WKB approximation for quantum theory on a lattice

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In this paper we use a WKB approximation to solve the lattice version of some quantum-mechanical models. This requires the solution of an infinite-order differential equation (an integral equation) in terms of a WKB series. The resulting WKB expansion on the lattice is complementary to the high-temperature expansion on the lattice.

I. INTRODUCTION

In previous papers[1-4] we developed a prescription for finding the strong-coupling expansion of various quantum-mechanical and quantum-field-theoretic models. Beginning with a functional-integral representation of the quantum theory we factored out the kinetic-energy terms from this integral. We then evaluated the remaining functional integral. This led to a set of graphical rules for computing the high-temperature expansion of the quantum theory on the lattice. This expansion (which is not the strong-coupling expansion) is a power series in a dimensionless parameter x.

For example, for the massless $g\phi^4$ quantum-field theory in one-dimensional space-time (the anharmonic oscillator), whose Lagrangian density is[5]

$$L=\tfrac{1}{2}\dot{q}^2+\tfrac{1}{4}gq^4\,, \tag{1.1}$$

the lattice expansion parameter is

$$x=\frac{1}{\sqrt{g}\,a^{3/2}}\,. \tag{1.2}$$

For this theory, on the lattice, the high-temperature expansion for the ground-state energy E has the form

$$E(x)=g^{1/3}x^{2/3}\left(-\tfrac{1}{2}\ln x+\sum_{n=0}^{\infty}\alpha_n x^n\right).$$

The continuum limit ($x\to\infty$) of the high-temperature expansion, obtained by a Padé-type extrapolation,[3] is the strong-coupling expansion.

In this paper we present an alternative to the high-temperature expansion of the lattice theory. Specifically, we use WKB techniques to obtain a sequence of approximations to the large-x expansion of the theory. Typically, the large-x expansion is a series in powers of $1/x$. For example, for the anharmonic oscillator in (1.1) the large-x expansion of the ground-state energy has the form

$$E(x)=\sum_{n=0}^{\infty}a_n x^{-4n/3}\,.$$

This expansion complements the high-temperature expansion in the same sense that the asymptotic expansion of a Bessel function is complementary to the Taylor expansion. Observe that unlike the high-temperature expansion, the large-x expansion has a smooth continuum limit [$E(\infty)=a_0$] and does not require any extrapolation techniques. [However, obtaining the above series for $E(x)$ from a WKB approximation sometimes requires a summation procedure. See Sec. V and the Appendix.]

Normally, WKB theory is used to solve a second-order differential equation in Schrödinger form

$$\epsilon^2 y''(u)=[V(u)-\lambda]y(u)\,.$$

However, in this paper we use it to solve for the eigenvalues of the transfer matrix, an infinite-order differential equation (an integral equation) which occurs in a natural way when we represent the quantum theory on a lattice:

$$e^{\epsilon^2 d^2/du^2}y(u)=e^{V(u)-\lambda}y(u)\,. \tag{1.3}$$

For the case of the anharmonic oscillator (1.1), $V(u)=u^4/4$ and the small dimensionless parameter ϵ that appears here is simply related to the lattice expansion parameter x that appears in Refs. 1–4:

$$\epsilon=(2x)^{-1/2}\,. \tag{1.4}$$

Observe that $\epsilon\to 0$ in the continuum limit $a\to 0$.

This paper is organized as follows: In Sec. II we show how to derive the integral equation in (1.3) for the case of the anharmonic oscillator

(1.1). In Sec. III we give a general discussion of the WKB solution of (1.3) for a general class of potentials $V(u)$. In Secs. IV and V we compare the WKB approximation for the ground-state eigenvalue with the high-temperature expansion of the ground-state eigenvalue for two models, the harmonic oscillator and the anharmonic oscillator.

II. DERIVATION OF THE INTEGRAL EQUATION (1.3)

We are interested in the ground-state eigenvalue of the Lagrangian L in (1.1) on a lattice. We showed in Ref. 2 that in the absence of external sources, the vacuum persistence function in Euclidean space is given by a path integral

$$\langle 0|0\rangle = \lim_{T\to\infty} \int Dq \exp\left[-\int_{-T/2}^{T/2} L(t)dt\right] \tag{2.1}$$

and that the ground-state energy E of the Hamiltonian $H=\frac{1}{2}p^2+\frac{1}{4}gq^4$ associated with this Lagrangian is given by

$$E=\lim_{T\to\infty} -\frac{1}{T}\ln\langle 0_+|0_-\rangle . \tag{2.2}$$

On the lattice, the path integral in (2.1) becomes a multiple integral,

$$\langle 0|0\rangle = \lim_{a\to 0}\int_{-\infty}^{\infty}\prod_{i=1}^{n}\frac{dq_i}{(2\pi a)^{1/2}}\exp\left[-\frac{1}{2a}(q_{i+1}-q_i)^2 - \tfrac{1}{4}agq_i^4\right], \tag{2.3}$$

where a is the lattice spacing, n is the number of lattice sites, and $T=na$ is the volume of space.

To obtain the high-temperature expansion we rescale the integration variables in (2.3) so that a small parameter appears multiplying the kinematic term. Thus, we let

$$q_i = z_i(ag)^{-1/4}$$

and obtain

$$\langle 0|0\rangle = \lim_{x\to 0}\int_{-\infty}^{\infty}\prod_{i=1}^{n}\frac{dz_i}{(2\pi/x)^{1/2}}\exp[-\tfrac{1}{2}x(z_{i+1}-z_i)^2 - \tfrac{1}{4}z_i^4], \tag{2.4}$$

where the dimensionless parameter x is given in (1.2).

In the conventional high-temperature expansion we actually treat x as a *small* parameter because a is held fixed and g is taken large. This allows an expansion in powers of x which must ultimately be extrapolated to $x=\infty$.

In this paper we take an alternative route; to wit, we define the integral operator Q by

$$Qf(u)=\int_{-\infty}^{\infty}\frac{dt}{(2\pi/x)^{1/2}}\exp[-\tfrac{1}{2}x(t-u)^2-\tfrac{1}{4}t^4]f(t). \tag{2.5}$$

Next we observe that the multiple integral in (2.4) is merely the repeated application of the operator Q. To see this, consider first the integral for $i=1$:

$$\int_{-\infty}^{\infty}\frac{dz_1}{(2\pi/x)^{1/2}}\exp[-\tfrac{1}{2}x(z_2-z_1)^2-\tfrac{1}{4}z_1^4].$$

This is just $Q1(z_2)$, where 1 means the function 1. Now include the integral over z_2. This is the composition $Q^21(z_3)$. Clearly, the full expression in (2.4) is

$$\langle 0|0\rangle = \lim_{x\to\infty}\left(\lim_{n\to\infty}\right)Q^n 1(z_{n+1}). \tag{2.6}$$

To examine (2.6) in the limit $n\to\infty$ we need to find the eigenvalues of the operator Q. Because we will compute these eigenvalues using WKB theory, it is convenient to write the eigenvalue equation in differential form. To do this we make use of the identity

$$\exp\left(\frac{1}{2x}\frac{d^2}{du^2}\right)\delta(t-u)=\frac{\exp[-\frac{1}{2}x(t-u)^2]}{(2\pi/x)^{1/2}} . \tag{2.7}$$

Now let $\phi(u)$ be an eigenfunction of the operator Q with eigenvalue $e^{-\lambda}$. Then $y(u)$ satisfies

$$Q\phi = e^{-\lambda}\phi . \tag{2.8}$$

Substituting (2.7) into (2.5) and doing the δ-function integration gives

$$\exp\left(\frac{1}{2x}\frac{d^2}{du^2}\right)\exp(-\tfrac{1}{4}u^4)\phi(u)=e^{-\lambda}\phi(u). \tag{2.9}$$

Finally, letting $\exp(-\frac{1}{4}u^4)\phi(u)=y(u)$ and defining the small parameter ϵ by

$$\epsilon=(2x)^{-1/2} \tag{2.10}$$

gives the eigenvalue problem

$$\exp(\epsilon^2 d^2/du^2)y(u)=\exp(\tfrac{1}{4}u^4-\lambda)y(u), \tag{2.11}$$

which is of the form (1.3).

The quantity λ is related to the ground-state energy on the lattice in a simple way. Since d^2/du^2 and $-u^4/4$ are both negative-definite operators, it follows that the spectrum of

$$Q=\exp(\epsilon^2 d^2/du^2)\exp(-u^4/4)$$

is bounded above by 1.[6] Thus, if $e^{-\lambda}$ is the maximum eigenvalue of the operator Q, then we can evaluate (2.6) to get the ground-state energy $E(x)$ on the lattice:

$$\begin{aligned} E(x) &= -(1/T)\ln\langle 0|0\rangle \\ &= -(1/T)\ln[Q^n 1(z_{n+1})] \\ &= n\lambda/T \\ &= \lambda/a \\ &= x^{2/3}g^{1/3}\lambda . \end{aligned} \tag{2.12}$$

In the next section we use WKB theory to calculate λ.

III. WKB SOLUTION OF (1.3)

In this section we show how to solve (1.3) to sixth order in a WKB approximation. We begin by transforming (1.3) to a more convenient form by substituting

$$Q(u)=V(u)-\lambda\,, \tag{3.1}$$

$$e^{\epsilon^2 d^2/du^2}y(u)=e^{Q(u)}y(u)\,. \tag{3.2}$$

The standard WKB form for the wave function $y(u)$ is

$$y(u)=e^{S(u)/\epsilon}\,. \tag{3.3}$$

The first thing to do is to find the form for the nth derivative of $\exp[S(u)/\epsilon]$. We express the result as a power series in ϵ (keeping terms of order ϵ^6) using the symbol

$$(n,k)\equiv n(n-1)(n-2)\cdots(n-k+1)\,, \tag{3.4}$$

$$\begin{aligned}
\epsilon^n\frac{d^n}{du^n}e^{S(u)/\epsilon}=e^{S(u)/\epsilon}(S')^n\Big\{1&+\epsilon\frac{(n,2)}{2}S''/(S')^2+\epsilon^2\left[\frac{(n,3)}{6}S'''/(S')^3+\frac{(n,4)}{8}(S'')^2/(S')^4\right]\\
&+\epsilon^3\left[\frac{(n,4)}{24}S^{(4)}/(S')^4+\frac{(n,5)}{12}S''S'''/(S')^5+\frac{(n,6)}{48}(S'')^3/(S')^6\right]\\
&+\epsilon^4\left[\frac{(n,5)}{120}S^{(5)}/(S')^5+\frac{(n,6)}{72}(S''')^2/(S')^6+\frac{(n,6)}{48}S''S^{(4)}/(S')^6\right.\\
&\qquad\left.+\frac{(n,7)}{48}(S'')^2S'''/(S')^7+\frac{(n,8)}{384}(S'')^4/(S')^8\right]\\
&+\epsilon^5\left[\frac{(n,6)}{720}S^{(6)}/(S')^6+\frac{(n,7)}{240}S''S^{(5)}/(S')^7+\frac{(n,7)}{144}S'''S^{(4)}/(S')^7\right.\\
&\qquad\left.+\frac{(n,8)}{144}S''(S''')^2/(S')^8+\frac{(n,8)}{192}(S'')^2S^{(4)}/(S')^8+\frac{(n,9)}{288}(S'')^3S'''/(S')^9+\frac{(n,10)}{3840}(S'')^5/(S')^{10}\right]\\
&+\epsilon^6\left[\frac{(n,7)}{5040}S^{(7)}/(S')^7+\frac{(n,8)}{1152}(S^{(4)})^2/(S')^8+\frac{(n,8)}{1440}S''S^{(6)}/(S')^8\right.\\
&\qquad+\frac{(n,8)}{720}S'''S^{(5)}/(S')^8+\frac{(n,9)}{1296}(S''')^3/(S')^9+\frac{(n,9)}{288}S''S'''S^{(4)}/(S')^9\\
&\qquad+\frac{(n,9)}{960}(S'')^2S^{(5)}/(S')^9+\frac{(n,10)}{576}(S'')^2(S''')^2/(S')^{10}\\
&\qquad\left.+\frac{(n,10)}{1152}(S'')^3S^{(4)}/(S')^{10}+\frac{(n,11)}{2304}(S'')^4S'''/(S')^{11}+\frac{(n,12)}{46\,080}(S'')^6/(S')^{12}\right]+\cdots\Big\}\,.
\end{aligned} \tag{3.5}$$

Next we evaluate the expression $\exp(\epsilon^2d^2/du^2)\exp[S(u)/\epsilon]$ by expanding the derivative operator, substituting the result in (3.5), and explicitly evaluating the infinite sums over n:

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{\epsilon^{2n}d^{2n}/du^{2n}}{n!}e^{S(u)/\epsilon}=e^{S(u)/\epsilon}e^{[S'(u)]^2}\{1&+\tfrac{1}{2}\epsilon S''P_2(S')+\epsilon^2[\tfrac{1}{6}S'''P_3(S')+\tfrac{1}{8}(S'')^2P_4(S')]\\
&+\epsilon^3[\tfrac{1}{24}S^{(4)}P_4(S')+\tfrac{1}{12}S''S'''P_5(S')+\tfrac{1}{48}(S'')^3P_6(S')]\\
&+\epsilon^4[\tfrac{1}{120}S^{(5)}P_5(S')+\tfrac{1}{72}(S''')^2P_6(S')+\tfrac{1}{48}S''S^{(4)}P_6(S')\\
&\qquad+\tfrac{1}{48}(S'')^2S'''P_7(S')+\tfrac{1}{384}(S'')^4P_8(S')]\\
&+\epsilon^5[\tfrac{1}{720}S^{(6)}P_6(S')+\tfrac{1}{240}S''S^{(5)}P_7(S')+\tfrac{1}{144}S'''S^{(4)}P_7(S')+\tfrac{1}{144}S''(S''')^2P_8(S')\\
&\qquad+\tfrac{1}{192}(S'')^2S^{(4)}P_8(S')+\tfrac{1}{288}(S'')^3S'''P_9(S')+\tfrac{1}{3840}(S'')^5P_{10}(S')]\\
&+\epsilon^6[\tfrac{1}{5040}S^{(7)}P_7(S')+\tfrac{1}{1152}(S^{(4)})^2P_8(S')+\tfrac{1}{1440}S''S^{(6)}P_8(S')+\tfrac{1}{720}S'''S^{(5)}P_8(S')\\
&\qquad+\tfrac{1}{1296}(S''')^3P_9(S')+\tfrac{1}{288}S''S'''S^{(4)}P_9(S')\\
&\qquad+\tfrac{1}{960}(S'')^2S^{(5)}P_9(S')+\tfrac{1}{576}(S'')^2(S''')^2P_{10}(S')\\
&\qquad+\tfrac{1}{1152}(S'')^3S^{(4)}P_{10}(S')+\tfrac{1}{2304}(S'')^4S'''P_{11}(S')+\tfrac{1}{46\,080}(S'')^6P_{12}(S')]+\cdots\}\,,
\end{aligned} \tag{3.6}$$

where P_n are polynomials given by

$$P_2(z)=2(1+2z^2)\,,$$
$$P_3(z)=4(3z+2z^3)\,,\quad P_4(z)=4(3+12z^2+4z^4)\,,$$
$$P_5(z)=8(15z+20z^3+4z^5)\,,\quad P_6(z)=8(15+90z^2+60z^4+8z^6)\,,$$
$$P_7(z)=16(105z+210z^3+84z^5+8z^7)\,,\quad P_8(z)=16(105+840z^2+840z^4+224z^6+16z^8)\,,$$
$$P_9(z)=32(945z+2520z^3+1512z^5+288z^7+16z^9)\,,\quad P_{10}(z)=32(945+9450z^2+12\,600z^4+5040z^6+720z^8+32z^{10})\,,$$
$$P_{11}(z)=64(10\,395z+34\,650z^3+27\,720z^5+7920z^7+880z^9+32z^{11})\,,$$
$$P_{12}(z)=64(10\,395+124\,740z^2+207\,900z^4+110\,880z^6+23\,760z^8+2112z^{10}+64z^{12})\,.$$

Next we make the conventional WKB series approximation which consists of representing the phase $S(u)$ as a power series in ϵ:[7]

$$S(u)=\sum_{n=0}^{\infty} S_n(u)\epsilon^n\,. \tag{3.7}$$

The final step consists of substituting (3.7) into (3.6) and reexpanding the entire expression as a series in powers of ϵ. This is an extremely lengthy calculation which we performed using the MACSYMA computer program. We have now calculated the left-hand side of (3.2).

Setting the expansion of the left-hand side of (3.2) equal to the right-hand side and matching coefficients of like powers of ϵ gives a sequence of algebraic equations for $S_0'(u), S_1'(u), S_2'(u), \ldots$. The first of these equations, which determines S_0', is

$$(S_0')^2=Q\,, \tag{3.8}$$

whose solutions are

$$S_0'(u)=Q^{1/2}(u) \tag{3.9}$$

and

$$S_0'(u)=-Q^{1/2}(u)\,. \tag{3.10}$$

Once the solution for $S_0'(u)$ is chosen, the solutions for $S_1'(u)$, $S_2'(u), \ldots$ are uniquely determined. This shows that although (3.2) is an infinite-order differential equation, it has only two linearly independent solutions having WKB expansions.

As in Ref. 7 we choose the solution in (3.10) and proceed to solve for the next six terms in the WKB series (3.7). The results are listed below:

$$S_1'=-\tfrac{1}{4}Q'/Q-\tfrac{1}{2}Q'\,, \tag{3.11}$$

$$S_2'=-\tfrac{1}{6}Q''Q^{1/2}-\tfrac{1}{8}Q''Q^{-3/2}-\tfrac{1}{24}(Q')^2Q^{-1/2}+\tfrac{5}{32}(Q')^2Q^{-5/2}\,, \tag{3.12}$$

$$S_3'=-\tfrac{1}{16}Q'''Q^{-2}-\tfrac{1}{24}Q'Q''/Q+\tfrac{9}{32}Q'Q''Q^{-3}+\tfrac{1}{48}(Q')^3Q^{-2}-\tfrac{15}{64}(Q')^3Q^{-4}+\tfrac{1}{12}Q'''\,, \tag{3.13}$$

$$\begin{aligned}S_4'=&\tfrac{1}{90}Q^{(4)}Q^{3/2}-\tfrac{1}{32}Q^{(4)}Q^{-5/2}+\tfrac{1}{30}Q'Q'''Q^{1/2}+\tfrac{7}{32}Q'Q'''Q^{-7/2}+\tfrac{1}{40}(Q'')^2Q^{1/2}+\tfrac{19}{128}(Q'')^2Q^{-7/2}+\tfrac{1}{720}(Q')^2Q''Q^{-1/2}\\&+\tfrac{1}{24}(Q')^2Q''Q^{-5/2}-\tfrac{221}{256}(Q')^2Q''Q^{-9/2}+\tfrac{1}{1152}(Q')^4Q^{-3/2}-\tfrac{25}{768}(Q')^4Q^{-7/2}+\tfrac{1105}{2048}(Q')^4Q^{-11/2}\,,\end{aligned} \tag{3.14}$$

$$\begin{aligned}S_5'=&-\tfrac{1}{60}Q^{(5)}Q-\tfrac{1}{64}Q^{(5)}Q^{-3}+\tfrac{5}{32}Q'Q^{(4)}Q^{-4}-\tfrac{1}{30}Q'Q^{(4)}+\tfrac{1}{48}Q''Q'''Q^{-2}+\tfrac{17}{64}Q''Q'''Q^{-4}-\tfrac{1}{18}Q''Q'''+\tfrac{1}{240}(Q')^2Q'''/Q\\&+\tfrac{1}{96}(Q')^2Q'''Q^{-3}-\tfrac{225}{256}(Q')^2Q'''Q^{-5}+\tfrac{1}{120}Q'(Q'')^2/Q-\tfrac{153}{128}Q'(Q'')^2Q^{-5}-\tfrac{1}{1440}(Q')^3Q''Q^{-2}\\&-\tfrac{7}{64}(Q')^3Q''Q^{-4}+\tfrac{1695}{512}(Q')^3Q''Q^{-6}-\tfrac{1}{576}(Q')^5Q^{-3}+\tfrac{5}{64}(Q')^5Q^{-5}-\tfrac{1695}{1024}(Q')^5Q^{-7}\,,\end{aligned} \tag{3.15}$$

$$\begin{aligned}S_6'=&-\tfrac{1}{945}Q^{(6)}Q^{5/2}+\tfrac{1}{120}Q^{(6)}Q^{1/2}-\tfrac{1}{128}Q^{(6)}Q^{-7/2}-\tfrac{1}{126}Q'Q^{(5)}Q^{3/2}+\tfrac{27}{256}Q'Q^{(5)}Q^{-9/2}+\tfrac{1}{240}Q'Q^{(5)}Q^{-1/2}\\&-\tfrac{61}{3780}Q''Q^{(4)}Q^{3/2}+\tfrac{1}{64}Q''Q^{(4)}Q^{-5/2}+\tfrac{55}{256}Q''Q^{(4)}Q^{-9/2}+\tfrac{1}{720}Q''Q^{(4)}Q^{-1/2}-\tfrac{53}{5040}(Q')^2Q^{(4)}Q^{1/2}\\&-\tfrac{1}{576}(Q')^2Q^{(4)}Q^{-3/2}+\tfrac{5}{768}(Q')^2Q^{(4)}Q^{-7/2}-\tfrac{815}{1024}(Q')^2Q^{(4)}Q^{-11/2}-\tfrac{1}{105}(Q''')^2Q^{3/2}+\tfrac{1}{64}(Q''')^2Q^{-5/2}\\&+\tfrac{69}{512}(Q''')^2Q^{-9/2}+\tfrac{1}{1440}(Q''')^2Q^{-1/2}-\tfrac{13}{420}Q'Q''Q'''Q^{1/2}-\tfrac{1}{480}Q'Q''Q'''Q^{-3/2}-\tfrac{7}{96}Q'Q''Q'''Q^{-7/2}-\tfrac{1391}{512}Q'Q''Q'''Q^{-11/2}\\&-\tfrac{49}{768}(Q')^3Q'''Q^{-9/2}+\tfrac{1055}{256}(Q')^3Q'''Q^{-13/2}-\tfrac{1}{1680}(Q')^3Q'''Q^{-1/2}-\tfrac{103}{15\,120}(Q'')^3Q^{1/2}+\tfrac{1}{960}(Q'')^3Q^{-3/2}\\&+\tfrac{19}{768}(Q'')^3Q^{-7/2}-\tfrac{631}{1024}(Q'')^3Q^{-11/2}-\tfrac{37}{3840}(Q')^2(Q'')^2Q^{-5/2}-\tfrac{133}{3072}(Q')^2(Q'')^2Q^{-9/2}\\&+\tfrac{34\,503}{4096}(Q')^2(Q'')^2Q^{-13/2}+\tfrac{1}{20\,160}(Q')^2(Q'')^2Q^{-1/2}-\tfrac{7}{34\,560}(Q')^4Q''Q^{-3/2}\\&+\tfrac{5}{9216}(Q')^4Q''Q^{-7/2}+\tfrac{5083}{12\,288}(Q')^4Q''Q^{-11/2}-\tfrac{248\,475}{16\,384}(Q')^4Q''Q^{-15/2}\\&-\tfrac{1}{27\,648}(Q')^6Q^{-5/2}+\tfrac{175}{36\,864}(Q')^6Q^{-9/2}-\tfrac{12\,155}{49\,152}(Q')^6Q^{-13/2}+\tfrac{414\,125}{65\,536}(Q')^6Q^{-17/2}\,.\end{aligned} \tag{3.16}$$

Now we impose the WKB quantization condition which will determine the eigenvalues λ of the equation (3.2). We use the same approximate quantization condition as was used in Ref. 7,

$$\frac{1}{2i\epsilon}\oint\sum_{n=0}^{\infty}\epsilon^n S_n'(u)du=K\pi\quad(K=0,1,2,\ldots)\,, \tag{3.17}$$

where K is the number of the eigenvalue and the contour encircles the branch cut joining the two turning points (we assume for simplicity that there are only two). The turning points are solutions of $Q(u)=0$.

As is the case with the ordinary WKB series[7] the integrals in (3.17) corresponding to odd values of n are trivial to evaluate. This is true because $S_1', S_3', S_5', \ldots$ are all representable as total derivatives:

$$S_1'=\frac{d}{du}\left(-\tfrac{1}{4}\ln Q-\tfrac{1}{2}Q\right), \tag{3.18}$$

$$S_3'=\frac{d}{du}\left[-\tfrac{1}{48}(Q')/Q-\tfrac{1}{16}Q''Q^{-2}+\tfrac{5}{64}(Q')^2Q^{-3}+\tfrac{1}{12}Q''\right], \tag{3.19}$$

$$\begin{aligned}S_5'=\frac{d}{du}\Big[&-\tfrac{7}{360}(Q'')^2-\tfrac{1}{60}Q^{(4)}Q-\tfrac{1}{60}Q'Q'''-\tfrac{1}{64}Q^{(4)}Q^{-3}\\&+\tfrac{7}{64}Q'Q'''Q^{-4}+\tfrac{1}{96}(Q'')^2Q^{-2}+\tfrac{1}{96}(Q')^2Q''Q^{-3}\\&-\tfrac{5}{256}(Q')^4Q^{-4}+\tfrac{1}{240}(Q')^2Q''/Q+\tfrac{1}{1152}(Q')^4Q^{-2}\\&+\tfrac{5}{64}(Q'')^2Q^{-4}-\tfrac{113}{256}(Q')^2Q''Q^{-5}+\tfrac{565}{2048}(Q')^4Q^{-6}\Big]\,.\end{aligned} \tag{3.20}$$

All terms being differentiated in (3.18)–(3.20) are single-valued except for $-\frac{1}{4}\ln Q$. Thus, the contour integration gives

$$\frac{1}{2i\epsilon}\oint\epsilon S_1'(u)du=-\frac{\pi}{2}\,,$$

$$\frac{1}{2i\epsilon}\oint\epsilon^3 S_3'(u)du=0\,,$$

$$\frac{1}{2i\epsilon}\oint\epsilon^5 S_5'(u)du=0\,.$$

[Evaluating $\ln Q(u)$ once around the contour gives $4\pi i$ because the contour encircles two simple zeros (turning points) of $Q(u)$.]

Substituting the above results into the quantization condition in (3.17) gives, to sixth order in powers of ϵ,

$$\frac{\pi}{2}=\frac{1}{2i\epsilon}\oint du\left[S_0'(u)+\epsilon^2S_2'(u)+\epsilon^4S_4'(u)+\epsilon^6S_6'(u)+\cdots\right], \tag{3.21}$$

where we have taken $K=0$ because we are only interested in the first eigenvalue λ. We show in Secs. IV and V how to use this general result to solve for the eigenvalues of the infinite-order differential equation in (3.2).

IV. APPLICATION OF THE GENERAL WKB FORMULA TO THE LATTICE HARMONIC OSCILLATOR

In Ref. 2 we derived the high-temperature expansion for the harmonic oscillator defined by the Lagrangian

$$L=\tfrac{1}{2}\dot{q}^2+\tfrac{1}{2}m^2q^2\,. \tag{4.1}$$

We showed that the graphical expansion for the ground-state energy E consists of summing over all closed polygon graphs (one graph in every order). The final result was[2]

$$E=\frac{m}{2}\left[-\sqrt{x}\ln x-\sqrt{x}\sum_{k=1}^{\infty}\frac{1}{k}(-x)^k\frac{(2k)!}{(k!)^2}\right], \tag{4.2}$$

where $x=m^{-2}a^{-2}$ and a is the lattice spacing.

Observe that this high-temperature expansion is a power series in x. This is typical of any high-temperature expansion. By contrast, the WKB expansion on the lattice is a series in powers of $1/x$. Our intention here is to reexpand (4.2) as a series in powers of $1/x$ and then to compare the result with the predictions of the WKB series in (3.21).

We begin by expressing (4.2) as an integral:

$$\begin{aligned}E&=\frac{m}{2}\left[-\sqrt{x}\ln x-\sqrt{x}\int_0^x\frac{dt}{t}\left(\frac{1}{(1+4t)^{1/2}}-1\right)\right]\\&=\frac{m}{2}\sqrt{x}\int_x^{\infty}\frac{dt}{t}\frac{1}{(1+4t)^{1/2}}\,.\end{aligned}$$

Next we expand this integral as a series in powers of $1/x$:

$$\begin{aligned}E&=\frac{m}{4}\sqrt{x}\int_x^{\infty}\frac{dt}{t^{3/2}}\sum_{k=0}^{\infty}\frac{(-1)^k(2k)!}{16^k(k!)^2t^k}\\&=\frac{m}{2}\sum_{k=0}^{\infty}\frac{(-1)^k(2k)!}{16^k(k!)^2x^k(2k+1)}\\&=\frac{m}{2}\left(1-\frac{1}{24x}+\frac{3}{640x^2}-\frac{5}{7168x^3}+\cdots\right).\end{aligned} \tag{4.3}$$

Next we turn to the WKB expansion of E. The integral equation corresponding to (2.11) is

$$e^{\epsilon^2d^2/du^2}y(u)=e^{(1/2)u^2-\lambda}y(u)\,, \tag{4.4}$$

where again $\epsilon=1/\sqrt{2x}$. Also, the equation corresponding to (2.12) which relates the ground-state energy E with the eigenvalue λ is

$$E=\sqrt{x}\,m\lambda\,. \tag{4.5}$$

We have just derived in (4.3) an expansion for E and the WKB series in (3.21) will give an expansion for λ. We must compare these two expansions.

Let us take $Q(u)$ to have the general form

$$Q(u)=A+Bu^2\,, \tag{4.6}$$

where A and B are constants. For this *ansatz* S_0' in (3.10), S_2' in (3.12), S_4' in (3.14), and S_6' in (3.16) reduce to

$$S_0' = -Q^{1/2}, \tag{4.7}$$

$$S_2' = -\tfrac{1}{2}BQ^{1/2} + \tfrac{1}{6}ABQ^{-1/2} + \tfrac{3}{8}BQ^{-3/2} - \tfrac{5}{8}ABQ^{-5/2}, \tag{4.8}$$

$$\begin{aligned} S_4' = {} & \tfrac{1}{8}B^2Q^{1/2} - \tfrac{7}{180}AB^2Q^{-1/2} + \left(\tfrac{1}{72}A^2B^2 - \tfrac{3}{16}B^2\right)Q^{-3/2} + \tfrac{17}{24}AB^2Q^{-5/2} \\ & + \left(\tfrac{297}{128}B^2 - \tfrac{25}{48}A^2B^2\right)Q^{-7/2} - \tfrac{663}{64}AB^2Q^{-9/2} + \tfrac{1105}{128}A^2B^2Q^{-11/2}, \end{aligned} \tag{4.9}$$

$$\begin{aligned} S_6' = {} & -\tfrac{1}{16}B^3Q^{1/2} + \tfrac{289}{15\,120}AB^3Q^{-1/2} + \left(\tfrac{9}{64}B^3 - \tfrac{29}{2160}A^2B^3\right)Q^{-3/2} + \left(\tfrac{1}{432}A^3B^3 - \tfrac{2081}{2880}AB^3\right)Q^{-5/2} \\ & + \left(\tfrac{515}{576}A^2B^3 - \tfrac{891}{256}B^3\right)Q^{-7/2} + \left(\tfrac{5555}{256}AB^3 - \tfrac{175}{576}A^3B^3\right)Q^{-9/2} + \left(\tfrac{50\,139}{1024}B^3 - \tfrac{26\,299}{768}A^2B^3\right)Q^{-11/2} \\ & + \left(\tfrac{12\,155}{768}A^3B^3 - \tfrac{386\,487}{1024}AB^3\right)Q^{-13/2} + \tfrac{745\,425}{1024}A^2B^3Q^{-15/2} - \tfrac{414\,125}{1024}A^3B^3Q^{-17/2}. \end{aligned} \tag{4.10}$$

An enormous simplification now takes place because Q is quadratic. Specifically, the contour integrals are

$$\oint Q^{n/2}du = 0 \quad (n = -3, -5, -7, -9, \ldots). \tag{4.11}$$

The only nonvanishing integrals are

$$\frac{1}{2i}\oint Q^{1/2}du = \tfrac{1}{2}\pi AB^{-1/2} \tag{4.12}$$

and

$$\frac{1}{2i}\oint Q^{-1/2}du = \pi B^{-1/2}. \tag{4.13}$$

For our case $Q(u) = \frac{1}{2}u^2 - \lambda$, so $A = -\lambda$ and $B = \frac{1}{2}$. Thus, after combining (4.7)–(4.13), the WKB series in (3.21) becomes

$$\begin{aligned} \frac{\pi}{2} = {} & \frac{1}{\epsilon}\left(\tfrac{1}{2}\pi\lambda\sqrt{2}\right) + \epsilon\left(\tfrac{1}{8}\pi\lambda\sqrt{2} - \tfrac{1}{12}\pi\lambda\sqrt{2}\right) \\ & + \epsilon^3\left(-\tfrac{1}{64}\pi\lambda\sqrt{2} + \tfrac{7}{720}\pi\lambda\sqrt{2}\right) \\ & + \epsilon^5\left(\tfrac{1}{256}\pi\lambda\sqrt{2} - \tfrac{289}{120\,960}\pi\lambda\sqrt{2}\right) + \cdots. \end{aligned}$$

Multiplying this equation by $\sqrt{2}\epsilon/(\pi\lambda)$ gives

$$\frac{\epsilon}{\lambda\sqrt{2}} = 1 + \tfrac{1}{12}\epsilon^2 - \tfrac{17}{1440}\epsilon^4 + \tfrac{367}{120\,960}\epsilon^6 + \cdots. \tag{4.14}$$

Finally, we use the relation

$$\epsilon = 1/\sqrt{2x}$$

and

$$\lambda = E/(m\sqrt{x})$$

to transform (4.14) to the form

$$\frac{m}{2E} = 1 + \frac{1}{24x} - \frac{17}{5760x^2} + \frac{367}{967\,680x^3} + \cdots. \tag{4.15}$$

To solve for E we must invert the power series on the right-hand side of (4.15). The result is

$$E = \frac{m}{2}\left(1 - \frac{1}{24x} + \frac{3}{640x^2} - \frac{5}{7168x^3} + \cdots\right), \tag{4.16}$$

which is precisely the series in (4.3).

Thus, WKB theory gives the exact series for the harmonic oscillator on the lattice in powers of $1/x$. WKB theory also gives the exact eigenvalues for the continuum harmonic oscillator. In Sec. V we consider the WKB expansion for the lattice anharmonic oscillator where we do not expect WKB theory to be exact, but to give a sequence of approximations to the exact expansion.

V. APPLICATION OF THE GENERAL WKB FORMULA TO THE LATTICE ANHARMONIC OSCILLATOR

From Refs. 1–4 the high-temperature expansion on the lattice for the ground-state energy of the anharmonic oscillator defined by (1.1) is[8]

$$\begin{aligned} E = g^{1/3}x^{2/3}\{ & \ln[2\Gamma(1/4)\sqrt{\pi/x}\,] + 2Rx - \tfrac{1}{2}x^2 + \left(-\tfrac{16}{3}R^3 + 2R\right)x^3 + \left(-14R^4 + 3R^2 - \tfrac{5}{8}\right)x^4 \\ & + \left(-\tfrac{48}{5}R^5 - 4R^3 + \tfrac{29}{15}R\right)x^5 + \left(64R^6 - 36R^4 + \tfrac{91}{15}R^2 - \tfrac{3}{4}\right)x^6 + \left(\tfrac{1920}{7}R^7 - 80R^5 + \tfrac{4}{5}R^3 + \tfrac{5}{3}R\right)x^7 \\ & + \left(396R^8 + 84R^6 - \tfrac{719}{10}R^4 + \tfrac{233}{20}R^2 - \tfrac{3547}{4032}\right)x^8 + \left(-\tfrac{8512}{9}R^9 + 1136R^7 - \tfrac{1444}{5}R^5 + \tfrac{109}{5}R^3 + \tfrac{65}{252}R\right)x^9 \\ & + \left(-6656R^{10} + 3280R^8 - \tfrac{1004}{5}R^6 - \tfrac{502}{5}R^4 + \tfrac{1889}{105}R^2 - \tfrac{899}{1008}\right)x^{10} \\ & + \left(-\tfrac{161\,280}{11}R^{11} - 64R^9 + \tfrac{44\,912}{15}R^7 - \tfrac{11\,852}{15}R^5 + \tfrac{119\,153}{1575}R^3 - \tfrac{3761}{1260}R\right)x^{11} \\ & + \left(\tfrac{28\,480}{3}R^{12} - 36\,128R^{10} + \tfrac{221\,524}{15}R^8 - \tfrac{5456}{3}R^6 - \tfrac{175\,799}{6300}R^4 + \tfrac{49\,369}{2520}R^2 - \tfrac{7751}{12\,096}\right)x^{12} + \cdots\}, \end{aligned} \tag{5.1}$$

where

$R=\Gamma(3/4)/\Gamma(1/4)\approx 0.337\,989\,120$.

Evaluating the coefficients in (5.1) numerically gives

$$\begin{aligned}E(x)=g^{1/3}x^{2/3}(&-\tfrac{1}{2}\ln x+2.553\,534\,65+0.675\,978\,24x-0.5x^2+0.470\,054\,28x^3-0.464\,990\,22x^4\\&+0.456\,659\,35x^5-0.431\,354\,22x^6+0.379\,547\,37x^7-0.294\,484\,96x^8\\&+0.173\,023\,54x^9-0.017\,163\,21x^{10}-0.164\,390\,76x^{11}+0.355\,112\,52x^{12}+\cdots).\end{aligned} \tag{5.2}$$

Equation (5.2) is the analog of (4.2) for the harmonic oscillator.

What does WKB theory predict about the large-x behavior of the function E? To answer this question we substitute

$$Q(u)=A+Bu^4, \tag{5.3}$$

where A and B are constants, into the expressions for S_0' in (3.10), S_2' in (3.12), S_4' in (3.14), and S_6' in (3.16). The results are

$$S_0'=-Q^{1/2}, \tag{5.4}$$

$$S_2'=Bu^2(-\tfrac{8}{3}Q^{1/2}+\tfrac{2}{3}AQ^{-1/2}+Q^{-3/2}-\tfrac{5}{2}AQ^{-5/2}), \tag{5.5}$$

$$\begin{aligned}S_4'=B[&\tfrac{68}{9}Q^{3/2}-8AQ^{1/2}+(\tfrac{14}{15}A^2-\tfrac{1}{3})Q^{-1/2}+(9A-\tfrac{2}{9}A^3)Q^{-3/2}+(14-17A^2)Q^{-5/2}\\&+(\tfrac{25}{3}A^3-\tfrac{501}{4}A)Q^{-7/2}+\tfrac{1989}{8}A^2Q^{-9/2}-\tfrac{1105}{8}A^3Q^{-11/2}],\end{aligned} \tag{5.6}$$

$$\begin{aligned}S_6'=B^2u^2[&-\tfrac{8528}{135}Q^{3/2}+\tfrac{7496}{135}AQ^{1/2}-(\tfrac{32}{9}A^2+\tfrac{44}{9})Q^{-1/2}+(\tfrac{164}{135}A^3-\tfrac{244}{9}A)Q^{-3/2}+(\tfrac{1342}{15}A^2-\tfrac{4}{27}A^4-\tfrac{152}{3})Q^{-5/2}\\&+(\tfrac{4555}{6}A-\tfrac{685}{9}A^3)Q^{-7/2}+(\tfrac{175}{9}A^4-2463A^2+671)Q^{-9/2}+(\tfrac{33\,371}{12}A^3-\tfrac{42\,959}{4}A)Q^{-11/2}\\&+(\tfrac{330\,141}{8}A^2-\tfrac{12\,155}{12}A^4)Q^{-13/2}-\tfrac{911\,075}{16}A^3Q^{-15/2}+\tfrac{414\,125}{16}A^4Q^{-17/2}].\end{aligned} \tag{5.7}$$

Next we evaluate the contour integrals in the quantization condition (3.21), taking $A=-\lambda$ and $B=\frac{1}{4}$. The results are

$$\frac{1}{2i}\oint S_0'(u)du=\frac{1}{3R}\sqrt{2\pi}\,\lambda^{3/4}, \tag{5.8}$$

$$\frac{1}{2i}\oint S_2'(u)du=R\sqrt{2\pi}\,(\tfrac{2}{5}\lambda^{5/4}-\tfrac{1}{8}\lambda^{-3/4}), \tag{5.9}$$

$$\frac{1}{2i}\oint S_4'(u)du=\frac{\sqrt{2\pi}}{R}(-\tfrac{61}{2520}\lambda^{7/4}+\tfrac{5}{192}\lambda^{-1/4}+\tfrac{11}{6144}\lambda^{-9/4}), \tag{5.10}$$

$$\frac{1}{2i}\oint S_6'(u)du=R\sqrt{2\pi}\,(\tfrac{2509}{8100}\lambda^{9/4}-\tfrac{733}{960}\lambda^{1/4}+\tfrac{83}{5120}\lambda^{-7/4}+\tfrac{4697}{245\,760}\lambda^{-15/4}). \tag{5.11}$$

Inserting (5.8)–(5.11) into the WKB series (3.21) and simplifying, we get

$$\begin{aligned}\frac{\epsilon\sqrt{\pi}}{2\sqrt{2}}=\frac{1}{3R}\lambda^{3/4}&\epsilon^2(\tfrac{2}{5}\lambda^{5/4}-\tfrac{1}{8}\lambda^{-3/4})R+\epsilon^4(-\tfrac{61}{2520}\lambda^{7/4}+\tfrac{5}{192}\lambda^{-1/4}+\tfrac{11}{6144}\lambda^{-9/4})/R\\&+\epsilon^6(\tfrac{2509}{8100}\lambda^{9/4}-\tfrac{733}{960}\lambda^{1/4}+\tfrac{83}{5120}\lambda^{-7/4}+\tfrac{4697}{245\,760}\lambda^{-15/4})R+\cdots,\end{aligned} \tag{5.12}$$

which is the analog of (4.14) for the harmonic oscillator.

Following the approach we took in Sec. IV for the harmonic oscillator we eliminate the expansion parameter ϵ in favor of x using

$$\epsilon=1/\sqrt{2x}$$

and we eliminate λ in favor of E using

$$\lambda=g^{-1/3}x^{-2/3}E.$$

[This last equation follows from (2.12).] For simplicity we define the dimensionless quantity

$$F=Eg^{-1/3}.$$

In terms of the new variable F and x, (5.12) becomes

$$\sqrt{\pi}=\frac{4}{3R}F^{3/4}+\left(-\frac{R}{4}F^{-3/4}+\frac{4R}{5}F^{5/4}x^{-4/3}\right)+\left(\frac{11}{6144R}F^{-9/4}+\frac{5}{192R}F^{-1/4}x^{-4/3}-\frac{61}{2520R}F^{7/4}x^{-8/3}\right)$$
$$+\left(\frac{4697}{491\,520}F^{-15/4}+\frac{83R}{10\,240}F^{-7/4}x^{-4/3}-\frac{733R}{1920}F^{1/4}x^{-8/3}+\frac{2509R}{16\,200}F^{9/4}x^{-4}\right)+\cdots, \tag{5.13}$$

which is the analog of (4.15).

Observe that (5.13) is much more complicated than (4.15) because the eigenvalue F appears implicitly rather than explicitly. Each term in parentheses corresponds to an additional order in WKB theory. Thus, if we solve (5.13) for F as an expansion in powers of $1/x$ (or, more precisely $x^{-4/3}$) then unlike (4.16), every coefficient of this series continues to change with each new order of WKB theory.[9] Specifically, solving (5.13) for F to 0th, 2nd, 4th, and 6th orders in WKB theory and multiplying by $g^{1/3}$ to obtain E gives

$$E_{\text{0th order in WKB}}=g^{1/3}(0.344\,126\,88+\cdots), \tag{5.14}$$

$$E_{\text{2nd order in WKB}}=g^{1/3}(0.389\,215\,68-0.020\,392\,45x^{-4/3}+\cdots), \tag{5.15}$$

$$E_{\text{4th order in WKB}}=g^{1/3}(0.377\,659\,78-0.046\,439\,09x^{-4/3}+0.006\,283\,07x^{-8/3}+\cdots), \tag{5.16}$$

$$E_{\text{6th order in WKB}}=g^{1/3}(0.312\,535\,48-0.212\,003\,19x^{-4/3}-1.030\,631\,08x^{-8/3}-12.917\,172\,9x^{-4}+\cdots). \tag{5.17}$$

Note that (5.14)–(5.17) are converging to a series representation for $E(x)$ in powers of $x^{-4/3}$:

$$E(x)=g^{1/3}\sum_{n=0}^{\infty}a_n x^{-4n/3}. \tag{5.18}$$

However, the convergence is in an asymptotic sense. That is, each coefficient of a given power of $x^{-4/3}$ in the series (5.14)–(5.17) approaches the corresponding coefficient in (5.18) for a while and then veers off. For example, the eigenvalue of the continuum anharmonic oscillator is $E(\infty)=a_0\approx 0.420\,805$. Second-order WKB theory gives the best approximation to this answer and as the order of WKB theory is increased the approximation to a_0 gets poorer. (Note that for eigenvalues larger than the ground-state energy the accuracy of the WKB series is dramatically improved. See Ref. 7.)

Thus, in order to extract better results from the WKB series, it is necessary to use a summation procedure. We discuss one such procedure which uses continued fractions in the Appendix. From the information we now have we believe that it is likely that the series (5.18) has the following properties: (i) It is alternating. (ii) It is rapidly convergent (a_1 is roughly $a_0/10$ and a_2 is roughly $a_1/10$). The series in (5.18) is the analog of (4.16) for the harmonic oscillator. It is also the complement of the series in (5.2) in the sense that it gives the large-x rather than the small-x behavior of $E(x)$.

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APPENDIX

In this Appendix we show how a summation method can improve the predictions of the WKB series. We investigate here only the WKB series for the continuum case (that is, the series with $x=\infty$). Using the results from Ref. 7 the WKB series in fourteenth order for the ground-state energy is

$$\sqrt{\pi}=\frac{4}{3R}F^{3/4}-\frac{R}{4}F^{-3/4}+\frac{11}{6144R}F^{-9/4}+\frac{4697R}{491\,520}F^{-15/4}$$
$$-\frac{390\,065}{234\,881\,024R}F^{-21/4}-\frac{53\,352\,893R}{1\,610\,612\,736}F^{-27/4}$$
$$+\frac{122\,528\,437\,805}{9\,070\,970\,929\,152R}F^{-33/4}$$
$$+\frac{37\,089\,126\,931\,059\,R}{57\,174\,604\,644\,352}F^{-39/4}-\cdots, \tag{A1}$$

where each new term on the right-hand side of (A1) corresponds to one new even order of WKB approximation.

The exact value of F is 0.420 805... . The WKB approximations to F are obtained by truncating (A1) and solving for F. The sequence of solutions for F ultimately diverges:

0th-order WKB $F\approx 0.34$,

2nd-order WKB $F\approx 0.39$,

4th-order WKB $F\approx 0.38$,

6th-order WKB $F\approx 0.31$,

8th-order WKB $F\approx 0.45$,

10th-order WKB $F\approx 0.56$,

12th-order WKB no positive roots,

14th-order WKB no positive roots.

The following summation procedure makes a dramatic improvement. We rewrite (A1) in the form of a power series in $F^{-3/2}$,

$$1+\sum_{n=1}^{\infty} A_n F^{-3n/2}=\frac{3R}{4}\sqrt{\pi}\,F^{-3/4}$$

and convert the left-hand side of this divergent series to a continued fraction

$$\cfrac{1}{1+\cfrac{C_1F^{-3/2}}{1+\cfrac{C_2F^{-3/2}}{1+\cfrac{C_3F^{-3/2}}{1+\cdots}}}}=\frac{3R}{4}\sqrt{\pi}\,F^{-3/4}. \tag{A2}$$

The first seven continued fraction coefficients in (A2) are

$$C_1=\frac{3R^2}{16}\approx 0.021\,419\,37\,,$$

$$C_2=\frac{11-288R^4}{1536R^2}\approx 0.041\,270\,31\,,$$

$$C_3=\frac{605+450\,912R^4}{-84\,480R^2+2\,215\,840R^6}\approx -1.021\,419\,1\,,$$

$$C_4\approx 2.281\,321\,5\,,$$

$$C_5\approx -2.129\,416\,8\,,$$

$$C_6\approx 2.226\,307\,6\,,$$

$$C_7\approx -5.639\,487\,3\,.$$

Next we solve for F by truncating the continued-fraction expansion. Now, rather than diverging, the solutions for F could well be slowly converging to the exact answer for F:

0th-order WKB continued fraction $F\approx 0.344\,127$,

2nd-order WKB continued fraction $F\approx 0.385\,748$,

4th-order WKB continued fraction $F\approx 0.380\,225$,

6th-order WKB continued fraction $F\approx 0.387\,760$,

8th-order WKB continued fraction $F\approx 0.377\,142$,

10th-order WKB continued fraction $F\approx 0.385\,461$,

12th-order WKB continued fraction $F\approx 0.379\,919$,

14th-order WKB continued fraction $F\approx 0.386\,271$.

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[1]C. M. Bender, F. Cooper, G. S. Guralnik, and D. H. Sharp, Phys. Rev. D 19, 1865 (1979).

[2]C. M. Bender, F. Cooper, G. S. Guralnik, R. Roskies, D. H. Sharp, and M. L. Silverstein, Phys. Rev. D 15, 1374 (1979).

[3]C. M. Bender, F. Cooper, G. S. Guralnik, R. Roskies, and D. H. Sharp, Phys. Rev. Lett. 43, 537 (1979).

[4]C. M. Bender, F. Cooper, G. S. Guralnik, H. Rose, and D. Sharp, Los Alamos Scientific Laboratory report (unpublished).

[5]Here we have replaced time t by it because we are working with Euclidean-space functional integrals.

[6]It is easy to show that this operator is a Hilbert-Schmidt operator and therefore has a discrete spectrum. Moreover, we are using the almost obvious facts that 1 is not orthogonal to $|0\rangle$ and $|0\rangle$ is nondegenerate. If $\phi(u)$ satisfies (2.9) and is not everywhere positive or everywhere negative, then

$$\int du\, e^{-u^4/4}\,|\phi(u)|Q|\phi(u)|>e^{-\lambda}\int du\, e^{-u^4/4}\,|\phi(u)|^2.$$

This means that ϕ cannot be an eigenfunction with $e^{-\lambda}$ maximal. Therefore, if ϕ satisfies (2.9) with $e^{-\lambda}$ maximal, ϕ is not orthogonal to 1 and also ϕ is unique up to a multiplicative constant.

[7]See J. L. Dunham, Phys. Rev. 41, 713 (1932) and C. M. Bender, K. Olaussen, and P. S. Wang, Phys. Rev. D 16, 1740 (1977). See also C. M. Bender and S. A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers* (McGraw-Hill, New York, 1978).

[8]Coefficients of x^{10}, x^{11}, and x^{12} obtained from R. Roskies (private communication).

[9]In the continuum limit $x\to\infty$ (5.13) reduces to Eq. (14) in Ref. 7 with R replaced by $1/R$ and $F^{3/4}$ replaced by $E^{3/4}/2$.