|e^{-HT}|\varphi_0\rangle\,. \tag{1.4}$$

The Green's function can be expanded in the complete set of eigenstates $|n\rangle$ of H as

$$\langle\eta|e^{-HT}|\varphi_0\rangle=\sum_n e^{-E_nT}\langle\eta|n\rangle\langle n|\varphi_0\rangle\,. \tag{1.5}$$

When $T\to\infty$ only the lowest-energy state survives which is the vacuum state $|\Omega\rangle$ with energy $E_\Omega=0$. Thus

$$\lim_{T\to\infty}F(T;\eta)=|\langle\varphi_0|\Omega\rangle|^2\,|\langle\eta|\Omega\rangle|^2\,. \tag{1.6}$$

The first factor is independent of $\eta(\vec{x})$ and the second factor is $\Psi^2(\eta)$. Therefore the vacuum functional can be written

$$\Psi^2(\eta)=N\int_{P(\eta)}D\varphi'(x)e^{-S(\varphi')}\,, \tag{1.7}$$

where $P(\eta)$ is the set of paths $\varphi'(x)$ that begin and end at φ_0 at $x_4=\pm\infty$ and pass through $\eta(\vec{x})$ at $x_4=0$, and N is a normalization factor chosen so that

$$\int D\eta(\vec{x})\Psi^2(\eta)=1\,. \tag{1.8}$$

The formula (1.7) for $\Psi^2(\eta)$ can alternatively be written

$$\Psi^2(\eta)=\frac{1}{Z}\int D\varphi'(x)e^{-S(\varphi')}\prod_{\vec{x}}\delta[\varphi'(\vec{x},0)-\eta(\vec{x})]\,, \tag{1.9}$$

where $\varphi'(\vec{x},0)$ is $\varphi'(x)$ at $x_4=0$ and

$$Z=\int D\varphi'(x)e^{-S(\varphi')}\,. \tag{1.10}$$

The formula (1.7) can be taken over directly into gauge theories apart from the additional complication of the need for gauge fixing. For example, consider the Coulomb-gauge version of the Yang-Mills theory, in which the gauge field $A_a^i(x)$ is required to be transverse $\partial_i A_a^i(x)=0$. Then the vacuum functional is a functional of time-independent transverse fields $A_a^i(\vec{x})$ and the formula (1.7) becomes

$$\Psi^2[A_a^i(\vec{x})]=N\int_{P(A_a^i)}DA_a'^{\mu}D\varphi D\varphi^* e^{-S_{\text{eff}}(A',\varphi,\varphi^*)}\,, \tag{1.11}$$

where φ and φ^* are Faddeev-Popov ghost fields and S_{eff} is the action plus gauge-fixing and ghost terms appropriate to the Coulomb gauge. Alternatively, consider the temporal gauge in which the (Euclidean) time component $A_a^4(x)$ of the gauge field is required to vanish, $A_a^4(x)=0$. The vacuum state is required to be invariant under the residual local gauge freedom associated with invariance under time-independent gauge transformations.[5] Again a path-integral formula can be written

$$\Psi_0{}^2(A_a^i)=N\int_{P_0(A_a^i)}DA_a'^i e^{-S(A_a'^i)}\,, \tag{1.12}$$

where the subscript 0 indicates the temporal-gauge theory. Here $P_0(A_a^i)$ is the set of paths that begin and end at arbitrary pure-gauge configurations and pass through A_a^i at $x_4=0$; the endpoints of the paths are left arbitrary so that $\Psi_0{}^2(A_a^i)$ is invariant under gauge transformations of $A_a^i(\vec{x})$. The functionals $\Psi^2(A_a^i)$ and $\Psi_0{}^2(A_a^i)$ are related by

$$\Psi_0{}^2(A_a^i)=\Psi^2(\tilde{A}_a^i)\,, \tag{1.13}$$

where $\tilde{A}_a^i$ is the transverse field gauge equivalent to A_a^i. Equation (1.11) can be derived from Eq. (1.12) by changing variables of integration in (1.12) from $A_a'^i$ to the transverse field gauge equivalent to $A_a'^i$; the Faddeev-Popov ghost fields give the Jacobian determinant as usual.

The functional $\Psi^2(A_a^i)$ is the probability distribution of vacuum fluctuations $A_a^i(\vec{x})$. It has been suggested that quark confinement in QCD is a consequence of the existence of vacuum fluctuations that are of long range.[4,5] A long-range field $A_a^i(\vec{x})$ is defined as one that decreases as $|\vec{x}|^{-1}$ as $|\vec{x}|\to\infty$. A simple indication of the importance of such fields is that in the covariant derivative

$$D^i_{ab} = \delta_{ab}\partial^i - g\epsilon_{abc}A^i_c(\vec{x}) ,$$

the gradient term and the field-dependent term both fall off as $|\vec{x}|^{-1}$ as $|\vec{x}| \to \infty$ if $A^i_a(\vec{x})$ is of long range. This implies, for example, that the instantaneous Coulomb potential of a pair of color charges in a long-range background field has a large-distance asymptotic form that decreases less rapidly than the ordinary Abelian Coulomb potential (i.e., with $A^i_a = 0$).[5] In contrast, if $A^i_a(\vec{x})$ is of short range then $D^i_{ab}(\vec{x})$ approaches $\partial_i \delta_{ab}$ as $|\vec{x}| \to \infty$ and the instantaneous Coulomb potential at large distances has the usual $1/|\vec{r} - \vec{r}'|$ asymptotic form.[12]

Long-range vacuum fluctuations do not occur in perturbation theory. That is, the vacuum functional $\Psi_{\rm PT}{}^2(A^i_a)$ calculated in perturbation theory vanishes if A^i_a is of long range. To be specific, in the unperturbed theory (i.e., with coupling constant $g=0$) each gauge field is equivalent to a free photon field. The vacuum functional for a free photon field in the Coulomb gauge (or the temporal gauge) is

$$\Psi^2(A^i) = N \exp\left[-\tfrac{1}{2}\int d^3x\, d^3y A^i(\vec{x}) \times D^{-1}_{ij}(\vec{x}-\vec{y})A^j(\vec{y})\right], \qquad (1.14)$$

where $D^{-1}{}_{ij}(\vec{x})$ is the inverse of the equal-time photon propagator

$$D^{-1}{}_{ij}(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot\vec{x}} 2|\vec{p}|\left(\delta_{ij} - \frac{p^ip^j}{\vec{p}^2}\right). \qquad (1.15)$$

In terms of the Fourier transform $\hat{A}^i(\vec{p})$ of the transverse field $A^i(\vec{x})$,

$$\Psi^2(A^i) = N \exp\left[-\tfrac{1}{2}\int \frac{d^3p}{(2\pi)^3} \hat{A}_i(\vec{p})\hat{A}^i(-\vec{p}) 2|\vec{p}|\right]. \qquad (1.16)$$

Now $\Psi^2(A^i)$ vanishes if $A^i(\vec{x})$ is of long range. For if $A^i(\vec{x})$ is of order $|\vec{x}|^{-1}$ as $|\vec{x}| \to \infty$ then $\hat{A}^i(\vec{p})$ is singular at $\vec{p} = 0$ and

$$\hat{A}^i(\vec{p}) \sim \frac{1}{p^2} f^i(\hat{p}) \text{ as } p^2 \to 0 , \qquad (1.17)$$

where $f^i(\hat{p})$ depends only on the angular variables of $\vec{p}$. Then the integral in Eq. (1.16) is logarithmically infrared divergent and so $\Psi^2(A^i) = 0$. Corrections to Eq. (1.16) for $\Psi^2(A^i_a)$ are, in perturbation theory, only polynomial factors, so this result holds to all finite orders of perturbation theory.

In principle, the path-integral formula (1.12) provides a way to determine $\Psi^2(A^i_a)$ for long-range fields $A^i_a(\vec{x})$ that goes beyond perturbation theory. To use the formula it is necessary to identify paths $A'^i_a(x)$ that are of long range at $x_4 = 0$. It will be shown presently that the meron solution is such a path. However, the classical action of any such path is infinite.[13] Thus a single isolated meron configuration, for instance, does not contribute to the path integral (1.12). Nevertheless, superpositions of merons with finite action do exist and would produce vacuum fluctuations that are superpositions of long-range fields over a large range although at *sufficiently* large distances the vacuum fluctuation decreases rapidly to zero. These vacuum fluctuations would significantly affect the quark-antiquark potential.[5]

The meron solution[1] is given by

$$A^\mu_a = \frac{1}{g}\eta^{\mu\nu}_a \frac{x^\nu}{x^2} , \qquad (1.18)$$

where the quantity $\eta^{\mu\nu}_a$ is antisymmetric in μ, ν and has components

$$\eta^{i4}_a = -\eta^{4i}_a = \delta_{ai} , \quad \eta^{ij}_a = \epsilon_{aij} . \qquad (1.19)$$

The indices μ,ν refer to a four-dimensional Euclidean space and A^μ_a ($a = 1,2,3$) are the gauge fields of a Yang-Mills theory with gauge group SU(2). The meron field configuration is a solution of the Euclidean field equations[14] but is singular at $x^2 = 0$. The singularity at $x^2 = 0$ can be smeared out without changing the remarks to be made here, which concern the large-x behavior of the field.

The meron solution can easily be transformed into the temporal gauge. In that gauge the meron can be written

$$A^4_a = 0 , \quad A^i_a = \frac{2}{g}\epsilon_{aij}\frac{x^j}{r^2} b(r, x_4) , \qquad (1.20)$$

where

$$b(r, x_4) = \tfrac{1}{2} + \tfrac{1}{2}\frac{x_4}{(r^2 + x_4{}^2)^{1/2}} . \qquad (1.21)$$

Note that the field (1.20) also obeys the Coulomb-gauge condition. Some limiting values of this field are important. When $x_4 \to -\infty$, $b \to 0$ and the field A^i_a approaches zero. When $x_4 \to +\infty$, $b \to 1$ and the field tends to a long-range field, but one that is a pure gauge; in particular its energy density vanishes. In fact, the configuration (1.20) with $b = 1$ is an example of a pure-gauge configuration that obeys the Coulomb-gauge condition pointed out by Gribov.[15] When $x_4 = 0$, $b = \frac{1}{2}$; then A^i_a is a long-range field that is not a pure gauge. More generally, $b \to \frac{1}{2}$ as $r \to \infty$ for any finite value of x_4. The field in Eq. (1.20) with $b = \frac{1}{2}$ is precisely the Wu-

Yang gauge-field configuration sometimes called a magnetic monopole configuration.[3]

Thus if the temporal-gauge meron solution is viewed as a path with Euclidean time in the space of gauge fields, it begins and ends at pure-gauge configurations that are related by a long-range gauge transformation, and passes through long-range monopole configurations at finite x_4. If an isolated meron could contribute to the path integral in Eq. (1.12) it would contribute to the probability of a long-range vacuum fluctuation $A_a^i(\vec{x})$, i.e., one that is of order $|\vec{x}|^{-1}$ for large $|\vec{x}|$.

But an isolated meron has infinite action. The action is infinite not only because of the singularity at $x^2=0$, but also precisely because of the long-range nature of the field. Merons can contribute to the path integral only if they are combined in a way such that the action of the resulting field configuration is finite.

For example, a meron-pair field can be written

$$A_a^\mu = \frac{1}{g}\eta_a^{\mu\nu}\left[\frac{(x-c)^\nu}{(x-c)^2}+\frac{(x+c)^\nu}{(x+c)^2}\right], \qquad (1.22)$$

where $\pm c$ are the centers of the merons. This field satisfies the Euclidean field equations. Its action is infinite because of the singularities at $x=\pm c$, but if these singularities are smeared out, as will be done below, then the action becomes finite. What is important is that there is no divergence of the action from the long-range behavior of the field, because when $x^2\to\infty$ the field A_a^μ approaches a pure-gauge configuration. Specifically, when $x^2\to\infty$, the meron-pair field approaches an instanton solution.[16]

A simple way to smear out the singularities of $A_a^\mu(x)$ at $x=\pm c$ is to define a new field $\bar{A}_a^\mu(x)$ as

$$\bar{A}_a^\mu(x) = \frac{2}{g}\eta_a^{\mu\nu} f_\nu(x), \qquad (1.23)$$

where

$$f_\nu(x) = \frac{(x-c)_\nu}{(x-c)^2+\epsilon^2}+\frac{1}{2}\frac{(x+c)_\nu}{(x+c)^2} \quad \text{if } (x-c)^2\le\epsilon^2,$$

$$f_\nu(x) = \frac{1}{2}\frac{(x-c)_\nu}{(x-c)^2}+\frac{(x+c)_\nu}{(x+c)^2+\epsilon^2} \quad \text{if } (x+c)^2\le\epsilon^2, \qquad (1.24)$$

$$f_\nu(x) = \frac{1}{2}\frac{(x-c)_\nu}{(x-c)^2}+\frac{1}{2}\frac{(x+c)_\nu}{(x+c)^2} \quad \text{otherwise.}$$

The smeared meron-pair field $\bar{A}_a^\mu(x)$ is equal to the unsmeared meron pair $A_a^\mu(x)$ outside of two small spheres of radius ϵ around $x=\pm c$, and is not singular at $x=\pm c$.[17]

The action of the smeared meron pair is

$$S=\tfrac{1}{4}\int d^4x \bar{F}_a^{\mu\nu}\bar{F}_a^{\mu\nu}, \qquad (1.25)$$

where $\bar{F}_a^{\mu\nu}$ is the covariant field tensor

$$\bar{F}_a^{\mu\nu} = \partial^\mu\bar{A}_a^\nu - \partial^\nu\bar{A}_a^\mu + g\epsilon_{abc}\bar{A}_b^\mu\bar{A}_c^\nu. \qquad (1.26)$$

The action is a dimensionless function of c^2 and ϵ^2, and is logarithmically divergent as $\epsilon^2/c^2\to 0$. Thus S can be written

$$S=\frac{3\pi^2}{g^2}\ln\frac{c^2}{\epsilon^2}+S_c, \qquad (1.27)$$

where S_c is convergent as $\epsilon^2/c^2\to 0$. The interaction of merons is proportional to the log of their separation.[1]

It is a straightforward calculation to transform the meron-pair field configuration to the temporal gauge if the centers $\pm c$ are on the x_4 axis, i.e., $c=(\vec{0},T)$. In that case a temporal-gauge version of the unsmeared meron pair is given again by Eq. (1.20) where now

$$b(r,x_4)=\frac{1}{2}-\frac{1}{2}\frac{r^2+(x_4+T)(x_4-T)}{[r^2+(x_4+T)^2]^{1/2}[r^2+(x_4-T)^2]^{1/2}}. \qquad (1.28)$$

The function b tends to zero when either $x_4\to\pm\infty$ (r fixed) or $r\to\infty$ (x_4 fixed). If the meron pair is viewed as a path in the space of gauge fields, it begins and ends at $A_a^i=0$ and passes through fields that are only of short range. In particular, the field at $x_4=0$ has

$$b(r,0)=\frac{T^2}{r^2+T^2}. \qquad (1.29)$$

This vacuum fluctuation $A_a^i(\vec{x})$ can be called a *large* gauge-field fluctuation in that its magnitude is of order $1/g$.[13]

Although the meron-pair solution produces only a short-range vacuum fluctuation, if T is large then the meron pair produces fluctuations that resemble those of an isolated meron over distances comparable to T. For if T is large and $x_4\simeq -T$, then b is approximately

$$b(r,x_4)\simeq\tfrac{1}{2}+\tfrac{1}{2}\frac{x_4+T}{[(x_4+T)^2+r^2]^{1/2}} \qquad (1.30)$$

provided that $r^2\ll T^2$. Alternatively, the energy density of the configuration $A_a^i(x_4,\vec{x})$, in which x_4 is treated as a parameter rather than a time variable, is

$$\mathcal{H}=\tfrac{1}{4}F_a^{ij}F_a^{ij},$$
$$\mathcal{H}=\frac{8}{g^2}\frac{T^4}{[r^2+(x_4+T)^2]^3[r^2+(x_4-T)^2]^3}\times\{3[r^2-(x_4+T)(x_4-T)]^2+4x_4^2r^2\}. \qquad (1.31)$$

If T is large, $x_4=-T$, and $r\ll T$ then

$$\mathcal{H}\simeq\frac{4}{g^2}\frac{1}{r^4}, \qquad (1.32)$$

which corresponds to a long-range field at least over the range $r \ll T$. On the other hand, the energy density at $x_4 = 0$ is

$$\mathcal{H} = \frac{24}{g^2}\frac{T^4}{(T^2+r^2)^4}, \tag{1.33}$$

which is of order T^{-4} for $r \ll T$ and falls off as r^{-8} for $r \gg T$. A meron pair with meron positions on the x_4 axis produces large gauge-field vacuum fluctuations (i.e., with magnitude of order $1/g$) that are of short range. It should be noted that instantons produce similar vacuum fluctuations.[12]

If the centers $\pm c$ of the merons in the meron-pair solution are not on the x_4 axis then the temporal-gauge expression for the field is not simple. However, it can be shown, for instance, that if $\pm c$ are located at equal times $c = (\vec{c}, 0)$ then the meron pair is like an instanton[10] in that it is a path from one pure-gauge configuration $A_1^i = g_1^{-1}\partial_i g_1$ at $x_4 = -\infty$ to another $A_2^i = g_2^{-1}\partial_i g_2$ at $x_4 = +\infty$, where g_1 and g_2 are in different homotopy classes.[18] In particular, the meron-pair field has Pontryagin index 1, as the instanton solution does. The pure-gauge configurations A_1^i and A_2^i differ by a large gauge transformation[13] in the sense that g_1 and g_2 are not homotopic.

The problem of finding general multimeron configurations has not been solved but examples of such configurations have been given.[1,2,19] If an ionized meron phase exists in QCD, i.e., a phase in which the path integral is dominated by multimeron fields in which the merons have large separations, then Eq. (1.12) for the vacuum functional indicates that typical vacuum fluctuations would be superpositions of long-range monopole fields. This picture of the vacuum state resembles the monopole vacuum described by Mandelstam.[4] It has been suggested[20,21] that the monopole vacuum may describe the confining phase of QCD. This suggestion is based on an analogy between the monopole vacuum and the superconducting state of a metal.[20,21] The analogy suggests that the presence of monopole fluctuations in the vacuum is just what is needed to make the disorder parameter defined by 't Hooft[22] large.

In the next section the XY model will be reviewed in some detail and the analogy between merons in QCD and spin-field vortices in the XY model described.

II. THE XY MODEL

A. Definitions[23]

The purpose of this section is to discuss the XY model in a way that makes the analogy with QCD as precise as possible. Well-known properties of the model,[6-8] specifically, the role of vortices in the creation of long-range spin-field fluctuations, will be reviewed for the sake of completeness. The similarity between these theories will be indicated by considering the analog of the vacuum functional in the XY model.

The XY model describes classical two-component spins on a two-dimensional square lattice with nearest-neighbor interactions. The Hamiltonian is

$$H = -J \sum_{\langle \vec{x}, \vec{x}' \rangle} \vec{S}(\vec{x}) \cdot \vec{S}(\vec{x}'), \tag{2.1}$$

where J is the coupling constant, $\vec{S}(\vec{x})$ the spin at lattice site $\vec{x}$, and $\langle \vec{x}, \vec{x}' \rangle$ stands for the link joining nearest neighbors $\vec{x}$ and $\vec{x}'$. The two-component spin $\vec{S}(\vec{x})$ is of unit length $S^2(\vec{x}) = 1$, but can point in any direction. The XY model in statistical mechanics can be thought of as a lattice version of a $(1+1)$-dimensional field theory, namely the O(2) σ model. To emphasize this correspondence the lattice site $\vec{x}$ will be given components (x_1, x_2) and x_2 identified with the Euclidean-time coordinate, x_1 the space coordinate.

The classical spins $\vec{S}(\vec{x})$ can be specified by their angles $\varphi(\vec{x})$ with respect to an arbitrary axis. In terms of the angle variable $\varphi(\vec{x})$, the Hamiltonian is

$$H = J \sum_{\langle \vec{x}, \vec{x}' \rangle} \{1 - \cos[\varphi(\vec{x}) - \varphi(\vec{x}')]\}. \tag{2.2}$$

For later purposes it should be noted that H is invariant under a global transformation and a local transformation. The global transformation is $\varphi(\vec{x}) \to \varphi(\vec{x}) + \alpha$ for all $\vec{x}$ where α is a constant angle. The local transformation is $\varphi(\vec{x}) \to \varphi(\vec{x}) + 2\pi$ independently at each point or set of points in the lattice. The invariance under these transformations arises because the field $\varphi(\vec{x})$ is an angle with respect to some axis. In this paper, the local transformation will be called a gauge transformation.

The partition function Z of the XY model is

$$Z = \int D\varphi(\vec{x})\, e^{-A(\varphi)}, \tag{2.3}$$

where the action $A(\varphi)$ is

$$A(\varphi) = K \sum_{\langle \vec{x}, \vec{x}' \rangle} \{1 - \cos[\varphi(\vec{x}) - \varphi(\vec{x}')]\} \tag{2.4}$$

and $K = J/k_B T$. The functional integral notation means

$$\int D\varphi(\vec{x}) \equiv \prod_{\vec{x}} \int_{-\pi}^{\pi} d\varphi(\vec{x}). \tag{2.5}$$

In analogy with the vacuum functional of field theory, a functional $\Psi^2(\eta)$ will be defined as in Eq. (1.9)

$$\Psi^2(\eta)=\frac{1}{Z}\int D\varphi(\vec{x})\,e^{-A(\varphi)}\prod_{x_1}\delta[\varphi(x_1,0)-\eta(x_1)]\,. \tag{2.6}$$

In statistical mechanics, $\Psi^2(\eta)$ is simply the reduced probability distribution in which the spin variable along the x_1 axis is specified to be $\eta(x_1)$. Here the field-theoretic terminology will be used and $\Psi^2(\eta)$ called the vacuum functional. Also, the argument $\eta(x_1)$ which is the specified spin field at time $x_2=0$ will be called a vacuum fluctuation of the spin field.

The vacuum functional $\Psi^2(\eta)$ can be used to evaluate expectation values of operators at $x_2=0$. For instance, the spin-spin correlation function on the x_1 axis defined by

$$F(x-x')=\langle\vec{S}(x,0)\cdot\vec{S}(x',0)\rangle \tag{2.7}$$

can be written

$$F(x-x')=\int D\eta(x_1)\Psi^2(\eta)\cos[\eta(x)-\eta(x')]\,. \tag{2.8}$$

As in field theory, a generating functional $Z[j(\vec{x})]$ can be defined by

$$Z[j(\vec{x})]=\frac{1}{Z}\int D\varphi(\vec{x})e^{-A(\varphi)}\exp\Big[i\sum_{\vec{x}}j(\vec{x})\varphi(\vec{x})\Big]\,. \tag{2.9}$$

In terms of $Z[j]$, the vacuum functional can be determined by

$$\begin{aligned}\Psi^2(\eta)&=\frac{1}{Z}\int D\varphi(x)e^{-A(\varphi)}C\int D\omega(x_1)\exp\Big\{i\sum_{x_1}\omega(x_1)[\varphi(x_1,0)-\eta(x_1)]\Big\}\\ &=C\int D\omega(x_1)\exp\Big[-i\sum_{x_1}\omega(x_1)\eta(x_1)\Big]Z[\omega(x_1)\delta(x_2)]\,,\end{aligned} \tag{2.10}$$

where C is a normalization constant.

B. The spin-wave approximation

The starting point for this discussion of the XY model will be the spin-wave (SW) approximation of the model.[6] In the SW approximation the action $A(\varphi)$ is approximated by expanding the cosine in Eq. (2.2) up to terms quadratic in $\varphi(\vec{x})$

$$A_{SW}(\varphi)=\tfrac{1}{2}K\sum_{\langle\vec{x},\vec{x}'\rangle}[\varphi(\vec{x})-\varphi(\vec{x}')]^2\,. \tag{2.11}$$

Furthermore the angle variable $\varphi(\vec{x})$ is allowed to range over the entire domain $-\infty\leqslant\varphi\leqslant\infty$. The SW approximation is valid if the sum over fields is dominated by small fluctuations of $\varphi(\vec{x})$. In field-theoretic language the SW approximation is lowest-order perturbation theory: $A_{SW}(\varphi)$ is the unperturbed action and the interaction terms $A(\varphi)-A_{SW}(\varphi)$ are neglected.

In the SW approximation all functional integrals are Gaussian integrals and can be evaluated.[6] It is useful to rewrite the action as

$$A_{SW}(\varphi)=\tfrac{1}{2}K\sum_{\vec{x}'}\sum_{\vec{x}''}\varphi(\vec{x}')L(\vec{x}'-\vec{x}'')\varphi(\vec{x}'')\,, \tag{2.12}$$

where

$$\begin{aligned}L(\vec{x}'-\vec{x}'')=\sum_{\vec{x}}\sum_{i=1,2}[&2\delta(\vec{x}',\vec{x})\delta(\vec{x},\vec{x}'')\\ &-2\delta(\vec{x}',\vec{x})\delta(\vec{x},\vec{x}''+a_0\hat{e}_i)]\,,\end{aligned} \tag{2.13}$$

where $\hat{e}_i$ is the unit vector in the ith direction and a_0 the lattice spacing. If this form of the action is used in the definition (2.9) of the generating functional, then $Z[j]$ becomes

$$Z_{SW}[j]=\exp\Big[-\frac{1}{2K}\sum_{\vec{x}}\sum_{\vec{x}'}j(\vec{x})G(\vec{x}-\vec{x}')j(\vec{x}')\Big]\,, \tag{2.14}$$

where the Green's function $G(\vec{x}-\vec{x}')$ is the inverse of $L(\vec{x}-\vec{x}')$:

$$\sum_{\vec{x}'}L(\vec{x}-\vec{x}')G(\vec{x}'-\vec{x}'')=\delta(\vec{x},\vec{x}'')\,. \tag{2.15}$$

The lattice Green's function $G(\vec{R})$ can be expressed as the Fourier transform

$$G(\vec{R})=\int_{-\pi}^{\pi}\frac{d^2q}{(2\pi)^2}\frac{e^{-i\vec{q}\cdot\vec{R}/a_0}}{4-2\cos q_1-2\cos q_2}\,. \tag{2.16}$$

In the continuum limit $R/a_0\to\infty$ the Green's function is *formally*[24]

$$G(\vec{R})=\int\frac{d^2\xi}{(2\pi)^2}e^{-i\vec{\xi}\cdot\vec{R}}\frac{1}{\xi^2}\,. \tag{2.17}$$

The functional integral in Eq. (2.10) for the vacuum functional is again Gaussian. Thus the SW approximation of the vacuum functional $\Psi_{SW}{}^2(\eta)$ can be determined to be

$$\Psi_{SW}{}^2(\eta)=C\exp\Big[-\tfrac{1}{2}K\sum_{x}\sum_{x'}\eta(x)M(x-x')\eta(x')\Big]\,, \tag{2.18}$$

where $M(x-x')$ is the inverse of the Green's function at equal times $G(x-x')$; explicitly,

$$M(R)=\int_{-\pi}^{\pi}\frac{dq_1}{2\pi}\,e^{iq_1R/a_0}\hat{M}(q_1)\,, \tag{2.19}$$

where

$$\frac{1}{\hat{M}(q_1)}=\int_{-\pi}^{\pi}\frac{dq_2}{2\pi}\,\frac{1}{4-2\cos q_1-2\cos q_2}\,. \tag{2.20}$$

In terms of the Fourier transform $\hat{\eta}(q)$ of $\eta(x)$, $\Psi_{SW}{}^2(\eta)$ can be reexpressed as

$$\Psi_{SW}{}^2(\eta)=C\exp\left[-\tfrac{1}{2}K\int_{-\pi}^{\pi}\frac{dq}{2\pi}\,\hat{\eta}(q)\hat{\eta}(-q)\hat{M}(q)\right]. \tag{2.21}$$

Thus in the SW approximation the probability distribution $\Psi_{SW}{}^2(\eta)$ is Gaussian centered at $\eta=0$ with width determined by $K\hat{M}(q)$.

The vacuum fluctuations with probability distribution $\Psi_{SW}{}^2(\eta)$ are sufficiently large to prevent strong ordering of the spins for any value of K.[6] To be precise, the spin-spin correlation function $F_{SW}(R)$ defined in Eqs. (2.7), (2.8) can be shown to be

$$F_{SW}(R)=\exp\left\{-\frac{1}{K}\left[G(0)-G(R)\right]\right\}. \tag{2.22}$$

In the continuum limit $R/a_0\to\infty$ the Green's function is asymptotically[24]

$$G(0)-G(R)\sim\frac{1}{2\pi}\ln\frac{R}{a_0}+\text{constant}\,, \tag{2.23}$$

so the asymptotic form of the correlation function as $R\to\infty$ is

$$F_{SW}(R)\sim c\left(\frac{a_0}{R}\right)^{1/2\pi K}, \tag{2.24}$$

where c is a constant. Since $F_{SW}(R)$ decreases to zero as $R\to\infty$, the spins are not ordered. On the other hand, $F_{SW}(R)$ decreases only as a power of R^{-1}, so there is no finite correlation length for spin alignment. Therefore the spins are said to be critically ordered. The exponent of R^{-1} is inversely proportional to the coupling constant $K=J/k_BT$. In the SW approximation, critical ordering occurs for all values of K.

The vacuum functional $\Psi^2(\eta)$ is the probability distribution for spin-field configurations $\eta(x_1)$ along the x_1 axis, i.e., for vacuum fluctuations of the spin field. As in QCD the question arises whether long-range vacuum fluctuations occur. In order to characterize vacuum fluctuations it is useful to introduce the winding number w of the configuration $\eta(x_1)$. The winding number w is the net change of $\eta(x_1)$ along the x_1 axis,

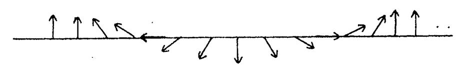

FIG. 1. A spin-field configuration along the x_1 axis with winding number $w=2\pi$.

$$w=\sum_{x_1}\left[\eta(x_1+a_0)-\eta(x_1)\right], \tag{2.25}$$

or, in continuum notation

$$w=\int_{-\infty}^{+\infty}dx_1\frac{\partial\eta}{\partial x_1}\,. \tag{2.26}$$

Figure 1 shows a vacuum fluctuation of the spin field with $w=2\pi$.

A long-range fluctuation $\eta(x_1)$ will be defined as one with nonzero winding number for which $\partial\eta/\partial x_1$ decreases to zero as $x_1{}^{-2}$ as $x_1\to\pm\infty$. Fields $\eta(x_1)$ with winding number zero or for which $\partial\eta/\partial x_1$ decreases more rapidly than $x_1{}^{-2}$ as $x_1\to\pm\infty$ will be said to be of short range.

In the SW approximation there are no long-range vacuum fluctuations. This is analogous to the fact that in QCD there are no long-range gauge-field fluctuations in perturbation theory. But in the XY model even more is true: There are no vacuum fluctuations with nonzero winding number of either short or long range. That is, if $\eta(x_1)$ has nonzero winding number then $\Psi_{SW}{}^2(\eta)=0$. To see that this is so, suppose that the winding number of $\eta(x_1)$ is w. Then the Fourier transform $\hat{\eta}(q)$ of $\eta(x)$ is singular at $q=0$ and

$$\hat{\eta}(q)=\sum_x\eta(x)e^{iqx/a_0}\sim 2\pi i\frac{w}{q}\ \text{as}\ q\to 0\,. \tag{2.27}$$

On the other hand, Eq. (2.20) for $\hat{M}(q)$ implies that

$$\hat{M}(q)\sim 2|q|\ \text{as}\ q\to 0\,. \tag{2.28}$$

Thus the integral in Eq. (2.21) for $\Psi_{SW}{}^2(\eta)$ is logarithmically divergent at $q=0$. This infrared divergence implies that $\Psi_{SW}{}^2(\eta)=0$ if $w\neq 0$.

Vacuum fluctuations $\eta(x)$ with nonzero winding number can be called large fluctuations in that $\eta(x)$ must be nonzero over an infinite part of the x_1 axis. But it should be noted that these fluctuations are not necessarily large when described in terms of the original spin variable $\vec{S}(x)$. For example, in the short-range fluctuation with $w=2\pi$ in Fig. 1, the spins are aligned except over a finite segment of the axis. The origin of the total suppression of these fluctuations in the SW approximation is the fact that the SW approximation does not respect the invariance under local gauge transformations. In particular, under the transformation $\varphi(\vec{x}) \to \varphi(\vec{x}) + 2\pi$ for all points $\vec{x}$ in an infinite region, $A(\varphi)$ is invariant but $A_{SW}(\varphi)$ changes by an infinite amount. It will be shown below that a path that produces a large vacuum fluctuation, i.e., one with nonzero winding number, is the analog of an instanton[10] in that it interpolates as $x_2 \to \pm\infty$ between different pure-gauge configurations.[25] These paths are totally suppressed in the SW approximation because $A_{SW}(\varphi)$ is infinite for these paths.[26]

C. Vortex configurations

Kosterlitz and Thouless pointed out that the SW approximation should be improved by including in the sum over states spin-field configurations that are superpositions of vortices and antivortices.[7] It will be shown below that inclusion of these configurations restores the symmetry under large gauge transformations, which was broken by the SW approximation, and leads to a vacuum functional $\Psi^2(\eta)$ with vacuum fluctuations with nonzero winding number. The most important effect of the vortices is that they create at high temperatures a disordered phase of the system that is not present in the SW approximation.

Figure 2 illustrates examples of vortex configurations. In Fig. 2(a), two examples of isolated vortex configurations are shown; the spins on the lattice are tangent to the indicated lines of flow. In Fig. 2(b) an antivortex configuration is shown. The configuration in Fig. 2(c) is a vortex-antivortex ($v\bar{v}$) pair with centers on the x_2 axis.

Vortex spin-field configurations are specified by the vorticity $m(\Gamma)$ of the field for arbitrary closed paths Γ. Let Γ be a closed path through n

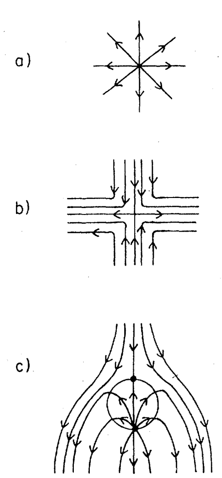

FIG. 2. Vortex configurations in the x_1, x_2 plane: (a) Examples of an isolated vortex; (b) an isolated antivortex; (c) a vortex-antivortex pair. The spins are parallel to the indicated lines of flow.

lattice points $\vec{x}_i$ $(i = 1, \ldots, n)$. The vorticity $m(\Gamma)$ of Γ in the spin field $\varphi(\vec{x})$ is defined by

$$m(\Gamma) = \sum_{i=1}^{n} [\varphi(\vec{x}_{i+1}) - \varphi(\vec{x}_i)], \tag{2.29}$$

or, in continuum notation

$$m(\Gamma) = \oint_\Gamma d\vec{x} \cdot \nabla\varphi(\vec{x}). \tag{2.30}$$

In terms of the spins $\vec{S}(\vec{x})$, the vorticity $m(\Gamma)$ is the angle through which the spin rotates as the loop Γ is traversed. Since the spins are single valued, $m(\Gamma)$ is an integer multiple of 2π for any Γ. The vorticity in the vortex configurations of Fig. 2(a) is 2π if Γ surrounds the vortex center; in the antivortex configuration of Fig. 2(b) it is -2π.

Explicit formulas for vortex configurations are expressed most simply in a notation that involves complex numbers. Let $z = x_1 + ix_2$ where x_1 and x_2 are the components of $\vec{x}$. Then the first vortex configuration shown in Fig. 2(a) is

$$\varphi(\vec{x}) = \arg z; \tag{2.31}$$

the second one is a global gauge transformation of the first

$$\varphi(\vec{x}) = \arg z + \tfrac{1}{2}\pi. \tag{2.32}$$

Similarly the antivortex configuration in Fig. 2(b) is

$$\varphi(\vec{x}) = -\arg z. \tag{2.33}$$

Finally the $v\bar{v}$ pair in Fig. 2(c) is

$$\varphi(\vec{x}) = \arg(z - c) - \arg(z + c) - \tfrac{1}{2}\pi, \tag{2.34}$$

where $c = c_1 + ic_2$ and $\vec{c} = (c_1, c_2)$ is the position of the vortex, which lies on the negative x_2 axis. A general multivortex configuration denoted $\bar{\varphi}(\vec{x})$ can be written

$$\bar{\varphi}(\vec{x}) = \sum_{i=1}^{n+} \arg(z - c_i) - \sum_{i=1}^{n-} \arg(z - \bar{c}_i), \tag{2.35}$$

where $n+$ $(n-)$ is the number of vortices (antivortices) and $\vec{c}_i$ $(\vec{\bar{c}}_i)$ are their positions. Of course a constant angle α can be added to $\bar{\varphi}(\vec{x})$ without changing either the vorticity or the action of the configuration.

The analogy between vortices in the XY model and merons in QCD follows at this point from an inspection of the vacuum fluctuations produced by vortices. It can be seen in Fig. 2(a) that the spin-field configuration along the x_1 axis (i.e., vacuum fluctuation) produced by an isolated vortex has winding number $+\pi$ if the vortex position $\vec{c}$ has $c_2 > 0$ and $-\pi$ if $c_2 < 0$. Furthermore, the vacuum fluctuation is of long range because by Eq. (2.31) for $\varphi(\vec{x})$, $\partial\varphi(x_1, x_2)/\partial x_1 \sim x_1^{-2}$ as $x_1 \to \infty$ with x_2

fixed. Also, as $x_2 \to \infty$ the vortex configuration approaches one in which all spins point in one direction, and as $x_2 \to +\infty$ one in which all spins point in the opposite direction. Thus if the isolated vortex is viewed as a path in the space of spin fields, it begins and ends at pure-gauge configurations that differ by a long-range gauge transformation, and it passes through long-range fields with winding number $\pm\pi$ at finite x_2. This is the basis of the analogy between vortices in the XY model and merons in QCD. Similarly the antivortex in Fig. 2(b) produces a long-range vacuum fluctuation with the opposite winding number.

The spin field along the x_1 axis produced by a closely-bound $v\bar{v}$ pair can be determined from Fig. 2(c). The winding number of the field $\varphi(x_1, x_2)$ with x_2 fixed is either 0 or $\pm 2\pi$ depending on the position and orientation of the $v\bar{v}$ pair. The field $\varphi(x_1, 0)$ in Fig. 2(c) has winding number 2π. Also this field is of short range by Eq. (2.34) which implies that $\partial\varphi(x_1,0)/\partial x_1$ decreases more rapidly than x_1^{-2} for $x_1 \gtrsim |\vec{c}|$.

The centers of the vortices in Fig. 2(c) lie on the x_2 axis. As another example of a $v\bar{v}$ pair, consider a $v\bar{v}$ pair with centers on the x_1 axis obtained by rotating the configuration in Fig. 2(c) by $\frac{1}{2}\pi$. Specifically, let

$$\varphi(\vec{x}) = \arg(z-c) - \arg(z+c)\,, \qquad (2.36)$$

where $c = c_1$ is real. This field plays the role of an instanton[10] in the XY model in that it interpolates between pure-gauge configurations that differ by a large gauge transformation. To be precise, at $x_2 = +\infty$, $\varphi(\vec{x}) = 0$; but at $x_2 = -\infty$,

$$\varphi(\vec{x}) = 2\pi\theta(c_1 - x_1)\theta(c_1 + x_1)\,.$$

That is, all the spins with $-c_1 \le x_1 \le c_1$ are rotated through -2π as x_2 goes from $-\infty$ to $+\infty$. The gauge transformation is called large because the field is rotated completely through 2π for $-c \le x_1 \le c$.

Thus isolated vortices are analogous to merons and $v\bar{v}$ pairs are analogous to meron pairs and instantons. The contribution of an isolated vortex to the path integral in Eq. (2.6) gives a vacuum fluctuation of long range with winding number π, the analog of a Wu-Yang monopole in QCD. These vacuum fluctuations have been called kinks.[27] The $v\bar{v}$ pairs, which tunnel between pure-gauge configurations that differ by a large gauge transformation, produce short-range vacuum fluctuations that have large magnitude, pairs of kinks and antikinks.[27] When multivortex configurations are included in the path integral, they restore the symmetry under large gauge transformations that was violated in the SW approximation. This similarity of vortices in the XY model and classical configurations like instantons and merons in QCD was also mentioned by Polyakov.[16]

Next, formulas for the contributions of vortices to the partition function and the vacuum functional $\Psi^2(\eta)$ will be derived as in Ref. 7. There it is assumed that the statistical sum over states is dominated by spin fields $\varphi(\vec{x})$ of the form

$$\varphi(\vec{x}) = \bar{\varphi}(\vec{x}) + \psi(\vec{x})\,, \qquad (2.37)$$

where $\psi(\vec{x})$ is small and where $\bar{\varphi}(\vec{x})$ is the multivortex configuration in Eq. (2.35). The dependence of $\bar{\varphi}(\vec{x})$ on the positions $\vec{c}_1, \ldots, \vec{c}_n$; $\bar{\vec{c}}_1, \ldots, \bar{\vec{c}}_n$ of the vortices is left implicit. The spin variables $\varphi(\vec{x})$ in Eq. (2.3) for the partition function are replaced by the collective coordinates $\vec{c}_i$ and $\bar{\vec{c}}_i$ that determine $\bar{\varphi}(\vec{x})$ and the small variations $\psi(\vec{x})$.[28] Actually only fields with an equal number of vortices and antivortices contribute to the sum because the action of $\varphi(\vec{x})$ is infinite otherwise; therefore in Eq. (2.35) for $\bar{\varphi}(\vec{x})$ take $n_+ = n_- = n$. Then the partition function becomes

$$Z = \sum_{n=0}^{\infty} \left(\frac{1}{n!}\right)^2 \sum_{c_i, \bar{c}_i} \int D\psi(\vec{x}) e^{-A(\bar{\varphi}+\psi)}; \qquad (2.38)$$

the factor $(n!)^{-2}$ is the Boltzmann counting factor needed because configurations that differ by interchanges of $\vec{c}_i$'s or $\bar{\vec{c}}_i$'s are identical.

The multivortex configuration $\bar{\varphi}(\vec{x})$ given in Eq. (2.35) is the minimum-action configuration with vorticity specified by the requirement that vortices occur at $\vec{c}_i$ and antivortices at $\bar{\vec{c}}_i$, $i = 1, \ldots, n$.[29] The action $A(\bar{\varphi}+\psi)$ is then approximately[7]

$$A(\bar{\varphi}+\psi) = A(\bar{\varphi}) + A_{\rm SW}(\psi)\,, \qquad (2.39)$$

where $A_{\rm SW}(\psi)$ is given in Eq. (2.11). Terms linear in ψ are absent because $\bar{\varphi}$ is a local minimum of $A(\varphi)$, and terms of higher than quadratic order in ψ are neglected. The action $A(\bar{\varphi})$ of the vortex configuration is[7]

$$A(\bar{\varphi}) = 2n\mu - 2\pi K \sum_{i<j}^{2n} q_i q_j \ln\frac{|\vec{r}_i - \vec{r}_j|}{a_0}\,, \qquad (2.40)$$

where $\vec{r}_i$ are the positions of vortices and antivortices, and q_i their vorticities. That is, for $i = 1, \ldots, n$, $\vec{r}_i = \vec{c}_i$ and $q_i = +1$; for $i = n+1, \ldots, 2n$, $\vec{r}_i = \bar{\vec{c}}_{i-n}$ and $q_i = -1$. Vortices interact by a logarithmic potential, attractive for a $v\bar{v}$ pair, repulsive for a vv or $\bar{v}\bar{v}$ pair. The parameter μ, which is the action of a $v\bar{v}$ pair separated by the minimum possible separation a_0 and can be identified as the chemical potential of a $v\bar{v}$ pair, is given approximately by $\mu \simeq \pi^2 K$.

Since the spin waves $\psi(\vec{x})$ do not interact with the vortices in this approximation, the partition function Z factorizes

$$Z = Z_{\rm SW} Z_v\,, \qquad (2.41)$$

where Z_{SW} is the SW approximation of the partition function and

$$Z_v = \sum_{n=0}^{\infty} \left(\frac{1}{n!}\right)^2 \sum_{c_i, \bar{c}_i} e^{-A(\bar{\varphi})} . \qquad (2.42)$$

Because spin waves and vortices do not interact, the vacuum functional has the simple form

$$\Psi^2(\eta) = \frac{1}{Z_v} \sum_n \left(\frac{1}{n!}\right)^2 \sum_{c_i, \bar{c}_i} e^{-A(\bar{\varphi})} \times \Psi_{SW}^2(\bar{\varphi}(x_1, 0) - \eta(x_1)) , \qquad (2.43)$$

where Ψ_{SW}^2 is given in Eq. (2.18). This formula makes it clear how vortices introduce vacuum fluctuations with nonzero winding number. $\Psi_{SW}^2(\bar{\varphi}(x_1, 0) - \eta(x_1))$ vanishes unless the winding numbers of $\bar{\varphi}(x_1, 0)$ and $\eta(x_1)$ are equal. The configuration $\bar{\varphi}(\vec{x})$ contains an equal number of vortices and antivortices. Therefore the winding number of $\bar{\varphi}(x_1, 0)$ along the x_1 axis is an integer multiple of 2π. More precisely, $\bar{\varphi}(\vec{x})$ approaches zero as $|\vec{x}| \to \infty$ in any direction [see Eq. (2.35)], so the vorticity of the curve that bounds the lower-half $\vec{x}$ plane with $x_2 < 0$ is equal to the winding number of $\bar{\varphi}(x_1, 0)$ along the x_1 axis, or

$$-w = 2\pi(\nu - \bar{\nu}) , \qquad (2.44)$$

where ν ($\bar{\nu}$) is the number of vortices (antivortices) with $x_2 < 0$. Thus multivortex configurations produce vacuum fluctuations with winding number $2\pi k$, k an integer.

The effect of the vacuum fluctuations produced by vortices on the spin-spin correlation function $F(R)$ can be determined from Eq. (2.8) for $F(R)$. If the form (2.43) for $\Psi^2(\eta)$ is used to calculate $F(R)$ the result is

$$F(R) = F_{SW}(R) F_v(R) , \qquad (2.45)$$

where $F_{SW}(R)$ is given in Eq. (2.22) and $F_v(R)$ is the contribution from vortices

$$F_v(x - x') = \frac{1}{Z_v} \sum_n \left(\frac{1}{n!}\right)^2 \sum_{c_i, \bar{c}_i} e^{-A(\bar{\varphi})} \times \cos[\bar{\varphi}(x, 0) - \bar{\varphi}(x', 0)] . \qquad (2.46)$$

At this point the question of the existence of long-range vacuum fluctuations in the XY model can be addressed. The multivortex field $\bar{\varphi}(\vec{x})$ contains equal numbers of vortices and antivortices. Closely bound $v\bar{v}$ pairs produce vacuum fluctuations of short range. If, however, the vortices and antivortices are widely separated then long-range fluctuations will be produced by the individual vortices. The question is whether the partition function Z_v is dominated by configurations in which the vortices are free or those in which they are bound in $v\bar{v}$ pairs.

The vortex partition function Z_v is equivalent to that of a two-dimensional gas of positive and negative charges that interact by a logarithmic Coulomb potential. Kosterlitz and Thouless[7] showed that this system exists in one of two phases depending upon the value of $K = J/k_B T$. If K is large, the system is in an ordered phase in which the vortices are bound in $v\bar{v}$ pairs. The mean square separation λ^2 between members of a pair can be estimated by the following simple free energy argument[7]:

$$\lambda^2 \simeq \frac{\int_{a_0}^{\infty} 2\pi r dr\, r^2 \exp[-2\pi K \ln(r/a_0)]}{\int_{a_0}^{\infty} 2\pi r dr \exp[-2\pi K \ln(r/a_0)]} = a_0^2 \frac{\pi K - 1}{\pi K - 2} . \qquad (2.47)$$

On the other hand, if K is small the system is in a disordered phase in which the vortices are free. The phase transition point K_* at which the $v\bar{v}$ pairs ionize can be estimated heuristically as follows. The energy of an isolated vortex is [see Eq. (2.40)] $E \sim \pi J \ln L$ where L is the linear size of the system. The phase space available to an isolated vortex is just the volume of the lattice space so its entropy is $S \sim k_B \ln L^2$. In the ordered phase the energy term dominates the entropy and vice versa in the disordered phase. The phase transition point is roughly the point at which energy and entropy balance $E \sim TS$; that is,

$$\pi J \ln L \sim 2 k_B T \ln L . \qquad (2.48)$$

Thus $K_* = J/k_B T_* \simeq 2/\pi$. A more careful estimate[7] gives the value $K_* = 2.24/\pi$.

In the ordered phase $K > K_*$ the closely-bound $v\bar{v}$ pairs produce only short-range vacuum fluctuations and play a role analogous to that of instantons and meron pairs in QCD as explained earlier. The density of $v\bar{v}$ pairs is proportional to $e^{-2\mu}$ where μ is the chemical potential $\mu \simeq \pi^2 K$. For $K \gg K_*$, μ is large and the $v\bar{v}$ pairs can be neglected so the SW approximation is an adequate one. As K decreases and approaches K_* the $v\bar{v}$ pairs affect the system, for example by changing the large-distance behavior of the spin-spin correlation function, Eq. (2.45). It has been shown[7,8] that in the ordered phase the correlation function at large distance is asymptotically of the form

$$F(R) \sim c\left(\frac{a_0}{R}\right)^{1/2\pi K_{eff}} , \qquad (2.49)$$

where K_{eff} depends on K. Comparison of this result to the SW approximation [Eq. (2.24)] shows that the effect of $v\bar{v}$ pairs is a renormalization of the coupling constant. The renormalization constant $z \equiv K_{eff}/K$ approaches 1 as $K\to\infty$. As K decreases to K_*, z decreases slowly; at $K=K_*$, $K_{eff}=2/\pi$ (Refs. 7 and 8) so from the earlier estimate of K_*, $z\simeq 2/2.24\simeq 0.88$. Thus the system remains critically ordered for $K>K_*$ but the exponent describing the asymptotic form of $F(R)$ is changed by the $v\bar{v}$ pairs. The change is rather small, $z\simeq 0.88$, because even at $K=K_*\simeq 2.24/\pi$ the chemical potential $\mu\simeq\pi^2K\simeq 7.0$ is still rather large.

On the other hand, in the disordered phase $K<K_*$ free vortices produce long-range vacuum fluctuations. These disorder the spins more strongly than spin waves and $v\bar{v}$ pairs do and produce a finite correlation length for spin alignment. That is, the correlation function $F(R)$ decreases rapidly as $R\to\infty$

$$F(R)\sim R^{-1/4}f(R/\xi)\,, \tag{2.50}$$

where $f(x)$ decreases rapidly for $x\gg 1$ and ξ is a finite correlation length.

The following picture of the vacuum functional of the XY model has emerged. In the disordered phase, ionized vortices produce contributions to $\Psi^2(\eta)$ in Eq. (2.6) where $\eta(x_1)$ is a superposition of long-range fluctuations with winding number $\pm\pi$, i.e., of kinks and antikinks. In the ordered phase, $v\bar{v}$ pairs produce large vacuum fluctuations that are superpositions of short-range fluctuations with winding number 0 (kink-antikink pairs) or $\pm 2\pi$ (pairs of kinks). This picture of $\Psi^2(\eta)$ has also been described by Fradkin and Susskind[27] in a discussion of the XY model in a transfer-matrix formulation of the problem.

Finally the above qualitative statements about the nature of vacuum fluctuations in the XY model can be illustrated quantitatively by a rough calculation of the probability of producing large vacuum fluctuations by vortices or $v\bar{v}$ pairs. Consider the winding number of $\bar{\varphi}(x_1,0)$ along the segment of the x_1 axis from $-r$ to r, defined by

$$w(r)=\int_{-r}^{r}dx_1\frac{\partial\bar{\varphi}(x_1,0)}{\partial x_1}\,. \tag{2.51}$$

The ordered phase consists of a gas of $v\bar{v}$ pairs of characteristic size λ given in Eq. (2.47). A $v\bar{v}$ pair makes a significant contribution to $w(r)$ only if the position of the pair is in the region within a distance λ of the segment $(-r,r)$; the area of this region is $4r\lambda$. Thus let $P(m)$ be the probability that m $v\bar{v}$ pairs lie in this region; the probability $P(m)$ is a measure of the effect of $v\bar{v}$ pairs on the vacuum functional for large vacuum fluctuations. For a rough approximation of $P(m)$ the $v\bar{v}$ pairs will be treated as noninteracting, i.e., as a perfect gas. Then the partition function is just

$$Z=\sum_{n=0}^{\infty}\frac{1}{n!}\rho^n\int d^2c_1\cdots d^2c_n=e^{\rho\Sigma}\,, \tag{2.52}$$

where Σ is the total area of the lattice; the density parameter ρ is, by Eq. (2.40),

$$\rho=\frac{1}{a_0^{\,2}}\exp\left(-2\mu+2\pi K\ln\frac{\lambda}{a_0}\right). \tag{2.53}$$

The probability $P(\sigma;m)$ that m $v\bar{v}$ pairs lie in a region with area σ is

$$P(\sigma;m)=\frac{1}{Z}\sum_n\frac{1}{n!}\rho^n\sigma^m(\Sigma-\sigma)^{n-m}\binom{n}{m}$$
$$=e^{-\sigma\rho}\frac{(\sigma\rho)^m}{m!}\,. \tag{2.54}$$

The probability $P(m)$ is $P(m)=P(4r\lambda;m)$ so

$$P(m)=e^{-4r\lambda\rho}\frac{(4r\lambda\rho)^m}{m!}\,. \tag{2.55}$$

If $4r\lambda\rho$ is large this probability distribution is sharply peaked at $m\sim 4r\lambda\rho$. More precisely, the mean value of m is

$$\langle m\rangle=\sum_{m=0}^{\infty}mP(m)=4r\lambda\rho\,. \tag{2.56}$$

Thus the mean number of $v\bar{v}$ pairs that lie within a distance λ of the x_1 axis *per unit length* is

$$\frac{\langle m\rangle}{2r}=2\lambda\rho\,. \tag{2.57}$$

These $v\bar{v}$ pairs produce large but short-range vacuum fluctuations along the x_1 axis, with density $2\lambda\rho$. Since the fluctuations are of short range their effect is just to renormalize the coupling constant as in Eq. (2.49). Since the density of fluctuations is small for $K\gtrsim K_*$, the renormalization is fairly small.

The disordered phase consists of a plasma of free vortices and antivortices. An isolated vortex significantly affects $w(r)$ if it lies in the region bounded by the circle whose diameter is the line segment $(-r,r)$. Specifically, the absolute value of the contribution to $w(r)$ from a vortex (or antivortex) inside the circle is greater than $\frac{1}{2}\pi$. A multivortex configuration contributes significantly if there is an imbalance in the number of vortices and the number of antivortices. A rough measure of the effect of vortices is given by the probability $P(m)$ that the vorticity of the lower-half circle $m(c_-)$ minus that of the upper-half circle $m(c_+)$ is m.

A curve Γ has vorticity k if $v_+-v_-=k$ where v_+ (v_-) is the number of vortices (antivortices) in the region bounded by Γ. Once again the vortex plasma will be treated as a perfect gas, so the partition function is

$$Z=\sum_{n_+,n_-}\frac{1}{n_+!}\frac{1}{n_-!}\rho^{n_++n_-}\int d^2c_1\cdots d^2c_{n_+}d^2\bar{c}_1\cdots d^2\bar{c}_{n_-}$$
$$=e^{2\rho\Sigma}, \quad (2.58)$$

where the density ρ is $\rho=(1/a_0^2)e^{-2\mu}$. The probability $P(\Gamma;k)$ that Γ has vorticity k is then

$$P(\Gamma;k)=\frac{1}{Z}\sum_{n_+,n_-}\frac{1}{n_+!}\frac{1}{n_-!}\rho^{n_++n_-}\times\sum_{v}\sigma^v(\Sigma-\sigma)^{n_+-v}\sigma^{v-k}(\Sigma-\sigma)^{n_--v+k}\times\binom{n_+}{v}\binom{n_-}{v-k}$$
$$=e^{-2\rho\sigma}I_k(2\rho\sigma), \quad (2.59)$$

where σ is the area of the region bounded by Γ, and I_k denotes the modified Bessel function of order k.[30]

The probability $P(m)$ is

$$P(m)=\sum_{k=-\infty}^{\infty}P(c_+;k)P(c_-;k-m). \quad (2.60)$$

The area of $c_\pm$ is $\frac{1}{2}\pi r^2$. Thus

$$P(m)=e^{-2\rho\pi r^2}\sum_{k=-\infty}^{+\infty}I_k(\rho\pi r^2)I_{k-m}(\rho\pi r^2)$$
$$=e^{-2\rho\pi r^2}I_m(2\rho\pi r^2). \quad (2.61)$$

The probability distribution $P(m)$ is peaked at $m\sim 2\rho\pi r^2$ when r is large. The effect of ionized vortices is much larger than that of $v\bar{v}$ pairs because they need not lie near the x_1 axis in order to produce a significant contribution to $w(r)$, but only within a region of area $\sim r^2$.

The probability $P(m)$ is even in m, $P(-m)=P(+m)$ so the expectation value of m is zero. But the mean-square fluctuation $\langle m^2\rangle$ is of order $(2\rho\pi r^2)^2$. This is a measure of the density of kinks in the vacuum.

III. SUMMARY

The analogy between merons in QCD and vortices in the XY model consists of the following similarities. Isolated merons and isolated vortices both create long-range fluctuations of the relevant fields in the subspace with Euclidean time equal to zero. Merons produce Wu-Yang monopoles; vortices produce long-range kinks with winding number $\pm\pi$. Also, closely bound meron pairs and $v\bar{v}$ pairs both produce large vacuum fluctuations of short range, and both serve as instantons[10] by interpolating between pure-gauge configurations that differ by large gauge transformations. In both cases what is meant by a large field is one whose magnitude is of order 1 in contrast to the gluon or spin-wave fluctuations of perturbation theory which are of order g or $K^{-1/2}$.

In both QCD and the XY model, the perturbation-theory vacuum functional describes Gaussian fluctuations away from the zero field, and fails to describe correctly the large vacuum fluctuations produced by merons or meron pairs and vortices or $v\bar{v}$ pairs. This effect is more striking in the XY model in which all fluctuations with nonzero winding number, even those of short range produced by $v\bar{v}$ pairs, are completely suppressed in the SW approximation; the short-range fluctuations produced by meron pairs in QCD are not completely suppressed in perturbation theory, but nor are they correctly described by perturbation theory. In both cases the vacuum functional vanishes for long-range fields because the exact action is infinite for paths such as single merons or vortices that produce long-range fluctuations.

Isolated merons and vortices both have infinite action. The action of a pair of merons or a $v\bar{v}$ pair is proportional to the log of the separation of the pair. In the XY model it is possible to write the general multivortex configuration, Eq. (2.35). Presumably there exist analogous multimeron configurations. In the ionized phase of the XY model the vacuum fluctuations are superpositions of long-range kinks. If there is an analogous ionized-meron phase in QCD (Ref. 1) then there should be vacuum fluctuations that are arbitrary superpositions of monopoles.

The XY model is an example of a theory in which the behavior of the system is controlled by different kinds of classical field configurations at different values of the coupling constant. When $K=J/k_BT$ is sufficiently large the SW approximation is valid. As K decreases, $v\bar{v}$ pairs begin to affect the system. Finally for K less than the phase transition point K_*, free vortices occur and disorder the system. It has been suggested that QCD is also an example of such a theory.[1]

In the ordered phase of the XY model near the phase transition point, i.e., for $K\gtrsim K_*$, instanton (i.e., $v\bar{v}$ pair) effects[10] are observable. Therefore the XY model differs from the generalized spin models considered by Witten.[9] In those models, which are two-dimensional models with higher-dimensional spins, contributions to the partition function from configurations consisting of a dilute gas of instantons are overwhelmed by the contributions of configurations with large quantum fluctuations, i.e., for which the supposedly small variations $\psi(\vec{x})$ are large. Perhaps this difference is not surprising. Indeed, Kosterlitz and Thouless argued[7] that the existence of the low-temperature ordered phase, in which instantons play a role, is a special property of the two-dimensional spin system with two-component spins; in contrast, they argued that such a phase would not occur in a two-dimensional system with three-component spins, the so-called isotropic O(3) Heisenberg model. Since this latter model is a lattice version of the Euclidean O(3) σ model, one of the models considered in Ref. 9, this would account for the absence of instanton effects in Witten's models.

The difference between the XY model and the O(3) Heisenberg model involves both topology and mechanics. Minima of the action of the O(3) Heisenberg model can be labeled by a topological invariant, the index q defined by[7,31]

$$q=\frac{1}{4\pi}\int d^2x\,\sin\theta\left(\frac{\partial\theta}{\partial x_1}\frac{\partial\varphi}{\partial x_2}-\frac{\partial\theta}{\partial x_2}\frac{\partial\varphi}{\partial x_1}\right), \tag{3.1}$$

where $\theta(\vec{x})$ and $\varphi(\vec{x})$ are the polar angles that define the three-component spin at $\vec{x}$. The minimum-action field with $q=1$ is[31]

$$\begin{aligned}&\varphi(\vec{x})=\arg z\,,\\&\tan\tfrac{1}{2}\theta(\vec{x})=\frac{\lambda}{|z|}\,,\end{aligned} \tag{3.2}$$

where $z=x_1+ix_2$ and λ is an arbitrary positive scale parameter. This field is the analog of an isolated vortex in the XY model, not of a $v\bar{v}$ pair, because it produces long-range fluctuations of the spins; the index q is the analog of the total vorticity (i.e., over all space).

The existence of the ordered phase in the XY model is a consequence of the fact that states with a given value of total vorticity are truly metastable: A change of the vorticity requires creation of a vortex which has large energy, of order $\log L$, where L is the size of the system. At sufficiently low temperature, only the $v\bar{v}$ pairs contribute to the partition function and these do not disorder the system. In contrast, in the O(3) Heisenberg model, states with a given value of q are not really metastable in that the action separating states with different values of q is only of order 1; more precisely, the lowest-action state with index q has action $8\pi q$.[31] Thus there is no action barrier to limit the effect of fluctuations such as that in Eq. (3.2) and the system is disordered at arbitrarily low temperature.

In this paper, instantons have been identified with classical fields such as $v\bar{v}$ pairs that produce short-range fluctuations and do not disorder the system. In this sense, the field in Eq. (3.2) does not qualify as an instanton in the O(3) Heisenberg model.

The analogy between merons in QCD and vortices in the XY model leads to the speculation that the phases of QCD resemble those of the XY model.[1,2] Comparison of the actions of the two theories [Eqs. (1.27) and (2.40)] implies the correspondence between coupling constants

$$\frac{4\pi}{g^2}=K\,. \tag{3.3}$$

Then the speculation is that when g^2 is small, perturbation theory is valid. As g^2 increases, effects of instantons and meron pairs appear. At sufficiently large g^2 there is a phase transition point at which meron pairs ionize. In the ionized-meron phase the vacuum state includes long-range fluctuations similar to Wu-Yang monopoles. Quark confinement might occur in this phase.

It is natural to try to push the analogy one step further and to repeat the heuristic estimate of the phase transition point [Eq. (2.48)] for meron ionization in QCD. By Eq. (1.27) the action of an isolated meron is $A\sim(3\pi^2/g^2)\ln R$ where R is the linear dimension of four-dimensional Euclidean space. The volume W of phase space available to a single meron is presumably just the volume of 4-space times some finite factor related to the volume of group space orientations, $W\sim cR^4$; the entropy is $S=\ln W=4\ln R$ apart from the finite term. The phase transition point $g_*{}^2$ is identified as the point at which action and entropy are equal, which is

$$\frac{g_*{}^2}{4\pi}=\frac{9\pi}{16}\,. \tag{3.4}$$

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[23]The notation to be used in this section follows Ref. 8; the coupling constant J is two times that of Kosterlitz and Thouless.
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[25]A pure-gauge configuration in the *XY* model means a local gauge transformation of the configuration $\varphi(x_1)=0$; in any pure-gauge configuration, the spins are aligned.
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