Explicit perturbative solution of multichannel wave equations

H. J. W. Müller-Kirsten and R. Müller
Department of Physics, University of Kaiserslautern, 6750 Kaiserslautern, West Germany
(Received 12 April 1979; revised manuscript received 14 August 1979)

It is shown that a general perturbative method can be developed for solving the coupled equations of the multichannel formalism. Assuming channel potentials for which Taylor expansions exist and weak channel coupling, we show that two pairs of eigensolutions can be constructed which belong to one and the same set of eigenvalues. The solutions can be matched and normalized, and the eigenvalues are calculated explicitly in the form of asymptotic expansions. Finally, generalizations and applications are discussed.

I. INTRODUCTION

Recently Dashen *et al.*[1] have given a detailed investigation of multichannel potential scattering with at least one permanently confined channel. In particular they established rigorously various fundamental properties of the wave operators, the S matrix, the spectrum of the Hamiltonian, and expansions in terms of eigenfunctions, and they formulated the problem in such a way that its extension of single-channel scattering theory is particularly transparent.

The success of nonrelativistic models in reproducing the observed mass spectrum of heavy quark-antiquark states has led to a revival of interest[2-5] in the multichannel formalism.[6,7] Thus the present investigation was motivated by the desire to explore the possibility of the existence of hadronic molecular states in the e^+e^- mass spectrum above the second radial charmonium state. The search for such states has been activated by the recent observation of a large number of narrow states in the baryon-antibaryon spectrum and their interpretation as baryonium excitations, and by the idea of exotic and cryptoexotic $Q^2\overline{Q}^2$ states.[8]

In e^+e^- scattering the region just above 3.7 GeV (center-of-mass energy) is of particular interest because this is where a $D\overline{D}^*$ molecular state[8] is most likely to show up (at around 3.85 GeV), and if the charmonium model is reasonably correct, this state could not be mistaken for the next radial excitation which is predicted to be at around 4.0 GeV. Single-channel potential theory normally leads to broad widths, but narrow widths can be generated by the weak coupling to a second channel.[7] It is therefore plausible to investigate the two-channel problem defined by the transitions $c\overline{c} \to D\overline{D}^* \to D\overline{D}^*$.

In the present investigation we shall not be concerned with a specific application. Instead we present a perturbative method for calculating the discrete eigenvalues and eigenfunctions for a large class of potentials, since this is a useful prerequisite for the determination of various composite states and their Regge trajectories as well as other applications. Our method is a direct generalization of the method applied previously to a large number of single-channel equations[9-12] and its extension to the multidimensional case.[13] However, in spite of its simplicity it will be seen that this generalization is by no means trivial, since the procedure depends crucially on the construction of an unperturbed "Hamiltonian" which is a multiple of the unit matrix and so commutes with each of the matrix coefficients of the perturbation. The latter has its counterpart in multidimensional perturbation theory, where a specific curvature constraint is required in order to permit a separation of the rotated variables in the leading order of the iteration scheme.[13]

We begin by considering the two-channel case for both channel masses equal to 1 and channel potentials containing a leading harmonic part and anharmonic perturbations. In Sec. II we derive one pair of solutions together with the associated eigenvalues. In Sec. III we derive a second pair of solutions (this is WKB-like and valid in a complementary domain), and we demonstrate explicitly that the eigenvalue expansion obtained is identical with the expansion obtained previously in conjunction with the first pair, thus verifying our calculations. In Sec. IV we consider the matching[14] of the solutions and show how the method can be generalized to the case of different channel masses. Finally, in Sec. V we summarize our conclusions and discuss the generalization to include scattering as well as possible applications.

II. A FIRST PAIR OF ASYMPTOTIC EIGENSOLUTIONS

We consider the two-channel problem defined by a system of coupled Schrödinger equations which we write in the form

$$\begin{bmatrix} \frac{d^2}{dy^2}+E-g^2y^2-V_{11}(y) & -V_{12}(y) \\ -V_{21}(y) & \frac{d^2}{dy^2}+E-g^2y^2-V_{22}(y) \end{bmatrix} \begin{bmatrix} \psi_1(y) \\ \psi_2(y) \end{bmatrix} = 0 . \tag{1}$$

Here E is the total energy of the system, which, of course, has to be the same[15] in both channels in order to permit the system of channel 1 to convert into the system of channel 2, and vice versa. We write the channel potentials as a harmonic term supplemented by anharmonic contributions, i.e., in channel i ($i=1,2$) we assume a potential

$$g^2y^2+V_{ii}(y), \quad V_{ii}(y)=\sum_{k=2}^{\infty} a_{ii}^{(k)}y^k . \tag{2}$$

We include a term $a_{ii}^{(2)}y^2$ in V_{ii} in order to allow for the possibility of different harmonic contributions in the two channels. It will be seen later that it is necessary to include this difference in the perturbative contributions. The coupling potentials will be assumed to possess the expansions (e.g., as entire functions)

$$V_{ij}(y)=\sum_{k=0}^{\infty} a_{ij}^{(k)}y^k , \tag{3}$$

where $i \neq j$.

As emphasized in the Introduction, the potentials we consider here are chosen so as to facilitate a simple and transparent presentation of our method. For this reason we ignore also the centrifugal terms (which would have to be incorporated in numerous specific applications), and for simplicity the reduced masses of our channels have been set equal to $\frac{1}{2}$. We will comment later on the possibility of generalizing our considerations. At this point it should be noted, that we do not assume the channel potentials or the coupling potentials to be the same. However, time-reversal invariance of the Hamiltonian implies $V_{12}=V_{21}$.[16]

Our first objective is to derive the approximate behavior of our energy eigenvalues E. For small perturbative contributions and weak channel coupling this behavior is determined by the dominant harmonic terms. For a convenient formulation of our method we proceed as follows. In (1) we make the substitutions

$$x=(2g)^{1/2}y , \tag{4}$$

$$\phi_i=2g\psi_i , \quad V_{ij}=2gv_{ij} . \tag{5}$$

Equation (1) can then be written

$$\begin{bmatrix} \frac{d^2}{dx^2}+\frac{E}{2g}-\frac{x^2}{4} & \cdot \\ \cdot & \frac{d^2}{dx^2}+\frac{E}{2g}-\frac{x^2}{4} \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} . \tag{6}$$

For small perturbative contributions and weak channel coupling the right-hand side of this equation may be neglected to a first approximation, i.e., the solution $\phi=\binom{\phi_1}{\phi_2}$ is given to zeroth order by

$$\phi^{(0)}=\begin{bmatrix} \phi_p \\ \phi_p \end{bmatrix} \equiv \phi_{pp} , \tag{7}$$

where $p=E/g$, and $\phi_p=D_{1/2(p-1)}(x)$ is a parabolic cylinder function. The requirement of square integrability of the wave functions over the entire interval $0 \leq y \leq \infty$ implies that $p=2n+1$, $n=0,1,2,\ldots$ (if, instead, the wave functions are required to vanish at some large but finite distance y_0 away from the origin, p is only approximately an odd integer, i.e., $p=2n+1+O((1/y_0))$. We now set

$$\frac{E}{2g}=\tfrac{1}{2}p+\frac{\Delta}{2g} . \tag{8}$$

The quantity Δ remains to be determined.

Next we substitute (8) into (6) and multiply the equation by -2. The resulting equation can be written

$$\mathfrak{D}_{pp}\phi=\frac{1}{g}U\phi , \tag{9}$$

where

$$\mathfrak{D}_{pp}=\begin{bmatrix} \mathfrak{D}_p & \cdot \\ \cdot & \mathfrak{D}_p \end{bmatrix} , \quad \mathfrak{D}_p \equiv -2\frac{d^2}{dx^2}-p+\tfrac{1}{2}x^2 , \tag{10}$$

and

$$U=\begin{bmatrix} \Delta-V_{11} & -V_{12} \\ -V_{21} & \Delta-V_{22} \end{bmatrix} . \tag{11}$$

Thus the zeroth-order solution (7) is given by

$$\mathfrak{D}_{pp}\phi_{pp}=0 . \tag{12}$$

Since

$$\mathfrak{D}_{p+i,p+i}\phi_{p+i,p+i}=0$$

and

$$\mathfrak{D}_{p+i,p+i}=\mathfrak{D}_{pp}-\mathcal{G}_{ii}\,, \tag{13}$$

where $\mathcal{G}_{ii}$ is the unit matrix multiplied by i, we have

$$\mathfrak{D}_{pp}\phi_{p+i,p+i}=\mathcal{G}_{ii}\phi_{p+i,p+i}\,. \tag{14}$$

This equation will be used in the development of our iteration procedure.

We now consider the right-hand side of Eq. (9). First we reexpress $U\phi_{pp}$ as a sum over various $\phi_{p+i,p+i}$, i.e., we write

$$U\phi_{pp}=\sum_i C(p,p+i)\phi_{p+i,p+i}\,, \tag{15}$$

where each coefficient C is a matrix. The explicit calculation of these coefficients will be considered at the end of this section. First we deal with the perturbation procedure. Thus the zeroth-order solution $\phi^{(0)}=\phi_{pp}$ leaves uncompensated on the right-hand side of (9) the contribution

$$R_{pp}^{(0)}=\frac{1}{g}U\phi_{pp}=\frac{1}{g}\sum_i C(p,p+i)\phi_{p+i,p+i}\,. \tag{16}$$

Using (14) we see that a term $C(p,p+i)\phi_{p+i,p+i}$ of this sum can be taken care of by adding to $\phi^{(0)}$ the contribution $\mathcal{G}_{ii}{}^{-1}C(p,p+i)\phi_{p+i,p+i}$ except, of course, when $i=0$. This means that the first-order contribution of ϕ is

$$\phi^{(1)}=\frac{1}{g}\sum_{i\neq 0} C(p,p+i)\mathcal{G}_{ii}{}^{-1}\phi_{p+i,p+i}\,. \tag{17}$$

At this point we make an important observation. The contribution $\phi^{(1)}$ is obtained only by virtue of the fact that $\mathfrak{D}_{pp}$ and $\mathcal{G}_{ii}$ are multiples of the unit matrix which commutes with $C(p,p+i)$. If instead we had formulated our procedure in terms of diagonal operators $\mathfrak{D}_{p_1,p_2}$, $\mathcal{G}_{ij}$, we would now be faced with the difficult problem of handling noncommuting operators for the purpose of deriving the first-order contribution. It is this difficulty which makes it practically unavoidable to proceed as we do, although superficially our considerations so far seem trivial.

We observe that the first-order contribution (17) leaves uncompensated in (16) the term in ϕ_{pp}. This will be used to determine Δ and hence E. Now, since $\phi^{(0)}=\phi_{pp}$ leaves uncompensated $R_{pp}^{(0)}$, the contribution $\phi^{(1)}$ leaves uncompensated

$$\begin{aligned}R_{pp}^{(1)}&=\frac{1}{g}\sum_{i\neq 0} C(p,p+i)\mathcal{G}_{ii}{}^{-1}R_{p+i,p+i}^{(0)}\\&=\frac{1}{g^2}\sum_{i\neq 0} C(p,p+i)\mathcal{G}_{ii}{}^{-1}\sum_j C(p+i,p+i+j)\phi_{p+i+j,p+i+j}\,.\end{aligned} \tag{18}$$

It follows that the next-order contribution $\phi^{(2)}$ becomes

$$\phi^{(2)}=\frac{1}{g^2}\sum_{i\neq 0} C(p,p+i)\mathcal{G}_{ii}{}^{-1}\sum_{j\neq -i}{}' C(p+i,p+i+j)\mathcal{G}_{i+j,i+j}{}^{-1}\phi_{p+i+j,p+i+j}\,. \tag{19}$$

It is now obvious how our method proceeds. Thus finally we have the iterated sum

$$\phi=\phi^{(0)}+\phi^{(1)}+\phi^{(2)}+\cdots\,. \tag{20}$$

This is a solution of our coupled equations, provided the sum of the coefficients of the terms containing ϕ_{pp} in $R_{pp}^{(0)},R_{pp}^{(1)},R_{pp}^{(2)},\ldots$, left uncompensated so far, is set equal to zero, i.e.,

$$0=\left(\frac{1}{g}C(p,p)+\frac{1}{g^2}\sum_{i\neq 0} C(p,p+i)\mathcal{G}_{ii}{}^{-1}C(p+i,p)+\cdots\right)\phi_{pp}\,. \tag{21}$$

This is the equation from which Δ and hence our eigenvalues E are determined.

We have thus found one solution of our set of coupled equations together with its associated eigenvalues. A second, linearly independent solution is obtained by changing the signs of p and g throughout (this change leaves the eigenvalues unaltered). A further pair of solutions is obtained by the interchanges

$$x\to\pm ix\,,\quad p\to -p\,,\quad g^{1/2}\to\pm i g^{1/2}\,.$$

The region of validity of these solutions is around the minimum of the potential at $x=0$, i.e., where $x\lesssim 0(1/g^{\alpha})$ for $\alpha>0$.

We now return to (15) and calculate Δ. For the evaluation of the matrix coefficients $C(p,p+i)$ we need the recurrence relation of the parabolic cylinder functions ϕ_p. We write this relation in the form

$$x\phi_p(x)=(p,p+2)\phi_{p+2}+(p,p-2)\phi_{p-2}\,, \tag{22}$$

where

$$(p,p+2)=1\,,\quad (p,p-2)=\tfrac{1}{2}(p-1)\,.$$

For higher powers we have

$$x^i\phi_p(x)=\sum_{j=2i}^{-2i} S_i(p,j)\phi_{p+j}\,. \tag{23}$$

The coefficients $S_i(p,j)$ satisfy the recurrence relation

$$S_i(p,j)=S_{i-1}(p,j+2)(p+j+2,p+j)$$
$$+S_{i-1}(p,j-2)(p+j-2,p+j) \tag{24}$$

under the conditions

$$S_0(p,0)=1\,,\quad S_i(p,j)=0 \text{ for } j>|2i|\,.$$

Using the relations (22) and (23) we obtain (for $i,j=1,2$)

$$\sum_k C_{ii}(p,p+k)\phi_{p+k}(x)=\Delta\phi_p(x)-\sum_{l=2}^{\infty} a_{ii}^{(l)}\frac{1}{(2g)^{l/2}}\sum_{k=2l}^{-2l} S_l(p,k)\phi_{p+k}(x)\,, \tag{25a}$$

and for $i\neq j$

$$\sum_k C_{ij}(p,p+k)\phi_{p+k}(x)=-\sum_{l=0}^{\infty} a_{ij}^{(l)}\frac{1}{(2g)^{l/2}}\sum_{k=2l}^{-2l} S_l(p,k)\phi_{p+k}(x)\,. \tag{25b}$$

The coefficients C can easily be read off from these relations, e.g.,

$$C_{ii}(p,p)=\Delta_i-\sum_{l=2}^{\infty} a_{ii}^{(l)}\frac{1}{(2g)^{l/2}}S_l(p,0)$$
$$=\Delta_i-\frac{1}{2g}a_{ii}^{(2)}S_2(p,0)$$
$$-\frac{1}{4g^2}a_{ii}^{(4)}S_4(p,0)$$
$$-\cdots$$
$$=\Delta_i-\frac{p}{2g}a_{ii}^{(2)}-\cdots$$

[since, e.g., $S_3(p,0)=0$] and for $i\neq j$

$$C_{ij}(p,p)=-\sum_{l=0}^{\infty} a_{ij}^{(l)}\frac{1}{(2g)^{l/2}}S_l(p,0)$$
$$=-a_{ij}^{(0)}-\frac{1}{2g}a_{ij}^{(2)}S_2(p,0)-\cdots$$
$$=-a_{ij}^{(0)}-\frac{p}{2g}a_{ij}^{(2)}-\cdots\,.$$

Inserting these coefficients into Eq. (21) we obtain

$$\Delta_i=a_{ij}^{(0)}+\frac{p}{2g}(a_{ii}^{(2)}+a_{ij}^{(2)})$$
$$+\frac{1}{8g^2}[3(p^2+1)(a_{ii}^{(4)}+a_{ij}^{(4)})-2a_{ij}^{(1)}a_{ji}^{(1)}]$$
$$+O\left(\frac{1}{g^3}\right)$$

i.e.,

$$E_i=pg+a_{ij}^{(0)}+\frac{p}{2g}(a_{ii}^{(2)}+a_{ij}^{(2)}$$
$$+\frac{1}{8g^2}[3(p^2+1)(a_{ii}^{(4)}+a_{ij}^{(4)})-2a_{ij}^{(1)}a_{ji}^{(1)}]$$
$$+O\left(\frac{1}{g^3}\right),$$

where $i,j=1,2$ and $i\neq j$. We have thus found the two eigenvalues of our 2×2 eigenvalue equation.

III. A SECOND PAIR OF ASYMPTOTIC EIGENSOLUTIONS

We now derive a second pair of large-g asymptotic expansions for the wave functions of our coupled equations. This pair is valid in regions of large y where the expansions obtained above are no longer applicable. Of course, the corresponding eigenvalue expansions are identical with (26) and (27) above.

Our starting point is Eq. (1) in which we insert the expression (8) in terms of the quantity Δ, which is again to be determined by iteration. We then have the equation

$$\begin{bmatrix} \frac{d^2}{dy^2}+pg+\Delta-\omega_{11}(y) & -V_{12}(y) \\ -V_{21}(y) & \frac{d^2}{dy^2}+pg+\Delta-\omega_{22}(y)\end{bmatrix}\begin{bmatrix}\psi_1(y)\\ \psi_2(y)\end{bmatrix}=0\,, \tag{28}$$

where we have set

$$\omega_{ii}(y)=g^2y^2+V_{ii}(y)\,. \tag{29}$$

In the following we will again require a matrix differential operator which commutes with an arbitrary matrix. For this reason we introduce a function $\omega(x)$ which is to be chosen such that it approximates both ω_{11} and ω_{22}. Then

$$\omega_{ii}(y)=\omega(y)+[\omega_{ii}(y)-\omega(y)].$$

Normally g^2y^2 will not be an approximation of $\omega_{ii}(y)$ for large y, so that in general $\omega(y)$ contains the dominant part of $V_{ii}(y)$.

Next we set

$$\psi_i(y)=\chi_i(y)\exp\left\{\pm\int^y dy[\omega(y)]^{1/2}\right\}. \tag{30}$$

The equation for $\chi=\binom{\chi_1}{\chi_2}$ can then be written

$$\mathfrak{D}_{pp}\chi=\frac{1}{g}U\chi, \tag{31}$$

where

$$\mathfrak{D}_{pp}=\begin{pmatrix}\mathfrak{D}_p & \cdot\\ \cdot & \mathfrak{D}_p\end{pmatrix},$$
$$\mathfrak{D}_p=\mp\frac{2}{g}\omega^{1/2}\frac{d}{dy}\mp\frac{1}{2g}\frac{\omega'}{\omega^{1/2}}-p, \tag{32}$$

and

$$U=\begin{pmatrix}\frac{d^2}{dy^2}+\Delta+\omega-\omega_{11} & -V_{12}\\ -V_{21} & \frac{d^2}{dy^2}+\Delta+\omega-\omega_{22}\end{pmatrix}. \tag{33}$$

We observe from (30), (32), and (33) that if we know one solution $\psi=\binom{\psi_1}{\psi_2}$ we can find another by changing the signs of $\omega^{1/2}$ throughout or, equivalently, by changing the signs of p and g [also in $\omega(y)$]. By construction the right-hand side of (31) is of $O(1/g)$ compared with the left-hand side. Hence to a first approximation we can neglect the terms on the right-hand side of (31) for $g\to\infty$ and write for the solution to that order

$$\chi^{(0)}=\begin{pmatrix}\chi_p\\ \chi_p\end{pmatrix}=\chi_{pp}, \tag{34}$$

where χ_{pp} is the solution of

$$\mathfrak{D}_{pp}\chi_{pp}=0$$

and so

$$\chi_p=\frac{g^{1/2}}{\omega^{1/4}}\exp\left\{\mp\frac{pg}{2}\int^y\frac{dy}{[\omega(y)]^{1/2}}\right\}, \tag{35}$$

apart from an overall multiplicative constant which we ignore in the following except in the context of normalization (the factor $g^{1/2}$ is inserted for convenience in later expressions). It will be observed that (31) is now in a form analogous to (9), although corresponding quantities are completely different. We can therefore proceed as before.

In particular we observe that

$$\mathfrak{D}_{pp}\chi_{p+i,p+i}=\mathcal{G}_{ii}\chi_{p+i,p+i}. \tag{36}$$

Again our next step is to express $gR_{pp}^{(0)}=U\chi_{pp}$ as

$$U\chi_{pp}=\sum_l C(p,p+l)\chi_{p+l,p+l} \tag{37}$$

(we use the same symbol C as in Sec. II, although they are different in the two cases). We defer the explicit calculation of the coefficients $C(p,p+l)$ to the end of this section. Thus the zeroth-order solution (34) leaves uncompensated on the right-hand side of (31) the contribution (37) multiplied by $1/g$. Using (36) we see that a term $(1/g)C(p,p+i)\chi_{p+i,p+i}$ can be taken care of by adding to $\chi^{(0)}$ the contribution

$$\frac{1}{g}\mathcal{G}_{ii}^{-1}C(p,p+i)\chi_{p+i,p+i}$$

except, of course, when $i=0$. This means that the first-order contribution of χ is

$$\chi^{(1)}=\frac{1}{g}\sum_{i\neq 0}C(p,p+i)\mathcal{G}_{ii}^{-1}\chi_{p+i,p+i}. \tag{38}$$

Again this contribution is obtained only by virtue of the fact that $\mathfrak{D}_{pp}$ and $\mathcal{G}_{ii}$ are multiples of the unit matrix which commutes with $C(p,p+i)$. The iteration method is now completely analogous to the procedure of Sec. II, so that we can skip further details. Thus finally we have the expansion

$$\chi=\chi^{(0)}+\chi^{(1)}+\chi^{(2)}+\cdots, \tag{39}$$

which represents a solution of (31) if again

$$0=\left[\frac{1}{g}C(p,p)+\frac{1}{g^2}\times\sum_{i\neq 0}C(p,p+i)\mathcal{G}_{ii}^{-1}C(p+i,p)+\cdots\right]\chi_{pp} \tag{40}$$

with the coefficients $C(p,p+i)$ now defined by (37). The relation (40) is the equation from which Δ and hence the eigenvalues E are determined.

We have thus found a second pair of asymptotic eigensolutions of our coupled equations. This pair is valid where $g^2y^2\gtrsim 1$, i.e., $y^2\gtrsim 1/g^2$ and so excludes the region in which our first pair is valid.

Finally, we check our considerations by calculating Δ in the context of this second pair of solutions. For this purpose we return to Eq. (37) and calculate the matrix coefficients $C(p,p+i)$. For the diagonal elements of U we have

$$\begin{aligned}d_{ii}&\equiv\left(\frac{d^2}{dy^2}+\Delta+\omega-\omega_{ii}\right)\chi_p\\&=\left(\frac{5}{16}\frac{\omega'^2}{\omega^2}+\frac{pg}{2}\frac{\omega'}{\omega^{3/2}}+\frac{p^2g^2}{4}\frac{1}{\omega}-\frac{\omega''}{4\omega}+\Delta+\omega-\omega_{ii}\right)\chi_p.\end{aligned} \tag{41}$$

For convenience we choose

$$\omega(y)=g^2y^2+V(y) \tag{42}$$

where $V(y)$ is the leading part of $V_{ii}(y)$ for both $i=1$ and 2 for large y. As in our previous investigations[11-14] we now expand $\omega(y)$ around a point y_0 for which $\omega(y_0)$ and $\omega'(y_0)=0$. Then

$$\omega(y)=\sum_{i=2}^{\infty}\frac{(y-y_0)^i}{i!}\,\omega^{(i)}(y_0)\,.$$

Substituting this expansion into Eq. (35) and reversing the resulting expansion we obtain

$$(y-y_0)^{1/2}=\sum_{i=0}^{\infty}d_{2i+1}\frac{\chi_{p-(2i+1)}}{\chi_p}$$

with easily calculable coefficients d_{2i+1}. This expansion can now be used in order to reexpress d_{ii} of Eq. (41) as a linear sum over various $\chi_{p-(2i+1)}$ with constant coefficients. The computation of these coefficients is very cumbersome and will not be repeated here; details of their calculation can be found in the treatment of very analogous cases considered in Refs. 11, 12, and 14. Thus, in order to bypass awkward algebraic expressions we content ourselves here with an approximation. Our approximation will consist in ignoring in the potentials all but the first terms in expansions (2) and (3) (recall that these are assumed to be the expansions of entire functions). The corresponding approximation of the solution χ_p will then be the dominant term of its expansion in the domain where it merges into the solutions derived in Sec. II. Then $\omega(y)\simeq g^2y^2$ and

$$\chi_p\simeq y^{-(1\pm p)/2}\,. \tag{43}$$

We emphasize that the approximation consists in ignoring all terms involving $a_{ii}^{(k)}$ for $k>2$. We designate such contributions (which in general depend on p) by $\epsilon_p(\chi)$.

Then, choosing the solution with the upper sign, we have

$$\begin{aligned}d_{ii}&=\left[\frac{(p+1)(p+3)}{4y^2}+\Delta-V_{ii}\right]\chi_p+\epsilon_p(\chi)\\&=\left[\frac{(p+1)(p+3)}{4}\chi_{p+4}-\sum_{k=2}^{\infty}a_{ii}^{(k)}\chi_{p-2k}+\Delta\chi_p\right]+\epsilon_p(\chi)\,.\end{aligned} \tag{44}$$

Similarly we obtain for the off-diagonal elements of U (i.e., $i\neq j$)

$$d_{ij}=-V_{ij}\chi_p=-\sum_{k=0}^{\infty}a_{ij}^{(k)}\chi_{p-2k}\,. \tag{45}$$

The coefficients $C(p,p+l)$ of (37) can now be read off (44) and (45) by comparison with

$$d_{ii}=\sum_l C_{ii}(p,p+l)\chi_{p+l}\,,$$

and (46)

$$d_{ij}=\sum_l C_{ij}(p,p+l)\chi_{p+l}\ \text{for } i\neq j\,.$$

Thus

$$C(p,p+4)=\frac{(p+1)(p+3)}{4}\begin{bmatrix}1&\cdot\\\cdot&1\end{bmatrix},$$

$$C(p,p+2)=O(\epsilon_p)\,,$$

$$C(p,p)=\begin{bmatrix}\Delta&-a_{12}^{(0)}\\-a_{21}^{(0)}&\Delta\end{bmatrix}+O(\epsilon_p)\,,$$

$$C(p,p-2)=\begin{bmatrix}\circ&-a_{12}^{(1)}\\-a_{21}^{(1)}&\circ\end{bmatrix}+O(\epsilon_p)\,,$$

and for $k\geq 2$

$$C(p,p-2k)=\begin{bmatrix}-a_{11}^{(k)}&-a_{12}^{(k)}\\-a_{21}^{(k)}&-a_{22}^{(k)}\end{bmatrix}+O(\epsilon_p)\,. \tag{47}$$

If we go as far as the second term in Eq. (40), we have apart from terms of $O(\epsilon_p)$

$$\begin{aligned}C(p,p)+\frac{1}{g}\sum_{\substack{i=4,2\\i\neq 0}}^{-\infty}C(p,p+i)\frac{1}{i}C(p+i,p)&=C(p,p)+\frac{1}{4g}[C(p,p+4)C(p+4,p)-C(p,p-4)C(p-4,p)]\\&=\begin{bmatrix}\Delta-\frac{p}{2g}a_{11}^{(2)}&-a_{12}^{(0)}-\frac{p}{2g}a_{12}^{(2)}\\-a_{21}^{(0)}-\frac{p}{2g}a_{21}^{(2)}&\Delta-\frac{p}{2g}a_{22}^{(2)}\end{bmatrix}.\end{aligned} \tag{48}$$

Setting the sums of the elements of each row of this matrix equal to zero, we see that the resulting expansions for Δ and hence E are identical with (26) or (27), although the method of calculation is completely different.

IV. MATCHING, NORMALIZATION, AND GENERALIZATION

In Secs. II and III we derived two pairs of linearly independent solutions of the matrix equation (1) which are valid in complementary domains of the independent variable. Their analytic continuation and normalization proceeds along the lines of our previous investigations.[14] We therefore do not go into extensive details here.

The solutions of Sec. II are

$$\psi_1(\omega^{1/2};p,g)=\frac{N_1(p,g)}{2g}\phi(\omega^{1/2};p,g) \tag{49}$$

and

$$\tilde{\psi}_1(\omega^{1/2};p,g)=\psi_1(\omega^{1/2};-p,-g),$$

and the solutions of Sec. III are

$$\tilde{\psi}_2(\omega^{1/2};p,g)=\tilde{N}_2(p,g)\exp\left\{\int^y dy[\omega(y)]^{1/2}\right\}\chi(y;p,g) \tag{50}$$

and

$$\psi_2(\omega^{1/2};p,g)=\tilde{\psi}_2(\omega^{1/2};-p,-g),$$

where we have introduced normalization constants N_1, N_2, and the variables and parameters are defined as above. The constant N_1 can be related to the constant N_2 by going to a region of common validity of our solutions. Thus inserting the large-x asymptotic expansion of $\phi_p(x)$ into ψ_1, i.e.,

$$\phi_p(x)=e^{-x^2/4}x^{(p-1)/2}\left[1-\frac{(p-1)(p-3)}{8x^2}+\cdots\right] \tag{51}$$

[recall that $x=(2g)^{1/2}y$ and g is assumed to be large], we see that ψ_1 behaves as

$$\psi_1(\omega^{1/2};p,g)=\frac{N_1(p,g)}{2g}e^{-gy^2/2}\times(2gy^2)^{(p-1)/4}\left[1+O\left(\frac{1}{g}\right)\right]. \tag{52}$$

Next we insert (42) and (43) into (50). Then

$$\tilde{\psi}_2(\omega^{1/2};p,g)=\tilde{N}_2(p,g)e^{gy^2/2}\times(y^2)^{-(p+1)/4}\left[1+O\left(\frac{1}{g}\right)\right]. \tag{53}$$

Thus comparing (52) and (53) we see that ψ_1 and ψ_2 represent the same solution in their common domain of validity, provided

$$\frac{N_1(p,g)}{2g}(2g)^{(p-1)/4}\left[1+O\left(\frac{1}{g}\right)\right]=\tilde{N}_2(-p,-g)\times[1+O(1/g)],$$

i.e.,

$$\tilde{N}_2(-p,-g)=(2g)^{(p-5)/4}N_1(p,g)\times[1+O(1/g)]. \tag{54}$$

By imposing the condition of square integrability and proceeding as in Ref. 14 we can determine N_1. We do not go into further details here.

We now consider the case of different reduced masses μ_1, μ_2 of the two channels. In this case the second derivatives in (1) will be multiplied by a factor $1/\mu_i$. In order to be able to construct a matrix operator which is a multiple of the unit matrix, we define a mean reduced mass μ by

$$\frac{1}{2}\left(\frac{1}{\mu_1}+\frac{1}{\mu_2}\right)=\frac{1}{\mu}. \tag{55}$$

The deviation

$$\left(\frac{1}{\mu}-\frac{1}{\mu_i}\right)\frac{d^2}{dy^2}$$

can then be treated as a perturbative contribution. Multiplying the equations through by μ, we have instead of (1)

$$\begin{pmatrix} \frac{d^2}{dy^2}+\mu E-g^2\mu y^2-\mu V_{11}(y)+\frac{\mu_2-\mu}{\mu_2}\frac{d^2}{dy^2} & -\mu V_{12}(y) \\ -\mu V_{21}(y) & \frac{d^2}{dy^2}+\mu E-g^2\mu y^2-\mu V_{22}(y)+\frac{\mu_1-\mu}{\mu_1}\frac{d^2}{dy^2} \end{pmatrix}\begin{pmatrix}\psi_1(y)\\ \psi_2(y)\end{pmatrix}=0. \tag{56}$$

Proceeding as before, we set

$$x=(4g^2\mu)^{1/4}y \tag{57}$$

and

$$\phi_i=(4g^2\mu)^{1/2}\psi_i,\quad V_{ij}=(4g^2\mu)^{1/2}v_{ij}. \tag{58}$$

Then

$$\mu E=pg\mu^{1/2}+\Delta, \tag{59}$$

and instead of Eq. (9) we have

$$\mathfrak{D}_{pp}\phi=\frac{1}{g\mu^{1/2}}U\phi, \tag{60}$$

where

$$U=\begin{bmatrix}\Delta - V_{11}+\frac{\mu_2-\mu}{\mu_2}(4g^2\mu)^{1/2}\frac{d^2}{dx^2} & -V_{12}\\ -V_{21} & \Delta - V_{22}+\frac{\mu_1-\mu}{\mu_1}(4g^2\mu)^{1/2}\frac{d^2}{dx^2}\end{bmatrix}. \tag{61}$$

From (15) and (25) it is clear that the coefficients $C_{ii}(p,p+k)$ of Sec. II now receive an additional contribution coming from

$$+(4g^2\mu)^{1/2}\frac{\mu_i-\mu}{\mu\mu_i}\frac{d^2}{dx^2}\phi_p. \tag{62}$$

Using the differential equation (12) we can rewrite this as

$$(g^2\mu)^{1/2}\frac{\mu_i-\mu}{\mu\mu_i}(-p+\tfrac{1}{2}x^2)\phi_p=(g^2\mu)^{1/2}\frac{\mu_i-\mu}{\mu\mu_i}[\tfrac{1}{2}S_2(p,4)\phi_{p+4}-\tfrac{1}{2}p\phi_p+\tfrac{1}{2}S_2(p,-4)\phi_{p-4}]. \tag{63}$$

Hence $C_{ii}(p,p)$ and $C_{ii}(p,p\pm 4)$ become

$$C_{ii}(p,p)=\Delta-\sum_{l=2}^{\infty}a_{ii}^{(l)}\frac{1}{(2g)^{l/2}}S_l(p,0)-(g^2\mu)^{1/2}\frac{\mu_i-\mu}{\mu\mu_i}\frac{p}{2} \tag{64a}$$

and

$$C_{ii}(p,p\pm 4)=-\sum_{l=2}^{\infty}a_{ii}^{(l)}\frac{1}{(2g)^{l/2}}S_l(p,\pm 4)+\frac{(g^2\mu)^{1/2}}{2}\frac{\mu_i-\mu}{\mu\mu_i}S_2(p,\pm 4), \tag{64b}$$

and other coefficients remain unchanged.

We have thus demonstrated that unequal channel masses can easily be handled in the formalism of Sec. II. It is not difficult to see that the appropriate generalizations can also be incorporated in the formalism of Sec. III, so we do not go into further details.

We close with some comments on the separability of our equations. In general the equations are not separable. However, we can search for a unitary transformation $U(\theta)$ (for some rotation angle θ) which diagonalizes the Hamiltonian H of Eq. (56) and so separates these equations. Thus setting

$$U(\theta)=e^{i\theta\sigma_2}$$

where σ_2 is the second Pauli matrix, and requiring UHU^{-1} to be diagonal, one finds for $V_{12}=V_{21}$

$$V_{12}=\tfrac{1}{2}\tan 2\theta\left[(V_{11}-V_{22})-\left(\frac{1}{\mu_1}-\frac{1}{\mu_2}\right)\frac{d^2}{dy^2}\right]$$

and so

$$UHU^{-1}=-\begin{bmatrix}\frac{1}{\mu_1}\frac{d^2}{dy^2}-g^2y^2-V_{11}-2g_{12}H_- & \cdot\\ \cdot & \frac{1}{\mu_2}\frac{d^2}{dy^2}-g^2y^2-V_{22}+2g_{12}H_-\end{bmatrix},$$

where $H_-=\frac{1}{2}(H_{11}-H_{22})$, H_{ii} being a diagonal element of H, and

$$\tan 2\theta=[(1+2g_{12})^2-1]^{1/2}.$$

Thus the difference between the original channel Hamiltonians, i.e., $2H_-$, appears as a contribution which again has to be treated as a perturbation. However, since the equations are now separable, one can solve the coupled system for channel states characterized by different quantum numbers p_1 and p_2. This is then another way to attack our problem in the special case of $V_{12}=V_{21}$.

V. CONCLUSIONS

In the preceding sections we have demonstrated that one can easily develop a perturbation theory for solving explicitly the eigenvalue problem defined by the coupled equations of the multichannel formalism. In particular we have shown that different types of solutions (valid in complementary domains) can be derived and matched in neighboring regions of validity. The eigenvalues are given explicitly by expansions (27) which have been derived by two independent and completely different methods, and thus serves as a verification of our procedure.

We have not dealt with the scattering problem here. However, this can be done in a manner analogous to the derivation of the S matrix for the Yukawa potential.[10] Put briefly, the procedure is to treat the secular equation (27) as an equation determining the auxiliary parameter p (which in the case of the discrete spectrum is $2n+1$, $n=0,1,\ldots$), i.e.,

$$p=\frac{1}{g}(E-a_{12}^{(0)})+O\left(\frac{1}{g^2}\right).$$

The solutions associated with the continuous spectrum can then be constructed from those of the discrete spectrum by replacing p by this expression.

We have shown above that our procedure depends crucially on the construction of a differential operator which is a multiple of the unit matrix and thus commutes with the matrix coefficients of the perturbation (this is reminiscent of the commutation property between a Hamiltonian and the generators of the transformation which leaves this Hamiltonian invariant). We have also shown that the difference between channel masses must be treated as a perturbation. Centrifugal terms can also be taken into account by following the method of Ref. 11. However, a difference between channel angular momenta or quantum numbers would again have to be treated as a perturbation.

The solutions we derived offer a simple way for investigating a number of interesting problems, particularly questions associated with the possible existence of $Q^2\overline{Q}^2$ composites. For instance, the approximate widths for such states should be calculable from our eigenvalue expansions by using the semiclassical formula[12] for the value of the wave function at the origin. If the centrifugal terms are taken into account, Regge trajectories and their distortion by channel coupling can be investigated. Some of these problems will be treated in a later publication.

ACKNOWLEDGMENT

This work was supported in part by the Deutsche Forschungsgemeinschaft.

[1]R. F. Dashen, J. B. Healy, and I. J. Muzinich, Phys. Rev. D 14, 2773 (1976); Ann. Phys. (N.Y.) 102, 1 (1976).

[2]G. v. Gehlen and V. Rittenberg, Phys. Lett. 71B, 373 (1977).

[3]B. R. Karlsson and B. Kerbikov, Nucl. Phys. B141, 241 (1978); C. Dullemond and E. van Beveren, Ann. Phys. (N.Y.) 105, 318 (1977).

[4]S. Nussinov and D. P. Sidhu, Fermilab Report No. 76/70-THY, 1976 (unpublished).

[5]D. Horn and D. E. Novoseller, Phys. Rev. D 17, 1763 (1978).

[6]The standard reference is R. G. Newton, *Scattering Theory of Waves and Particles* (McGraw-Hill, New York, 1966).

[7]H. Feshbach, Ann. Phys. (N.Y.) 5, 357 (1958); K. Gottfried, in *Elementary Particle Physics and Scattering Theory*, 1967 Brandeis Summer Institute in Theoretical Physics, edited by M. Chretien and S. S. Schweber (Gordon and Breach, New York, 1970), Vol. II, p. 148. A more recent treatment has been given by Cao-Xuan-Chuan and H. van Regemorter, ICTP Report No. 78/98, 1978 (unpublished).

[8]A. De Rújula and R. L. Jaffe, MIT Report No. CTP 658, 1977 (unpublished); A. De Rújula, H. Georgi, and S. L. Glashow, Phys. Rev. Lett. 38, 317 (1977); R. L. Jaffe, Phys. Rev. D 15, 281 (1977); W. Celmaster, Harvard University Report No. HUTP-77/A022, 1977 (unpublished); H. J. Lipkin, Phys. Lett. 70B, 113 (1977); S. F. Tuan, Phys. Rev. D 15, 3478 (1977); Ch. de la Vaissiere, CERN Report No. 77-33, 1977 (unpublished); D. Robson, Liverpool University Report No. LTH 35, 1977 (unpublished); F. Gutbrod, G. Kramer, and C. Rumpf, Z. Phys. (to be published).

[9]The method was first developed for the simple case of a periodic potential in R. B. Dingle and H. J. W. Müller, J. f. reine u. angew. Math. 211, 11 (1962); 216, 123 (1964); and H. J. W. Müller, *ibid.* 211, 179 (1962); the extension to the more complicated spheroidal and ellipsoidal wave equations is given in H. J. W. Müller, *ibid.* 211, 33 (1962); 212, 26 (1963); and Math. Nachr. 32, 157 (1966).

[10]The wave equation for Yukawa potentials has been treated in H. J. W. Müller, Ann. Phys. (Leipz.) 15, 395 (1965); H. J. W. Müller and K. Schilcher, J. Math. Phys. 9, 255 (1968); H. J. W. Müller-Kirsten and N. Vahedi-Faridi, *ibid.* 14, 1291 (1973).

[11]The wave equation for the Gauss potential has been treated in H. J. W. Müller, J. Math. Phys. 11, 355 (1970).

[12]The wave equations for positive power and logarithmic potentials have been solved in H. J. W. Müller-Kirsten and S. K. Bose, J. Math. Phys. (to be published); H. J. W. Müller-Kirsten, G. E. Hite, and S. K. Bose, J. Math. Phys. (to be published).

[13]H. J. W. Müller-Kirsten, Phys. Lett. 70A, 383 (1979).

[14]This is similar to the matching and normalization of the solutions given in Ref. 12. This has been done in R. S. Kaushal and H. J. W. Müller-Kirsten, J. Math. Phys. (to be published).

[15]R. G. Newton, Ann. Phys. (N. Y.) 4, 29 (1958); J. Math. Phys. 2, 188 (1961).

[16]R. G. Newton, J. Math. Phys. 1, 319 (1960).