Real-space renormalization-group scheme for spin and gauge systems

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We present a real-space renormalization-group scheme for both spin and gauge systems within a Hamiltonian formalism. The approximation, in particular, preserves gauge invariance at every step of the calculation. We apply this scheme to the $(1+1)$-dimensional Ising model in a transverse field and to the $(2+1)$-dimensional Ising gauge theory. We find reasonable results for the critical coupling and for those critical exponents which are related to energy gaps. We also obtain the correct qualitative behavior for order and disorder parameters and correlation functions. In particular, the calculation yields exponential decay for correlation functions in the disordered phase. However, the critical indices we find for spacelike quantities are not good. This defect of the approximation is related to the asymmetric scaling of space and time under the renormalization group.

I. INTRODUCTION

Our purpose in this paper is to present a real-space renormalization-group method applicable to lattice gauge theories[1,2] in a Hamiltonian formalism.[3] The main feature of this technique is that it is gauge invariant at all stages of the calculation. Satisfying this constraint is precisely the problem that makes it hard to develop renormalization-group methods for lattice gauge theories. To our knowledge, the only other gauge-invariant renormalization-group scheme is the Migdal-Kadanoff approach.[4,5]

The method we devise can be applied to both spin and gauge systems. It is a block-spin procedure[6-9] in which each block is considered in conjunction with all possible boundary conditions. We apply the technique to the $(1+1)$-dimensional [or sometimes referred to as the one- (space) dimensional] Ising model in a transverse field[10] and to the $(2+1)$-dimensional Ising gauge theory[11,12] in the Hamiltonian form.[13] The Ising model is discussed just to illustrate the technique in a simple situation. In this case we obtain recursion relations for the coupling constant, energy gaps, magnetization, and equal-time correlation functions. One important feature we find is that our technique predicts the correct qualitative behavior of the operators in the different phases. In particular, the method yields exponentially decaying correlation functions in the disordered phase. The numerical results, in this simplest scheme, however, are not particularly good. Indeed we get, at best, 20% accuracy (see Table I). A characteristic feature of the approximation is that the critical exponents ν_t for the correlation length and ν_G for the energy gap are different. This is not the case in the exact solution[10] since, in the critical region, the correlation length ξ is the inverse of the energy gap G. This follows from the fact that this model is the Hamiltonian limit[13] of the two-dimensional Ising model which is rotationally ("space-time") symmetric at the critical point.[14] Our approximation explicitly violates this symmetry.

The exponents ν_t and ν_G are given by

$$\nu_t = \frac{\ln 2}{\ln(1+\beta_c')} , \tag{1.1a}$$

$$\nu_G = \frac{\ln(1/Z_c)}{\ln(1+\beta_c')} , \tag{1.1b}$$

where β_c' and Z_c are, respectively, the slope of the β function and the change in the energy scale at the critical point. Since this technique provides a variational approximation to the ground-state energy, it is not surprising that ν_G is found to be much better than ν_t (see Table I).

In the case of the $(2+1)$-dimensional Z_2 gauge theory, we compute the critical coupling and the critical exponents for the energy gaps, string tension, and monopole ground-state expectation value. These results are shown in Table II. We compare them with the results obtained in Refs. 9, 15, and 16 for the analogous quantities in the dual model, the two-dimensional Ising model in a transverse field. We get remarkably good results for the critical coupling (as compared with Pfeuty and Elliott's[15] perturbation-theory calculations) and the energy-gap exponent.

We argue that techniques such as ours, as well as other similar ones,[6-9] will give good results for the ground-state energy and the energies of local excitations. As they stand, however, these tech-

TABLE I. (1+1)-dimensional Ising model with a transverse field.

	λ_c	ν_G	ν_ξ	β	η	Z_c
Calculated value	1.277	0.817	1.482	0.075	0.10	0.68
Ref. 9 first order	0.784	···	1.48	0.408	···	0.68
Ref. 9 second order	0.913	···	0.882	0.170	···	0.52
Ref. 8 7 spins/cell	0.947	0.86	1.16	0.145	0.275	0.60
Exact result Ref. 10	1	1	1	0.125	0.25	0.5

niques are too crude to produce reliable results for "spacelike" quantities such as correlation lengths and, in general, any spatial correlation function.

The paper is organized as follows: Section II deals with the one-dimensional Ising model in a transverse field. The Ising gauge theory in 2 + 1 dimensions is discussed in Sec. III.

II. ONE-DIMENSIONAL ISING MODEL IN A TRANSVERSE FIELD

A. The model

The one-dimensional Ising model in a transverse field is a quantum-mechanical spin system with a Hamiltonian

$$H = -\epsilon \sum_{i=1}^{N} \sigma_1(i) - \Delta \sum_{i=1}^{N} \sigma_3(i)\sigma_3(i+1), \qquad (2.1)$$

where i runs over the N sites of a linear chain with periodic boundary conditions. The operators $\sigma_1(i)$ and $\sigma_3(i)$ are the standard Pauli matrices

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad (2.2)$$

at site i.

The parameters ϵ and Δ are dimensional coupling constants. Indeed any one of them, say ϵ, defines the *scale of energies* and the ratio

$$\lambda = \Delta/\epsilon \qquad (2.3)$$

is a *dimensionless coupling constant*.

On a one-dimensional lattice this model is exactly soluble (see Pfeuty[10]). The critical coupling is $\lambda_c = 1$ and at this point the ground state becomes degenerate. The energy gap is given by the expression

$$G = 2\epsilon|1-\lambda|, \qquad (2.4)$$

which yields a critical exponent ν_G for the gap equal to one.

TABLE II. (2+1)-dimensional Ising gauge theory.

	λ_c	ν_G	ν_ξ	β	ν_T	Z_c
Calculated value	3.280	0.622	1.202	0.91	1.82	0.349
Ref. 16(a) 3D Ising model high-temp. exp.	···	···	0.639	0.3125	···	···
Ref. 15 2+1D Ising model with a transverse field perturbative calc.	3.11 ± 0.03	0.63	···	0.32	···	···
Ref. 16(b) 3D Ising model RSRG calc.	···	···	0.6289	0.3243	···	$\frac{1}{4}$
Ref. 9 2+1D Ising model with a transverse field RSRG calc.	···	···	0.724	0.394	···	···

The one-dimensional Ising model in a transverse field belongs to the same universality class as the two-dimensional classical Ising model. Actually, the former is the time continuous (or Hamiltonian) limit of the latter.[13]

B. The method

The technique we use is a variation of real-space renormalization-group methods for Hamiltonian systems used previously in the literature by Drell *et al.*[6] and other authors[7-9]. Like other schemes, it is an algorithm for selecting a variational ground state for the theory. The algorithm makes use of the essential fact that it is better to approximate the ground state for a few number of degrees of freedom at any one stage of the calculation than to try to guess at one shot the structure of the ground-state wave function for the entire system. Hence, one typically breaks up the lattice into blocks, solves for the low-lying states within a block, and then truncates the Hilbert space to the subspace spanned by these states. One then has a new renormalized Hamiltonian, for length scales larger than the size of the block, which can always be written in terms of spin operators and renormalized coupling constants. This procedure is then iterated until the effective Hamiltonian for large length scales has approached some limiting form which can be solved perturbatively. Procedures of this type, however, generally assume the blocks to be free of boundary conditions, i.e., the blocks are disconnected from the rest of the system. This then leads to strong edge effects for the block which eventually induce long-range correlations in the ground state of the whole system.[8] As a consequence, correlation functions, which should decay exponentially in the disordered phase, invariably exhibit power-law behavior. This defect of the approximation is well known and there are several approaches in the literature for improving the method.[8,9,17,18]

The method we propose, however, eliminates this problem from the very beginning. The blocks are no longer disjoint. In fact, they are coupled to one another by a set of boundary variables which provide internal boundary conditions for the block. In its simplest form the procedure is the following.

Consider a linear chain of spins and its two sublattices, the even and the odd sites (Fig. 1). A block will consist of a spin at a given odd site coupled to its two (even) neighbors. The block Hamiltonian, at the odd site $2i+1$, is given by

$$H_{2i+1} = -\epsilon\sigma_1(2i+1) - \Delta\sigma_3(2i+1)[\sigma_3(2i)+\sigma_3(2i+2)]. \quad (2.5)$$

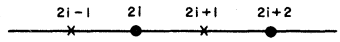

FIG. 1. The two sublattices of the linear chain. The odd sites (crosses) are the center of the blocks. The even sites (dots) are the remaining degrees of freedom.

The full Hamiltonian (2.1) may be written as

$$H = H_0 + H_1, \quad (2.6a)$$

where

$$H_0 = \sum_{i=1}^{N/2} H_{2i-1} \quad (2.6b)$$

and

$$H_1 = -\epsilon \sum_{i=1}^{N/2} \sigma_1(2i). \quad (2.6c)$$

The procedure can be generalized to larger blocks in an obvious manner. What is important to observe here is that the Hamiltonian for the block at the site $2i-1$ depends on the state of the spins at the two nearby even sites, $\sigma_3(2i)$ and $\sigma_3(2i-2)$. These spins act effectively like arbitrary boundary magnetic fields with strength Δ. Since their transverse fields are not included in the block Hamiltonian then, from the point of view of the block, they behave like classical variables (i.e., commute with H_{2i-1}).

The strategy is first to fix an arbitrary configuration at the even sites and then to diagonalize all the block Hamiltonians in that configuration. We will then truncate the Hilbert space to the set of states of the form

$$|\psi(\sigma)\rangle = \prod_{i=1}^{N/2} \psi_{2i-1}(\sigma_3(2i-2);\ \sigma_3(2i))\,|\sigma_1(2i)\rangle, \quad (2.7)$$

where ψ_{2i-1} is the ground-state wave function of the block $2i-1$ and $|\sigma_1(2i)\rangle$ is a complete set of states that we represent as eigenstates of σ_1, the transverse field operator. Notice that ψ is an operator when acting on the even sites, since it depends on the σ_3's at these sites.

The renormalized Hamiltonian will be the old one restricted to the subspace (2.7)

$$H^{\text{ren}} \equiv \langle\psi(\sigma)|H|\psi(\sigma')\rangle, \quad (2.8)$$

where $|\psi(\sigma)\rangle$ and $|\psi(\sigma')\rangle$ are two states of the form (2.7).

Let us now evaluate the ground-state energy E and wave function ψ for the block. This can easily be done. The ground-state energy, for arbitrary boundary conditions, is found to be

$$E_{2i-1}(\sigma_3(2i-2);\ \sigma_3(2i)) = E_0 - \epsilon B(\lambda)\sigma_3(2i)\sigma_3(2i-2), \quad (2.9)$$

where

$$B(\lambda)=\tfrac{1}{2}[(1+4\lambda^2)^{1/2}-1] \tag{2.10a}$$

and

$$E_0(\lambda,\epsilon)=-\tfrac{1}{2}\epsilon[1+(1+4\lambda^2)^{1/2}]. \tag{2.10b}$$

Notice that the ground-state energy is explicitly invariant under the inversion of the spins at the boundaries. This is a consequence of the invariance of the block Hamiltonian itself under an overall spin flip. Moreover, we emphasize that the ground state is uniquely specified for all values of λ and for all boundary conditions.

The (normalized) ground-state wave function of the block has the form

$$\psi_{2i-1}(\sigma_3(2i-2);\ \sigma_3(2i))$$

$$=\frac{1}{[1+a^2(s,s')]^{1/2}}\,[\,|\uparrow\rangle+a(s,s')|\downarrow\rangle\,], \tag{2.11}$$

where $s\equiv\sigma_3(2i)$, $s'\equiv\sigma_3(2i-2)$, and $|\uparrow\rangle$ and $|\downarrow\rangle$ are the eigenstates of the operator $\sigma_1(2i-1)$. The amplitude $a(s,s')$ is given by

$$a(s,s')=\frac{\lambda}{1+(1+4\lambda^2)^{1/2}}\,(s+s'). \tag{2.12}$$

We see that the ground-state wave function of the block depends on the configuration of the spins at the boundaries and hence is an operator on the Hilbert space of the even sites.

We now proceed with the renormalization prescription stated above by keeping only the ground state of the blocks and defining the new Hamiltonian using (2.7) and (2.8). We find

$$H^{\rm ren}=\tfrac{1}{2}NE_0(\lambda,\epsilon)-\epsilon'(\lambda)\sum_{i=1}^{N/2}\sigma_1(2i)$$

$$-\Delta'(\lambda)\sum_{i=1}^{N/2}\sigma_3(2i)\sigma_3(2i+2). \tag{2.13}$$

Note that the renormalized Hamiltonian does not contain any new interactions. The renormalized couplings $\epsilon'(\lambda)$ and $\Delta'(\lambda)$ are given by

$$\epsilon'(\lambda)=\epsilon Z(\lambda), \tag{2.14a}$$

$$\Delta'(\lambda)=\Delta\,\frac{B(\lambda)}{\lambda}, \tag{2.14b}$$

$$\lambda'=\frac{B(\lambda)}{Z(\lambda)}. \tag{2.14c}$$

The renormalization constant $Z(\lambda)$ for the *energy scale* ϵ (or transverse field) is given by the relation

$$Z(\lambda)\equiv(\psi_{2i-1}(s,s')|\psi_{2i-1}(-s,s'))^2$$

$$=\left[1+\left(\frac{2\lambda}{1+(1+4\lambda^2)^{1/2}}\right)^2\right]^{-1}. \tag{2.15}$$

It is important to stress that Z depends only on the effective coupling λ and that, unlike ψ, it does not depend on the configuration (s,s') of the even sites. It is this property which guarantees that no new couplings are generated by the renormalization.

Clearly this process can now be iterated such that after n iterations we find

$$H_n=\sum_{i=1}^{N/2^n}[-\epsilon_n\sigma_1(2^ni)-\Delta_n\sigma_3(2^ni)\sigma_3(2^n(i+1))$$

$$+E_n(\lambda_n,\epsilon_n)], \tag{2.16}$$

where

$$\epsilon_n=Z(\lambda_{n-1})\epsilon_{n-1}, \tag{2.17a}$$

$$\Delta_n=\frac{B(\lambda_{n-1})}{\lambda_{n-1}}\,\Delta_{n-1}, \tag{2.17b}$$

$$\lambda_n=\frac{B(\lambda_{n-1})}{Z(\lambda_{n-1})}, \tag{2.17c}$$

$$E_n=C_n+2E_{n-1}, \tag{2.17d}$$

$$C_n=-\tfrac{1}{2}\epsilon_n[1+(1+4\lambda_n^2)^{1/2}]. \tag{2.17e}$$

The renormalized Hamiltonian (2.16) determines within our variational approximation the dynamics for length scales greater than 2^na.

C. Phases of the model

Using the recursion relations (2.14) we can now discuss the phases of the model. Let us define the β function

$$\beta(\lambda)\equiv\frac{B(\lambda)}{Z(\lambda)}-\lambda \tag{2.18}$$

such that $\lambda_{n+1}-\lambda_n\equiv\beta(\lambda_n)$. The zeros of the function $\beta(\lambda)$ are the fixed points of our renormalization group. We find two infrared-stable fixed points at $\lambda=0,\ \infty$ which determine the long-distance properties of the model and a critical coupling (an infrared-unstable fixed point) at $\lambda_c=1.277$, which is the boundary between the two phases $\lambda>\lambda_c$ and $\lambda<\lambda_c$ (Fig. 2).

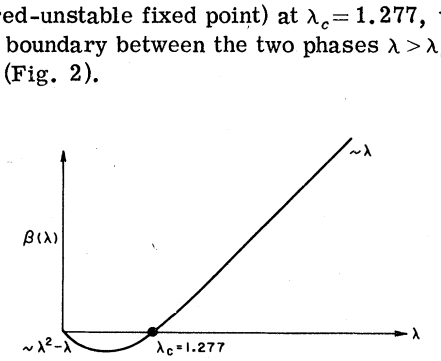

FIG. 2. The β function for the one-dimensional Ising model in a transverse field.

For $\lambda < \lambda_c$ we find that

$$\left.\begin{aligned} &\lim_{n\to\infty} \lambda_n = 0 \\ &\lim_{n\to\infty} \epsilon_n \neq 0 \\ &\lim_{n\to\infty} \Delta_n = 0 \end{aligned}\right\} \lambda < \lambda_c . \tag{2.19}$$

Thus, this is a disordered phase with an energy gap G_s for a single spin-flip excitation

$$G_s \equiv 2 \lim_{n\to\infty} \epsilon_n . \tag{2.20}$$

For $\lambda > \lambda_c$ we get

$$\left.\begin{aligned} &\lim_{n\to\infty} \lambda_n = \infty \\ &\lim_{n\to\infty} \epsilon_n = 0 \\ &\lim_{n\to\infty} \Delta_n \neq 0 \end{aligned}\right\} \lambda > \lambda_c . \tag{2.21}$$

The ground state of the fixed-point Hamiltonian is degenerate with all the spins aligned and pointing in either one of the two possible directions. Hence in this phase $G_s = 0$. But there is a gap in this phase which is related to the energy needed to excite a kink, antikink configuration[13] and is given by the relation

$$G_k = 4 \lim_{n\to\infty} \Delta_n . \tag{2.22}$$

This phase is characterized by a spontaneous breaking of the global spin-flip symmetry with a local order parameter M, the magnetization, which is the ground-state expectation value of the operator

$$O \equiv \frac{1}{N} \sum_{i=1}^{N} \sigma_3(i) .$$

D. Critical exponents for the correlation length and the energy gap

The energy gaps G_s and G_k may be computed using the recursion formulas (2.17a) and (2.17b),

$$G_s = 2\epsilon \lim_{n\to\infty} \prod_{j=0}^{n-1} Z(\lambda_j) , \tag{2.23a}$$

$$G_k = 4\Delta \lim_{n\to\infty} \prod_{j=0}^{n-1} \frac{B(\lambda_j)}{\lambda_j} , \tag{2.23b}$$

with $\lambda_0 \equiv \lambda$.

Near the critical point these gaps behave like

$$\begin{aligned} G_s &\approx c_s |\lambda - \lambda_c|^{\nu_{G_s}} , \quad \lambda \lesssim \lambda_c \\ G_k &\approx c_k |\lambda - \lambda_c|^{\nu_{G_k}} , \quad \lambda \gtrsim \lambda_c . \end{aligned} \tag{2.24}$$

The exponents ν_{G_s} and ν_{G_k} can be computed graphically or, alternatively, by means of the following scaling arguments: Consider the spin-flip gap G_s. The function $Z(\lambda)$ changes very slowly in a neighborhood of the critical point λ_c. Thus given an initial value of $\lambda \lesssim \lambda_c$, it follows that, for a number of interations $n < n_{\max}$, Z remains essentially constant, i.e., $Z(\lambda_n) \simeq Z(\lambda) \simeq Z(\lambda_c)$. Here $n_{\max}$ is the number of iterations necessary to make the coupling λ_n weak, i.e.,

$$|\lambda_{n_{\max}} - \lambda_c| \sim 1 . \tag{2.25}$$

For values of n greater than $n_{\max}$, $Z(\lambda_n)$ rapidly approaches its asymptotic limit

$$\lim_{n\to\infty} Z(\lambda_n) = 1 \quad \text{for} \quad \lambda < \lambda_c . \tag{2.26}$$

We can obtain an expression for $n_{\max}$ by linearizing the β function (2.18). We obtain

$$\begin{aligned} \lambda_c - \lambda_{n_{\max}} &= (1 + \beta_c')^{n_{\max}} (\lambda_c - \lambda) \\ &\quad + O((\lambda_c - \lambda)^2) , \end{aligned} \tag{2.27}$$

where $\beta_c' = d\beta/d\lambda|_{\lambda_c}$ is the slope of the β function at the critical point. Thus $n_{\max}$ is determined by the relation

$$n_{\max} = \frac{\ln |(\lambda_c - \lambda_{n_{\max}})/(\lambda_c - \lambda)|}{\ln(1 + \beta_c')} . \tag{2.28}$$

Hence, the energy gap G_s, close to λ_c, is given by the expression

$$G_s(\epsilon, \lambda) \approx 2\epsilon [Z(\lambda_c)]^{n_{\max}} = c(\lambda) |\lambda_c - \lambda|^{\nu_{G_s}} , \tag{2.29}$$

where $c(\lambda)$ is a smooth function of λ,

$$c(\lambda) \simeq 2\epsilon (\lambda_c - \lambda_{n_{\max}})^{-\nu_{G_s}} , \tag{2.30}$$

and the gap exponent ν_{G_s} is found to be equal to

$$\nu_{G_s} = \frac{\ln[1/Z(\lambda_c)]}{\ln(1 + \beta_c')} . \tag{1.1b'}$$

Analogous arguments show that the gap G_k behaves like (2.23b) with an exponent ν_{G_k}:

$$\nu_{G_k} = \frac{\ln[\lambda_c / B(\lambda_c)]}{\ln(1 + \beta_c')} . \tag{2.31}$$

However, the fixed-point relation

$$\lambda_c = \frac{B(\lambda_c)}{Z(\lambda_c)} \tag{2.32}$$

implies that both exponents are identical, i.e., $\nu_{G_s} = \nu_{G_k}$. Using the values $\beta_c' = 0.596$ and $Z_c = 0.682$ we find $\nu_G = 0.817$. This result is to be compared to the exact result found by Pfeuty,[10] $\nu_G = 1$. Thus we get about 20% error for this quantity.

The correlation-length exponent ν_t can be calculated by standard scaling arguments.[19] One assumes that the correlation length, near the critical point, has the functional form

$$\xi \simeq \frac{a}{|\lambda - \lambda_c|^{\nu_\xi}} . \tag{2.33}$$

However, after one interation of the renormalization group (RG) we obtain the same laboratory value of ξ expressed in terms of the new length scale $a' = 2a$ and new coupling $\lambda' = \lambda + \beta(\lambda)$. Equating the two expressions for ξ one immediately obtains

$$\nu_\xi = \frac{\ln 2}{\ln(1+\beta_c')} . \tag{1.1a}$$

This result can be also understood by noticing that the correlation length is the distance at which correlation functions begin to deviate from their scaling behavior. In terms of the renormalization group, this is the distance $\xi = 2^n a$ at which the effective coupling λ_n is small (or large), i.e., it satisfies (2.25). But this is exactly the definition of $n_{\max}$ [Eq. (2.28)]. Thus Eq. (1.1a) follows. If we plug in our numerical results, we get $\nu_\xi = 1.482$, which deviates by about 50% from the exact result $\nu_\xi = 1$.[10]

It is easy to convince oneself that the two exponents ν_ξ and ν_G should be equal to each other. It is true in the exact solution.[10] But, more important, they should be equal because the Ising model in a transverse field *in any dimension* is the Hamiltonian limit of a classical (or "Euclidean") Ising model. We thus expect both models to be in the same universality class. At the critical point the Ising model exhibits rotational invariance.[14] This implies that the energy gap should be equal to the inverse of the correlation length, near the critical point. Thus both exponents ν_G and ν_ξ should be identical.

The source of this discrepancy is found in the unequal space-time scaling generated by our renormalization group. Indeed after one iteration, lengths scale by a factor of 2, while energies, on the other hand, scale by a factor of Z_c. Thus time scales like $1/Z_c$ which is not equal to 2 in our case.

This result shall haunt us throughout the paper. In general, we shall find that energy gaps, i.e., quantities which scale with dimensions of time, are generally better within our approximation than quantities which scale with dimensions of length, e.g., magnetization or correlation functions. It is worth noting, however, that the method of Hirsch and Mazenko[9] apparently provides a way of correcting this asymmetry.[20]

E. RG equations for operators

We shall now consider the behavior of operators under our RG. For that purpose we need to know how the true ground state is constructed using our prescription. Equation (27) tells us which states we are keeping after renormalization. But it also says how a given state, for instance the ground state, is mapped under the RG, i.e., if $|0\rangle_\lambda$ represents the ground state for a coupling constant λ, after one iteration we obtain

$$|0\rangle_\lambda = \prod_{i=1}^{N/2} \psi_{2i-1}(\sigma)\,|0\rangle_{\lambda'} , \tag{2.34}$$

where λ and λ' are related by the β function $\beta(\lambda)$. Thus if θ is any operator, its ground-state expectation value ${}_\lambda\langle 0|\,\theta\,|0\rangle_\lambda$ at a coupling λ renormalizes as

$${}_\lambda\langle 0|\,\theta\,|0\rangle_\lambda = {}_{\lambda'}\langle 0|\,\pi\psi(\sigma)\theta\pi\psi(\sigma')\,|0\rangle_{\lambda'} . \tag{2.35}$$

Clearly to evaluate this expression we must understand how the operator θ renormalizes. We shall define the renormalized operator $\theta^{\rm ren}$ by the expression

$$\theta^{\rm ren} \equiv \langle \psi(\sigma)|\,\theta\,|\psi(\sigma')\rangle , \tag{2.36}$$

i.e., θ restricted to the subspace (2.7).

1. Magnetization

Let us consider the magnetization M, which is the ground-state expectation value of the operator

$$\theta = \frac{1}{N}\sum_{i=1}^{N} \sigma_3(i) .$$

The renormalized operator θ is given by (2.36). To compute $\theta^{\rm ren}$ it is useful to write θ as follows:

$$\theta = \frac{1}{N}\sum_{i=1}^{N/2} \sigma_3(2i) + \frac{1}{N}\sum_{i=1}^{N/2} \sigma_3(2i-1) . \tag{2.37}$$

Making explicit use of the wave functions (2.11) we find

$$\theta^{\rm ren} = \frac{1}{N}\sum_{i=1}^{N/2} \sigma_3(2i) + \left\langle \psi(\sigma)\left|\, \frac{1}{N}\sum_{i=1}^{N/2} \sigma_3(2i-1)\,\right|\psi(\sigma')\right\rangle . \tag{2.38}$$

After some algebra we get

$$\theta^{\rm ren} = \left(\frac{1}{2} + \frac{Z(\lambda)B(\lambda)}{\lambda}\right)\left(\frac{2}{N}\sum_{i=1}^{N/2} \sigma_3(2i)\right) . \tag{2.39}$$

As a result of the definition of the magnetization

$$M = {}_\lambda\left\langle 0\left|\, \frac{1}{N}\sum_{i=1}^{N} \sigma_3(i)\,\right|0\right\rangle_\lambda \tag{2.40}$$

and Eqs. (2.35), (2.36), (2.39), we find that the magnetization satisfies the recursion relation

$$M(\lambda) = \left(\frac{1}{2} + \frac{Z(\lambda)B(\lambda)}{\lambda}\right) M(\lambda') . \tag{2.41}$$

After an infinite number of iterations we obtain

$$M(\lambda)=\lim_{n\to\infty}\left[\prod_{j=0}^{n-1}\left(\frac{1}{2}+\frac{Z(\lambda_j)B(\lambda_j)}{\lambda_j}\right)\right]M(\lambda_n) \tag{2.42}$$

for appropriate boundary conditions.

Let us now evaluate this expression. In the disordered phase $\lambda<\lambda_c$, λ iterates to zero, Z iterates to one, and $B(\lambda)/\lambda$ iterates to zero like λ. Thus the magnetization is unstable in this phase, i.e.,

$$M(\lambda)=\lim_{n\to\infty}\frac{1}{2^n}M(\lambda_n)=0. \tag{2.43}$$

In the ordered phase, however, $(\lambda>\lambda_c)$, the situation is reversed. Here λ iterates to infinity, $Z(\lambda)$ iterates to $\frac{1}{2}$, and $B(\lambda)/\lambda$ iterates to one. Then $\frac{1}{2}+Z(\lambda)B(\lambda)/\lambda$ rapidly approaches one and the magnetization is stable. Thus we obtain a finite, nonvanishing result.

Using (2.41) we can now calculate the critical exponent β. Analogous arguments to those used to compute the energy-gap exponent may also be used here. Combining Eqs. (2.28) and (2.41) we find the scaling form for the magnetization

$$M(\lambda)=m(\lambda)|\lambda-\lambda_c|^{\beta}, \tag{2.44}$$

where the magnetization exponent

$$\beta=-\frac{\ln(\frac{1}{2}+Z_cB_c/\lambda_c)}{\ln(1+\beta_c')} \tag{2.45}$$

and $m(\lambda)$ is a smooth function of λ, i.e.,

$$m(\lambda)=|\lambda_n-\lambda_c|^{-\beta}M(\lambda_{n_{\max}}). \tag{2.46}$$

Note $M(\lambda_{n_{\max}})$ may be computed in perturbation theory since $\lambda_{n_{\max}}$ is large enough. Using (2.45) and our numerical results, we obtain $\beta=0.075$ which is to be compared with the exact result $\beta=0.125$ (see Table I).

2. *Correlation functions*

Let us discuss the important case of the equal-time correlation function $C(R)$,

$$C(R)=\langle 0|\sigma_3(0)\sigma_3(R)|0\rangle. \tag{2.47}$$

We will use the procedure just described for the magnetization. Define the operator $\theta(R)$ to be

$$\theta(R)=\frac{1}{N}\sum_{i=-N/2}^{N/2-1}\sigma_3(i)\sigma_3(i+R). \tag{2.48}$$

Then $C(R)$ is just ${}_\lambda\langle 0|\theta(R)|0\rangle_\lambda$. The renormalized operator $\theta^{\rm ren}(R)$ is again the old operator acting on the restricted Hilbert space (2.7) [Eq. (2.36)]. Let us now explicitly compute $\theta^{\rm ren}(R)$.

After one iteration we find, for R even,

$$\theta^{\rm ren}(R)=\left[\left(\frac{Z(\lambda)B(\lambda)}{\lambda}\right)^2+\frac{1}{2}\right]\frac{2}{N}\sum_{i=-N/4}^{N/4-1}\sigma_3(2i)\sigma_3(2i+R)$$
$$+\frac{1}{2}\left(\frac{Z(\lambda)B(\lambda)}{\lambda}\right)^2\frac{2}{N}\sum_{i=-N/4}^{N/4-1}\{\sigma_3(2i)[\sigma_3(2i+2+R)+\sigma_3(2i-2+R)]\} \tag{2.49a}$$

and, for R odd,

$$\theta^{\rm ren}(R)=\left(\frac{Z(\lambda)B(\lambda)}{\lambda}\right)\frac{2}{N}\sum_{i=-N/4}^{N/4-1}\{\sigma_3(2i)[\sigma_3(2i+1+R)+\sigma_3(2i-1+R)]\}. \tag{2.49b}$$

Hence we obtain the following renormalization-group equations for $C_\lambda(R)$ (R even):

$$C_\lambda(R)=\left[\frac{1}{2}+\left(\frac{Z(\lambda)B(\lambda)}{\lambda}\right)^2\right]C_{\lambda'}\left(\frac{R}{2}\right)+\left(\frac{Z(\lambda)B(\lambda)}{\lambda}\right)^2\frac{1}{2}\left[C_{\lambda'}\left(\frac{R}{2}+1\right)+C_{\lambda'}\left(\frac{R}{2}-1\right)\right], \tag{2.50a}$$

$$\tfrac{1}{2}[C_\lambda(R+1)+C_\lambda(R-1)]=\left(\frac{Z(\lambda)B(\lambda)}{\lambda}\right)C_{\lambda'}\left(\frac{R}{2}\right)+\left(\frac{Z(\lambda)B(\lambda)}{\lambda}\right)\frac{1}{2}\left[C_{\lambda'}\left(\frac{R}{2}+1\right)+C_{\lambda'}\left(\frac{R}{2}-1\right)\right]. \tag{2.50b}$$

In a compact matrix form (2.50a), (2.50b) read

$$V_\lambda(R)=Q_\lambda V_{\lambda'}(\tfrac{1}{2}R), \tag{2.51}$$

where $V_\lambda(R)$ is the two-component vector

$$V_\lambda(R)=\begin{bmatrix}C_\lambda(R)\\ \frac{1}{2}[C_\lambda(R+1)+C_\lambda(R-1)]\end{bmatrix} \tag{2.52}$$

and Q_λ is the 2×2 matrix

$$Q_\lambda = \begin{bmatrix} \frac{1}{2} + \left(\frac{Z(\lambda)B(\lambda)}{\lambda}\right)^2 & \left(\frac{Z(\lambda)B(\lambda)}{\lambda}\right)^2 \\ \frac{Z(\lambda)B(\lambda)}{\lambda} & \frac{Z(\lambda)B(\lambda)}{\lambda} \end{bmatrix} . \tag{2.53}$$

Let us consider the recursion relations (2.51) in the disordered phase ($\lambda < \lambda_c$). For $\lambda \ll \lambda_c$, Q_λ is given by

$$Q_\lambda \simeq \begin{bmatrix} \frac{1}{2} & 0 \\ \lambda & \lambda \end{bmatrix} \tag{2.54}$$

to order λ^2. If $R = 2^n$, then by iterating (2.50) n times we obtain

$$C_\lambda(R) \simeq \frac{1}{2^n} C_{\lambda_n}(1) . \tag{2.55}$$

But from perturbation theory we know that

$$C_{\lambda_n}(1) \simeq \lambda_n + O(\lambda_n^2) . \tag{2.56}$$

Thus, in this limit, $C_\lambda(R)$ is equal to

$$C_\lambda(R) \simeq \frac{1}{2^n} \lambda_n \simeq \frac{\lambda^{2^n}}{2^n} = \exp(-R|\ln\lambda|)/R , \tag{2.57}$$

where we have used the result $\lambda_n \approx \lambda_{n-1}^2$ for λ small (see Fig. 2). Thus our recursion relations yield an exponential falloff for the correlation function in the disordered phase, in contrast with other calculations of the same type.[6,8]

We may now use (2.51) to obtain the behavior of $C(R)$ near the critical point λ_c. In the regime $\lambda \lesssim \lambda_c$, $C_\lambda(R)$ is expected to behave like

$$C_\lambda(R) \approx \frac{\exp[-Ra/\xi(\lambda)]}{R^\eta} , \quad \text{as } R \to \infty \tag{2.58}$$

where $\xi(\lambda)$ is the correlation length at a coupling λ and η is the anomalous dimension. Near λ_c, ξ diverges like $a|\lambda - \lambda_c|^{-\nu_t}$.

Using the recursion relations (2.51) we obtain (for $R \gg 1$ and $\lambda \lesssim \lambda_c$)

$$\frac{\exp[-Ra/\xi(\lambda)]}{R^\eta} \begin{pmatrix} a \\ b \end{pmatrix}_\lambda = Q_\lambda \begin{pmatrix} a \\ b \end{pmatrix}_{\lambda'} \frac{\exp[-Ra'/2\xi(\lambda')]}{(R/2)^\eta} . \tag{2.59}$$

If we choose the vector $\binom{a}{b}_\lambda$ to be the eigenvector of Q_λ that belongs to the *largest* eigenvalue ϵ_λ, we obtain the following scaling relations:

$$\begin{aligned} &\xi(\lambda) = \xi(\lambda') , \\ &2^\eta \epsilon_{\lambda_c} = 1 . \end{aligned} \tag{2.60}$$

From (2.60) we see that the anomalous dimension η is equal to

$$\eta = -\frac{\ln\epsilon_{\lambda_c}}{\ln 2} \tag{2.61}$$

and ν_t is given by (1.1a), as it should. The eigenvalue ϵ_{λ_c} is readily found to be

$$\epsilon_{\lambda_c} = \frac{1}{2}\left(\left[\frac{1}{4} + \left(\frac{1}{2} + \frac{Z_c B_c}{\lambda_c}\right)^2\right] + \left\{\left[\frac{1}{4} + \left(\frac{1}{2} + \frac{Z_c B_c}{\lambda_c}\right)^2\right]^2 - 2\frac{Z_c B_c}{\lambda_c}\right\}^{1/2}\right) . \tag{2.62}$$

The numerical value of η, as given by (2.61) and (2.62), is $\eta = 0.100$. The exact result is $\eta = 0.25$.[10]

In the ordered phase ($\lambda \gtrsim \lambda_c$) the correlation function behaves asymptotically like

$$C_\lambda(R) \simeq M^2(\lambda) + \frac{\text{const}}{R^\eta} \exp[-Ra/\xi(\lambda)] . \tag{2.63}$$

Using Eq. (2.51) we find an exponent β equal to

$$\beta = -\frac{1}{2}\frac{\ln\epsilon_{\lambda_c}}{\ln(1+\beta_c')} \tag{2.64}$$

instead of (2.45). Note that expressions (2.45) and (2.64) are not analytically equivalent, however, both results yield the same numerical value to within three decimal places.[21] Finally, using (2.61) and (2.64) we arrive at the scaling relation

$$2\beta = \nu_t \eta , \tag{2.65}$$

which is also satisfied by the exact result.

III. Z_2 GAUGE THEORY IN 2+1 DIMENSIONS

A. The model

Following Ref. 13 we define the Hamiltonian for the Z_2 gauge theory, in the temporal gauge, as

$$H = -\epsilon \sum_{(\vec{r},\mu)} \sigma_1^\mu(\vec{r}) - \Delta \sum_{(\vec{r},\mu\nu)} \sigma_3^\mu(\vec{r})\sigma_3^\nu(\vec{r}+\hat{e}_\mu)\sigma_3^\mu(\vec{r}+\hat{e}_\nu)\sigma_3^\nu(\vec{r}) , \tag{3.1}$$

where $(\vec{r},\mu)$ and $(\vec{r},\mu\nu)$ label links and plaquettes of a two-dimensional square lattice ($\mu,\nu = 1,2$). σ_1 and σ_3 are standard Pauli matrices.

The Hamiltonian (3.1) is invariant under the local gauge transformation

$$\begin{aligned} &\sigma_3^\mu(\vec{r}) \to \tau(\vec{r})\sigma_3^\mu(\vec{r})\tau(\vec{r}+\hat{e}_\mu) , \\ &\sigma_1^\mu(\vec{r}) \to \sigma_1^\mu(\vec{r}) , \end{aligned} \tag{3.2}$$

where $\tau(\vec{r}) = \pm 1$. The local transformation (3.2) has a generator $G(\vec{r})$ (at site $\vec{r}$) given by

$$G(\vec{r}) = \prod_{\mu(\vec{r})} \sigma_1^\mu(\vec{r}) , \tag{3.3}$$

where $\mu(\vec{r})$ is the set of links that emerges from the site $\vec{r}$. Therefore the invariance of H reads

$$H = G(\vec{r})HG^\dagger(\vec{r}) \quad \text{for all } \vec{r} . \tag{3.4}$$

In the temporal gauge the states of H are subject to the constraint[13] (valid in the absence of sources)

$$G(\vec{r})|\psi_{\text{phys}}\rangle = |\psi_{\text{phys}}\rangle \quad \text{for all } \vec{r}, \tag{3.5}$$

which serves to define the space of gauge-invariant states. Notice that G can be diagonalized simultaneously with the Hamiltonian since they commute [Eq. (3.4)].

An important consequence of Eq. (3.5) is that the ground state $|0\rangle$ is a gauge-invariant state.[13] Accordingly, the local symmetry is never broken[22] and only gauge-invariant operators may have a nonvanishing expectation value. Thus this theory *does not have a local order parameter.*

B. Dual transformation

In 2 + 1 dimensions, however, the Z_2 *gauge theory has a disorder parameter.*[13,23] This is most clearly seen in the dual theory, *the* (2 + 1)*-dimensional Ising model in a transverse field.* The dual transformation[11,12,24] may be constructed by methods analogous to those of Ref. 13 for the (3 + 1)-dimensional model. The dual model has a Hamiltonian

$$H = -\tilde{\epsilon}\sum_{\tilde{r}} \mu_1(\tilde{r}) - \tilde{\Delta}\sum_{(\tilde{r},\mu)} \mu_3(\tilde{r})\mu_3(\tilde{r}+\hat{e}_\mu), \tag{3.6}$$

where $\tilde{\epsilon}$ and $\tilde{\Delta}$ are the dual couplings and $\tilde{r}$ labels the sites of the dual lattice (Fig. 3). The dual Pauli operators μ_1 and μ_3 are related to the original σ's by the expression

$$\mu_3(\tilde{r}) = \prod_{-\infty}^{r} \sigma_1^{(1)}(r'), \tag{3.7a}$$

where the product runs over the set of vertical ("1") links to the left of the plaquette centered at the dual site $\tilde{r}$ [see Fig. 4(a)]. Notice that this definition is not unique. In fact there exist an infinite set of operators, which in the subspace of gauge-invariant states, are equivalent to [3.7(a)] and realize the dual symmetry. For example,

$$\mu_3'(\tilde{r}) = \prod_{-\infty}^{r} \sigma_1^{(2)}(r'), \tag{3.7b}$$

where the product now runs over all the horizontal ("2") links below the plaquette $\tilde{r}$ [see Fig. 4(b)], can be obtained from $\mu_3(\tilde{r})$ by the operation

$$\mu_3'(\tilde{r}) = \mu_3(\tilde{r})U, \tag{3.8}$$

where $U = \pi G(r)$ is the product of the gauge generators $G(r)$ over all the sites inside the shaded area of Fig. 4(c). Note that in the gauge-invariant sector (3.5), $U \equiv 1$. For $\mu_1(\tilde{r})$ we have

$$\mu_1(\tilde{r}) = \sigma_3^\mu(r)\sigma_3^\nu(r+\hat{e}_\mu)\sigma_3^\mu(r+\hat{e}_\nu)\sigma_3^\nu(r). \tag{3.9}$$

The dual transformation (3.7)–(3.9) implies that dual and original couplings are related by

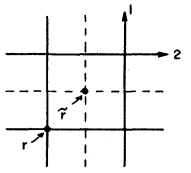

FIG. 3. The square lattice and its dual. The sites of the square lattice are indicated by r and the sites of the dual by $\tilde{r}$.

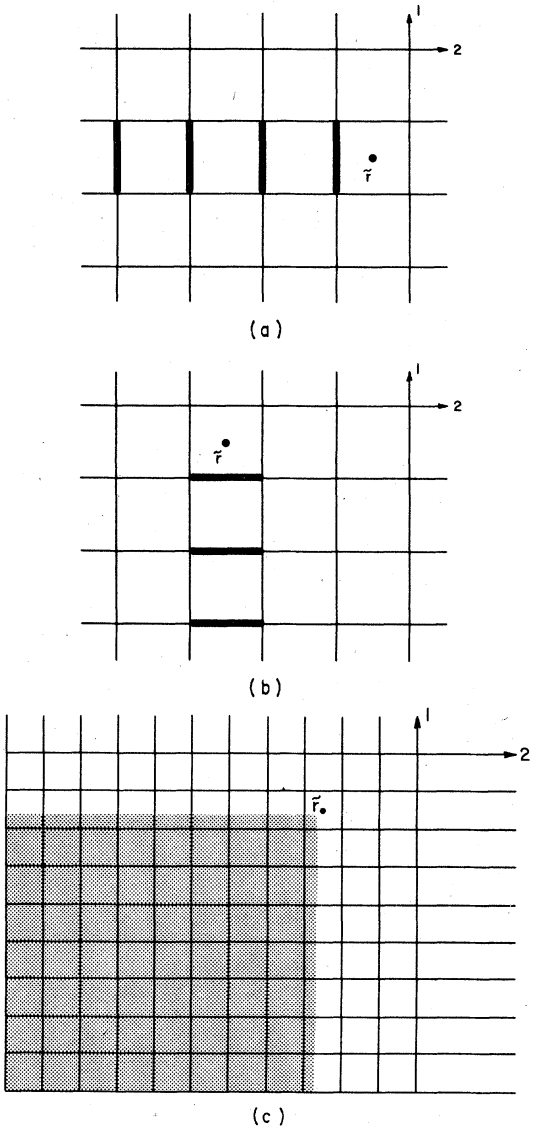

FIG. 4. The monopole creation operators σ_1^μ of Eqs. (3.7a) and (3.7b) reside on the heavy lines. (a) and (b) represent two equivalent definitions of the monopole creation operator. The shaded area in (c) represents the set of sites where the operator U [Eq. (3.8)] is defined.

$$\tilde{\Delta}=\epsilon\,,\qquad \tilde{\epsilon}=\Delta\,. \tag{3.10}$$

It should be noticed that it is the gauge-invariant sector of the gauge theory that dualizes onto the (2 + 1)-dimensional Ising model in a transverse field.

The ground-state expectation value of $\mu_3(\tilde{r})$ is the order parameter of the (2 + 1)-dimensional transverse Ising model. The dual of it

$$M(r)=\left\langle 0\left|\prod_{r'=-\infty}^{r}\sigma_1^{(1)}(r')\right|0\right\rangle \tag{3.11}$$

is the disorder parameter of the gauge theory.

The (2 + 1)-dimensional Ising model in a transverse field belongs to the same universality class as the three-dimensional classical Ising model.[10,13,15] Pfeuty and Elliott[15] have studied the former model by means of a Raleigh-Schrödinger perturbation theory improved with Padé approximants. Their results, as well as those of the high-temperature expansion in the three-dimensional classical (3D) Ising model, are shown in Table II.

C. Phases and states

We find it useful to discuss the qualitative properties of the theory before considering its renormalization properties. It has two phases:

(a) *A confining phase* ($\Delta \ll \epsilon$), which is dual to the *ordered* phase of the Ising model, and

(b) *A free phase* ($\Delta \gg \epsilon$), which is dual to the *disordered* phase of the Ising model.

In the confining regime the ground state, to zeroth order in perturbation theory, is the state that has $\sigma_1^{\mu}(r)=1$ for all links. In this state there is no electric flux. The first excited state should have at least one excited link, that is, there should be electric flux "running" through it. However, the constraint Eq. (3.5) implies that if a link entering into a given lattice site is excited, then there must be another excited link leaving that site. Thus the constraint Eq. (3.5) is effectively a law of conservation of electric flux, Gauss's law.[25] We conclude therefore that the first excited state is the smallest possible closed loop of flux, a box excitation (Fig. 5). For a state with open ends (Fig. 6) to exist it is necessary to have two sources at its end points. This means

$$G(0)\,|\psi\rangle=G(R)\,|\psi\rangle=-|\psi\rangle \tag{3.12}$$

at these points. This state is created by the operator

$$\hat{P}(R)\equiv\prod_{r_{0,R}}\sigma_3^{\mu}(r)\,, \tag{3.13}$$

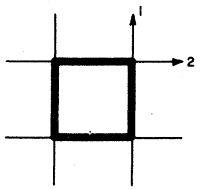

FIG. 5. The smallest possible strong-coupling gauge-invariant excitation: a box excitation.

which represents a string of electric flux. It is clear that the energy of such a state, with respect to the vacuum without sources, grows linearly with their separation

$$E(R)=2\epsilon R \tag{3.14}$$

to zeroth order in perturbation theory. Thus static sources are confined in this regime yielding a string tension $\tau=2\epsilon$ to this order. It can also be checked, either by duality or explicitly by a perturbation-theory calculation, that the disorder parameter (3.11) is nonvanishing in this phase.

In the free phase, on the other hand, the ground state has no magnetic energy. The *field strength*, or "frustration,"[26,27] at the plaquette $(r, \mu\nu)$ is measured by the operator $\Phi_{\mu\nu}(r)$ defined by

$$\Phi_{\mu\nu}(r)=\sigma_3^{\mu}(r)\sigma_3^{\nu}(r+\hat{e}_{\mu})\sigma_3^{\mu}(r+\hat{e}_{\nu})\sigma_3^{\nu}(r)\,. \tag{3.15}$$

Thus the ground state has $\Phi_{\mu\nu}(r)=1$, for all plaquettes. This state can be obtained by considering the linear superposition of the state $\sigma_3^{\mu}(r)=1$ (for all links) with all its gauge transformations. Consider now the excited states. The state that has $\Phi_{\mu\nu}(r)=-1$ at only one plaquette has finite energy and is created by the operator (3.7a). It represents a *magnetic charge* or *monopole*. A state with two magnetic charges is created by the operator

$$C(0,R)=\prod_{r'=0}^{R}\sigma_1^1(r')\equiv\mu_3(0)\mu_3(R)\,, \tag{3.16}$$

whose expectation value is the *correlation function of the dual model*. It is also easy to see that the state created by (3.13) costs no energy at all in

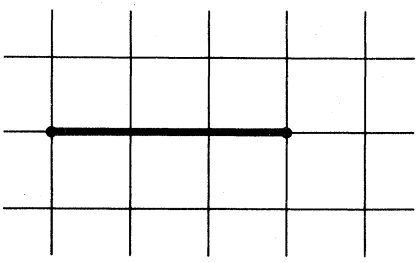

FIG. 6. A strong-coupling state with sources.

this regime, i.e., static sources are free.

To summarize, we see that the dual relation between the gauge theory and the Ising model in a transverse field is nothing more that the electric-magnetic duality discussed previously by 't Hooft[28] and Mandelstam.[29] In the confining phase, static electric sources are confined and magnetic monopoles form a condensate, whereas in the free phase a monopole excitation costs a finite amount of energy and static sources are free.

D. Renormalization

The first problem one encounters when applying block-spin techniques to gauge theories stems from the gauge constraint (3.5). As a result of that constraint the variables $\sigma_3^\mu(r)$ and $\sigma_1^\mu(r)$, for all links, are not independent in the subspace $|\psi_{\rm phys}\rangle$. There are essentially two paths one may then follow. Either one solves for the independent degrees of freedom in the subspace $|\psi_{\rm phys}\rangle$ or one keeps the constraint. For the Z_2 gauge theory in $2+1$ dimensions, solving the constraint equation is equivalent to performing the dual transformation (3.7)–(3.9).[30] Certainly in this model this would be the simplest procedure to follow since the resulting spin system is much easier to handle than the original model. However, for more relevant theories, such as non-Abelian gauge theories in $3+1$ dimensions, it is certain that the system one would obtain by solving the constraint equation would be far more complex than the original one (see, for example, Jackiw and Goldstone, Ref. 31). Moreover, this procedure, which is in fact a complete gauge fixing, obscures the role of the local symmetry. Thus if the renormalization group generates additional couplings, there is no way in principle to recover a gauge-invariant Hamiltonian. Thus we have chosen the latter alternative.

In the real-space renormalization group (RSRG) we shall now define, gauge invariance will be maintained at each stage of the calculation. The procedure is the following. We divide the space into elementary blocks of four links distributed uniformly through the lattice (see Fig. 7). The Hamiltonian then takes the form

$$H=-\epsilon\sum_{(r,\mu)\notin B}\sigma_1^\mu(r)+\sum_B H_B\,, \tag{3.17}$$

where B denotes the set of blocks and H_B is the block Hamiltonian. In general H_B has the form

$$H_B=-\epsilon\sum_{i=1}^{4}\sigma_1(i)-\Delta\sum_{i=1}^{4}\sigma_3(i)\sigma_3(i+1)A_{i,i+1}\,, \tag{3.18}$$

where

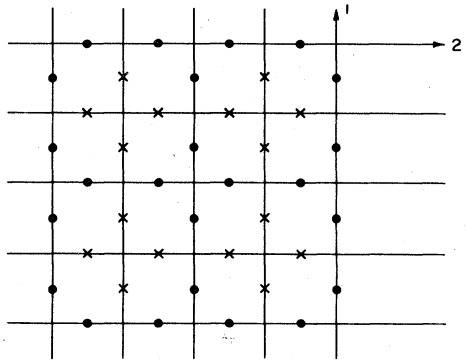

FIG. 7. Definition of the blocks for the (2+1)-dimensional Ising gauge theory. Each block includes four links, indicated with crosses, meeting at a site, and the four plaquettes which share those links.

$$\sigma_3(5)\equiv\sigma_3(1)\,,$$

$$\sigma_1(5)\equiv\sigma_1(1)\,,$$

and $A_{i,i+1}$ is the product of the σ_3^μ operators for the two links at the corner of the boundary of the block (see Fig. 8). Thus we have now a setup similar to that of Sec. II. We shall thus solve for the ground-state energy and wave function of H_B for an arbitrary configuration of the degrees of freedom at the boundary. Then, as discussed in Sec. II, we truncate the Hilbert space to the subspace

$$\prod_B \psi_g(B)\,|\sigma_1^\mu\rangle_{\rm boundaries}\,, \tag{3.19}$$

where the states at the boundaries are chosen to be eigenstates of the electric flux, i.e., of $\sigma_1^\mu(r)$. Note that as before $\psi_g(B)$ and $\epsilon_g(B)$ are really functions of the σ_3^μ operators at the boundaries. Thus the block wave functions and energies are effectively operators when acting upon the boundary states. As in Sec. II, we define the renormalized Hamiltonian to be

$$H^{\rm ren}\equiv\langle\psi(\sigma)\,|H|\,\psi(\sigma')\rangle\,, \tag{3.20}$$

where $|\psi(\sigma)\rangle$ are arbitrary states in the subspace (3.19).

Consider now the block Hamiltonian (3.18). We first observe that, once again, the boundary de-

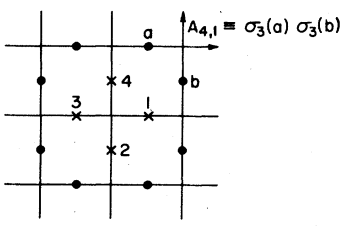

FIG. 8. An elementary block.

grees of freedom may be regarded as classical variables since their electric fields are not included in the block Hamiltonian. Thus (3.18) can be diagonalized for an arbitrary configuration of the "bond variables" $A_{i,i+1}$.

On each block there are four links and 16 states. Only eight of these states are gauge invariant at the common vertex. In order to enforce gauge invariance we shall restrict ourselves to this gauge-invariant subspace. H_B is, in this subspace, an 8×8 matrix of the form

$$H_B=\begin{pmatrix} A & B \\ B & 0 \end{pmatrix}, \tag{3.21}$$

where A and B are the 4×4 matrices

$$A=-\epsilon\begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 \end{pmatrix},$$

$$B=-\Delta\begin{pmatrix} A_{12} & A_{23} & A_{41} & A_{34} \\ A_{23} & A_{12} & A_{34} & A_{41} \\ A_{41} & A_{34} & A_{12} & A_{23} \\ A_{34} & A_{41} & A_{23} & A_{12} \end{pmatrix}. \tag{3.22}$$

We look for an eigenstate of the form

$$\psi=\begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \tag{3.23}$$

where α and β are four-component state vectors. The eigenvalue equation

$$\begin{pmatrix} A & B \\ B & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}=\epsilon\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{3.24}$$

reduces to

$$(A\epsilon+B^2)\,\alpha=\epsilon^2\alpha \tag{3.25a}$$

and

$$\beta=B\alpha/\epsilon \quad \text{for } \epsilon\neq 0. \tag{3.25b}$$

Solving explicitly the eigenvalue equation

$$\det[A\epsilon+B^2-\epsilon^2 I]=0, \tag{3.26}$$

which is a fourth-order polynomial, we obtain the ground-state energy of the block E_g and the ground-state wave function ψ_g. Note that there is a unique ground state for all boundary conditions. As a first consequence of restricting ourselves to the gauge-invariant sector, we find that E_g depends on the operators σ_3^μ on the boundary links only through the gauge-invariant product

$$\Phi_B=\prod_{\text{boundary}}\sigma_3^\mu \tag{3.27}$$

which is the product of all the boundary operators σ_3^μ around the complete boundary (see Fig. 8). The ground-state wave function also acquires simple properties due to gauge invariance. The block Hamiltonian (3.16) is invariant under the transformation at a given site i:

$$\begin{aligned} &\sigma_3(i)\to-\sigma_3(i), \\ &A_{i,i+1}\to-A_{i,i+1}, \\ &A_{i-1,i}\to-A_{i-1,i}. \end{aligned} \tag{3.28}$$

Note the flipping of the bond variables $A_{i,i+1}$ and $A_{i-1,i}$ can be obtained by flipping the two boundary σ_3^μ operators that have the same vertex in common with the ith link. Thus as a result of this symmetry, the ground-state wave function obeys the identity

$$\sigma_1(i)\psi_B(-A_{i-1,i};-A_{i,i+1})=\psi_B(A_{i-1,i};A_{i,i+1}), \tag{3.29}$$

as can be checked by explicit calculation. The relation (3.29) will be used later when discussing gauge invariance at the boundaries of the block.

Using the block ground-state energy and wave function we obtain the renormalized Hamiltonian

$$H^{\text{ren}}=\sum_{\text{blocks}}E_g(\Phi_B)-\epsilon\sum_{\text{boundaries}}\langle\psi(\sigma)|\,\sigma_1^\mu(r)\,|\psi(\sigma')\rangle. \tag{3.30}$$

Since $\Phi_{\text{block}}=\pm 1$ we can write

$$E_g(\Phi_B)=-E_0-\Delta'\Phi_B, \tag{3.31}$$

where we find explicitly

$$\begin{aligned} E_0&=-\tfrac{1}{2}[E_g(+1)+E_g(-1)] \\ &=\epsilon\{1+(1+\lambda^2)^{1/2}+[2+2\lambda^2+2(1+\lambda^4)^{1/2}]^{1/2}\}, \end{aligned} \tag{3.32a}$$

$$\begin{aligned} \Delta'&=-\tfrac{1}{2}[E_g(+1)-E_g(-1)] \\ &=\epsilon\{[2+2\lambda^2+2(1+\lambda^4)^{1/2}]^{1/2}-1-(1+\lambda^2)^{1/2}\}, \end{aligned} \tag{3.32b}$$

and $\lambda\equiv\Delta/\epsilon$.

Special care is needed for the links at the boundaries. At each link on the boundary of the blocks we find

$$\langle\psi_g(\sigma)|\,\sigma_1^\mu(r)\,|\psi_g(\sigma')\rangle\equiv\epsilon Z(\lambda)\sigma_1^\mu(r), \tag{3.33}$$

where

$$Z(\lambda)=(\psi_g(\sigma_3^\mu(r),\sigma_3)|\,\psi_g(-\sigma_3^\mu(r),\sigma_3))^2 \tag{3.34}$$

as in Sec. II.

At this point we have succeeded in eliminating the degrees of freedom *inside* the blocks. We still have, however, two degrees of freedom for each link of the new lattice (Fig. 9). These degrees of freedom are, however, not independent

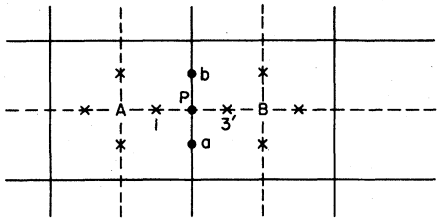

FIG. 9. Two nearby blocks that share the links a and b.

as a result of gauge invariance. To see this let us calculate the renormalized generator of gauge transformations at a site P on the boundary of a block. Let A and B be the two blocks that share that boundary site. Then the gauge generator G_p at P is

$$G_p=\sigma_1(1)\sigma_1(3')\sigma_1(a)\sigma_1(b)\,, \tag{3.35}$$

which restricted to the subspace (3.19) reads

$$G_p^{\rm ren}\equiv(\psi_A,\psi_B|\,G_p\,|\psi_A,\psi_B)\,. \tag{3.36}$$

Using the fact that σ_1 and σ_3 anticommute and Eq. (3.29), we get

$$G_p^{\rm ren}=\sigma_1(a)\sigma_1(b)\,. \tag{3.37}$$

Note that Eqs. (3.30), (3.31), and (3.33) imply $[H^{\rm ren},G_p^{\rm ren}]=0$. Thus if we are to preserve gauge invariance [Eqs. (3.4), (3.5)] we must demand

$$G_p^{\rm ren}=\sigma_1(a)\sigma_1(b)\equiv 1 \tag{3.38}$$

when acting on the reduced Hilbert space (3.19). Clearly the link variables $\sigma_1(a)$, $\sigma_1(b)$ are not independent. More than that, (3.38) implies that only two of the four states of the operator $\sigma_1(a)+\sigma_1(b)$ are possible. In particular, states for which electric flux is either created or destroyed at the point P are to be eliminated. We are now able to define link operators on the new lattice. Let us define σ_3^μ and σ_1^μ on a new link $(\vec{r},\mu)$ to be

$$\sigma_3^\mu(r)\equiv\sigma_3^\mu(a)\sigma_3^\mu(b) \tag{3.39}$$

and

$$\sigma_1^\mu(r)\equiv\tfrac{1}{2}[\sigma_1^\mu(a)+\sigma_1^\mu(b)] \tag{3.40}$$

(see Fig. 9). Clearly as a result of (3.39) and (3.40), σ_3^μ and σ_1^μ satisfy

$$\begin{aligned}&[\sigma_3^\mu(r)]^2=1\,,\\ &\{\sigma_1^\mu(r),\sigma_3^\mu(r)\}=0\,.\end{aligned} \tag{3.41}$$

Moreover, the constraint Eq. (3.38) implies

$$[\sigma_1^\mu(r)]^2=1\,. \tag{3.42}$$

Hence σ_1^μ, σ_3^μ reproduce the Pauli algebra on the new links. Note that, as a result of (3.38), it is also true that the new σ_1^μ is equal to either of the old σ_1^μ's residing on the old links. Finally, the renormalized Hamiltonian is

$$\begin{aligned}H^{\rm ren}=&-\epsilon'\sum_{(\vec{r},\mu)}\sigma_1^\mu(\vec{r})\\ &-\Delta'\sum_{(\vec{r},\mu\nu)}\sigma_3^\mu(\vec{r})\sigma_3^\nu(\vec{r}+\hat{e}_\mu)\sigma_3^\mu(\vec{r}+\hat{e}_\nu)\sigma_3^\nu(\vec{r})\\ &-\tfrac{1}{4}NE_0\end{aligned} \tag{3.43}$$

for a lattice spacing $a'=2a$, where the couplings ϵ and Δ satisfy the renormalization-group equations

$$\begin{aligned}&\epsilon'=2\epsilon Z(\lambda)\,,\\ &\Delta'=\epsilon B(\lambda)\,.\end{aligned} \tag{3.44}$$

Z is given in the Appendix. Once again, we may regard ϵ as an energy scale and $\lambda=\Delta/\epsilon$ as an effective coupling constant which renormalizes according to the β function

$$\beta(\lambda)=\lambda'-\lambda=\frac{B(\lambda)}{2Z(\lambda)}-\lambda\,. \tag{3.45}$$

E. Results

1. Fixed point

We will now use the recursion relations (3.43)–(3.45) to discuss the physics of this model. The β function (3.45) has a zero at $\lambda_c=3.28$. This unstable fixed point is the critical point (Fig. 10). In addition, we find that the slope of the β function at λ_c,

$$\beta_c'=\left.\frac{d\beta}{d\lambda}\right|_{\lambda=\lambda_c}\,,$$

is given by

$$\begin{aligned}&\beta_c'=0.78\,,\\ &\lambda_c=3.28\,.\end{aligned} \tag{3.46}$$

The critical point λ_c given above is in good agreement with the calculations of Pfeuty and Elliott[15] for the (2+1)-dimensional transverse Ising model.

2. Energy gap

As we have already discussed in Sec. III C, there are two types of excitations: monopoles

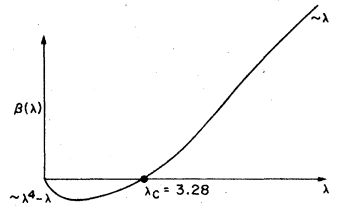

FIG. 10. The β function for the (2+1)-dimensional Ising gauge theory.

(free phase) and box excitations (confinement phase). In addition, in the confinement regime we can also calculate the string tension τ. Let us first consider the energy of a box excitation ($\lambda < \lambda_c$). If $\lambda \ll \lambda_c$ we can apply perturbation theory and we find $E_{\text{box}} = 8\epsilon$ to zeroth order in a perturbation expansion in λ. Thus if we begin with a value of $\lambda \equiv \lambda_0 \lesssim \lambda_c$ and we iterate enough times the renormalized coupling, λ_n will eventually become small enough to apply perturbation theory. At that point

$$E_{\text{box}} = 8\epsilon_n , \tag{3.47a}$$

where ϵ_n is given by

$$\epsilon_n = \epsilon \prod_{j=0}^{n-1} [2Z(\lambda_j)] . \tag{3.47b}$$

Using Eq. (3.47), which is valid for all values of λ, we can now calculate the critical exponent ν_b. Once again ν_b may be evaluated graphically or by scaling arguments as discussed previously in Sec. II. It can be verified that $Z(\lambda)$ (see Appendix) rapidly approaches 1 as $\lambda \to 0$. Thus we are able to use the same approximation as in (2.29). As a result we obtain

$$E_{\text{box}} = 8\epsilon [2Z(\lambda_c)]^{n_{\max}} \tag{3.48}$$

with $n_{\max}$ given by Eq. (2.28). Thus the "box excitation gap" vanishes like

$$E_{\text{box}} \simeq c(\lambda) |\lambda - \lambda_c|^{\nu_b} , \tag{3.49}$$

where the gap exponent ν_b is given by

$$\nu_b = -\frac{\ln[2Z(\lambda_c)]}{\ln(1+\beta_c')} \tag{3.50}$$

The function $c(\lambda) \simeq 8\epsilon |\lambda_n - \lambda_c|^{-\nu_b}$ is a smooth function of the coupling λ. Equation (3.50) together with the numerical result $Z_c = 0.349$ yields a gap exponent $\nu_b = 0.622$.

As we argued in the framework of the (1+1)-dimensional Ising model in a transverse field, this exponent should be equal to the correlation length exponent ν_t given by the relation

$$\nu_t = \frac{\ln 2}{\ln(1+\beta_c')} .$$

In this case this is not true since $Z(\lambda_c) \neq \frac{1}{4}$, whereas a symmetric space-time scaling suggests ($2Z_c = \frac{1}{2}$). Once again the gap exponent ν_b is numerically better than ν_t. In fact it agrees fairly well with the correlation length exponent calculated in the 3D classical Ising model by means of the high-temperature expansion (see Table II).

Let us now evaluate the monopole gap E_{monopole} for $\lambda \equiv \lambda_0 \gtrsim \lambda_c$. After n iterations λ_n becomes large and $E_{\text{monopole}} = 2\Delta_n$. Then using the recursion relations (3.44) we obtain

$$E_{\text{monopole}} = 2\Delta \prod_{j=0}^{n-1} \frac{B(\lambda_j)}{\lambda_j} . \tag{3.51}$$

It is now a simple exercise to find the gap exponent ν_m. Using the fact that $\beta(\lambda) \sim \lambda$ for large λ, we can again apply our approximation and obtain the result

$$E_{\text{monopole}} = p(\lambda) |\lambda - \lambda_c|^{\nu_m} \quad \text{for} \quad \lambda \gtrsim \lambda_c ,$$

where $p(\lambda)$ is a smooth function of λ and

$$\nu_m = -\frac{\ln[B(\lambda_c)/\lambda_c]}{\ln(1+\beta_c')} . \tag{3.52}$$

However, since λ_c is the fixed point of (3.45) we have

$$\lambda_c = \frac{B(\lambda_c)}{2Z(\lambda_c)} \tag{3.53}$$

and thus $\nu_m = \nu_b$, as might be expected by scaling arguments.

3. *String tension*

Let us consider the string tension τ for $\lambda \lesssim \lambda_c$. This is defined as the change in the ground-state energy per unit length due to two static sources separated by a distance R. Thus in principle one should first construct via the block-spinning procedure a ground state satisfying the constraint (3.12) and then calculate the expectation value of H in that state. This would be a very difficult calculation to do since the shape of the string is not preserved by the renormalization procedure. Instead we shall compute an approximation to the string tension τ. Imagine that the coupling λ is small enough so that perturbation theory is applicable. Then a string of p lattice units in length will have an energy over the vacuum given by

$$E_{\text{string}}(p) = 2\epsilon p . \tag{3.54}$$

We may now boost (3.54) to the critical region by assuming that we have really started with $\lambda \lesssim \lambda_c$ and after n iterations λ_n becomes small enough so that $E_{\text{string}}(p) = 2\epsilon_n p$ where p is now the number of lattice units as measured in the nth iterated lattice. Equivalently, we have $p = 2^{-n}R$, where R is the separation between the two static sources as measured in the original lattice unit. Finally using (3.47b) we obtain

$$E_{\text{string}}(R) = 2\epsilon \left(\prod_{j=0}^{n-1} [2Z(\lambda_j)] \right) 2^{-n} R \tag{3.55}$$

or

$$\tau = 2\epsilon \prod_{j=0}^{n-1} Z(\lambda_j) . \tag{3.56}$$

Applying the same approximation as we have used previously we can now obtain the behavior

of τ near the critical point. We find

$$\tau(\lambda)=g(\lambda)|\lambda-\lambda_c|^{\nu_\tau}, \tag{3.57}$$

where ν_τ is the *exponent of the tension* and is found to be equal to

$$\nu_\tau=-\frac{\ln Z(\lambda_c)}{\ln(1+\beta_c')} . \tag{3.58}$$

The numerical value is $\nu_\tau=1.82$. In order to compare this result with Ising-type quantities let us discuss a bit more the meaning of the string tension τ. We argue that the tension is just the surface tension of a domain wall in the dual Ising model. In fact, as it was already pointed out in Sec. III B, only the gauge-invariant sector (i.e., with no external sources) of the gauge theory dualizes onto the transverse Ising model. If now we introduce a pair of static sources then the ground state $|\psi\rangle$ must satisfy

$$\begin{aligned} &G|\psi\rangle=-|\psi\rangle \quad \text{at the sources}, \\ &G|\psi\rangle=|\psi\rangle \quad \text{otherwise}. \end{aligned} \tag{3.59}$$

Thus we must modify our dual transformation, which enforces $G=1$ everywhere, in order to satisfy (3.59). The simplest way of doing so is to define

$$\sigma_1^1(r)=-\mu_3\mu_3 \tag{3.60}$$

for all the links of the dual lattice that are pierced by a line that goes from one source to the other (see Fig. 11). As a result we end up with a $(2+1)$-dimensional transverse Ising model *with a string of flipped bonds*, that is, with a *piece of a domain wall*. Then τ is just the surface tension of the wall.

However, the exponent of the surface tension ν_τ and the correlation-length (or gap) exponent ν_b are known to satisfy the scaling relation

$$\nu_\tau=2\nu_b \tag{3.61}$$

due to Widom.[32] This scaling relation is not satisfied by (3.58) and (3.50) except when $Z_c=\frac{1}{4}$. Thus we can also understand this discrepancy as due to an asymmetric space-time scaling. This type of disagreement, we hope, will be solved by computing corrections using the method of Hirsch and Mazenko.[9]

4. *Disorder operator*

Finally, let us compute explicity the ground-state expectation value of the disorder operator, the monopole creation operator (3.11). As we pointed out in Sec. III C $M(r)$ is nonzero in the confinement phase $\lambda<\lambda_c$.

In order to compute $M(r)$ we define a translation-invariant version of $\mu_3(\tilde{r})$,

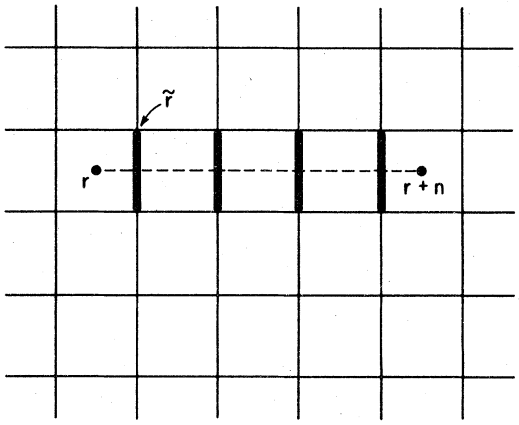

FIG. 11. The dual of the gauge theory with *two* sources at sites r and $r+n$ is a two-dimensional Ising model in a transverse field with a set of antiferromagnetic bonds (heavy links).

$$Q=\frac{1}{N}\sum_{\tilde{r}}\mu_3(\tilde{r}), \tag{3.62}$$

where $\tilde{r}$ runs over all the N sites of the dual lattice and $M(r)=\langle 0|Q|0\rangle$. We evaluate M recursively. We define the renormalized Q as

$$Q_1\equiv\langle\psi(\sigma)|Q|\psi(\sigma')\rangle, \tag{3.63}$$

i.e., the old Q restricted to the subspace (3.18). By direct calculation we obtain

$$Q_1=Z^{1/2}(\lambda)Q', \tag{3.64}$$

where Q' is the old operator Q acting on the decimated lattice with $N/4$ sites.

Then after n iterations we get

$$Q_n=\left[\prod_{j=0}^{n-1}Z^{1/2}(\lambda_j)\right]Q' \tag{3.65}$$

and M is given by

$$M=\pm\lim_{n\to\infty}\prod_{j=0}^{n-1}Z^{1/2}(\lambda_j), \tag{3.66}$$

with the overall sign being determined by the boundary conditions. In the critical region we can use our approximation to get

$$M\sim|\lambda-\lambda_c|^{\beta} \tag{3.67}$$

with an exponent β, i.e., the magnetization exponent of the dual Ising model given by

$$\beta=-\frac{\ln Z_c^{1/2}}{\ln(1+\beta_c')}\equiv\tfrac{1}{2}\nu_\tau. \tag{3.68}$$

Thus we find an exponent $\beta=0.91$ which is in fact off by a large amount (see Table II).

IV. DISCUSSION

We have presented a real-space renormalization-group scheme for a Hamiltonian formalism and

applied it to two models. We find three essential and worthy features of the scheme:

(1) It is applicable to gauge theories and preserves local gauge invariance at every stage of the calculation.

(2) We obtain the correct asymptotic behavior for correlation functions in both ordered and disordered phases.

(3) It yields reasonable results for the critical coupling and for critical indices for energy gaps.

An undesirable feature of the scheme is the asymmetric scaling of space and time. We believe, however, this property of the approximation can be remedied by using a formalism similar to that of Hirsh and Mazenko (Ref. 9).

Finally, we feel this scheme can be generalized to other interesting models:

(a) Certainly the most direct generalization of the scheme would be to $Z(N)$ spin systems in $1+1$ dimensions and to $Z(N)$ gauge systems in $2+1$ dimensions.[33]

(b) In general the scheme can, in principle, be applied to any model whose Hamiltonian can be written in the form

$$H = T(\{p\}) + V(\{q\}),$$

where $\{p\}$ and $\{q\}$ are two sets of commuting operators which do not commute with each other. It is clear, however, that any generalization either to larger groups, whether discrete or continuous, or to higher dimensions, will necessarily involve the generation, via the renormalization-group transformation, of new operators in H. This problem may not be serious but it can only be addressed in individual case studies.

ACKNOWLEDGMENTS

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Added note. When this work was near completion we learned that J. Gallardo and D. C. Mattis have done a RG calculation on the $(2+1)$-dimensional Ising model in a transverse field which yields the same numerical results as our RG for the gauge theory. We wish to thank them for communicating their results to us.

APPENDIX

If we define

$$\epsilon_+ = -[8(1+\lambda^2) + (1+\lambda^4)^{1/2}]^{1/2},$$

$$\epsilon_- = -2[1 + (1+\lambda^2)^{1/2}],$$

$$N_+^{\,2} = \left[\left(1+\frac{4\lambda^2}{\epsilon_+^{\,2}}\right)\left(1+2\,\frac{(4+\epsilon_+)^2}{\epsilon_+^{\,2}} + \frac{(\epsilon_+^{\,3}+4\epsilon_+^{\,2}-12\lambda^2\epsilon_+-32\lambda^2)^2}{16\lambda^4\epsilon_+^{\,2}}\right)\right.$$
$$+\frac{2}{\epsilon_+^{\,2}}\left(8\lambda^2\,\frac{(4+\epsilon_+)}{\epsilon_+} + \frac{\epsilon_+^{\,3}+4\epsilon_+^{\,2}-12\lambda^2\epsilon_+-32\lambda^2}{\epsilon_+}\right.$$
$$\left.\left.+4\lambda^2\,\frac{(4+\epsilon_+)^2}{\epsilon_+^{\,2}} + 2(4+\epsilon_+)\,\frac{\epsilon_+^{\,3}+4\epsilon_+^{\,2}-12\lambda^2\epsilon_+-32\lambda^2}{\epsilon_+^{\,2}}\right)\right],$$

and $N_-^{\,2} = (2\lambda^2-\epsilon_-)/(\lambda^2-\epsilon_-)$, then

$$Z(\lambda) = (N_+N_-)^{-2}\left(1+\frac{\epsilon_++4}{2\epsilon_-}\right)^2.$$

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