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Relativistic Bethe-Salpeter harmonic oscillator

John R. Henley
Department of Physics, B-019, University of California, San Diego, La Jolla, California 92093
and Lockheed Palo Alto Research Laboratory, 3251 Hanover Street, Palo Alto, California 94304
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A kernel is proposed for the Bethe-Salpeter equation for bosons that leads to an ordinary second-order linear differential equation in momentum space. The weak-coupling limit of this equation explicitly reduces to the Schrödinger equation for the simple harmonic oscillator. The relative energy dependence of the Bethe-Salpeter amplitude does not admit timelike excitations, nor are there anomalous solutions of any kind. We conclude that this model can be interpreted as a relativistic harmonic oscillator without spin.

I. INTRODUCTION

The harmonic oscillator is the simplest and most pervasive system in physics. However, what happens to this ubiquitous system when the spring becomes very strong, or when the mass travels at relativistic speeds is still largely unstudied on a quantum scale. This is understandable because a spring of such a strength that its ground state shows relativistic effects would be very strong indeed, and none that we knew of existed in nature. Nevertheless, recent developments in non-Abelian gauge theories[1,2] (i.e., quantum chromodynamics) have given rise to the notion that quarks are confined in a steeply rising "potential" where relativistic effects may be measurable. The phenomenology of the ψ/J particles has tended to support this, and efforts have been made to postulate such a potential for use in the Schrödinger equation[3] and, in more sophisticated approaches, an interaction kernel for use in the Bethe-Salpeter equation.[4] Thus, the concept of relativistic oscillators no longer seems so farfetched.

For relativistic problems, the Bethe-Salpeter formalism is appealing because it is explicitly covariant and has at least formal connection with quantum field theory and perturbation theory. It is, however, quite difficult to use for two reasons. First, if one is given a Lagrangian, it is by no means a simple matter to establish the proper kernel to use in the Bethe-Salpeter equation. Even in the case of positronium, it is only recently that this formalism has yielded to analysis.[5,6]

Of course, the inverse problem also exists. If one solves a Bethe-Salpeter equation with a "realistic" kernel for the interaction, he cannot be too certain what kind of Lagrangian has led to his results. These problems confine such work to the realm of phenomenology, which in itself is some deterrent to this approach. But the main obstacle, even if one accepts the limitation of not knowing the exact Lagrangian, is that the equation usually will not separate into independent equations in the four variables. That is, the timelike and spacelike parts of the equation are hopelessly mixed up so that one must solve an at least two-dimensional equation.

Nevertheless, it should be of some interest to find a solution to the Bethe-Salpeter equation which it can be said describes a relativistic harmonic oscillator. In the present work, we have discovered a kernel having this virtue, as well as being reducible to a one-dimensional problem. There have, of course, been others who have studied soluble kernels which one might also interpret as oscillators.[7] We present this work, therefore, as one possibility among others. Our approach has an advantage in that there is no question of timelike excitations, the interpretation of which has presented some difficulty in other models.[8] However, we must concede that we can actually solve the equation analytically only in the nonrelativistic limit, though numerical integration of the fully relativistic equation is quite trivial.

In Sec. II, we discuss the fundamental Yukawa interaction, its angular decomposition, and the reason that it is difficult to solve analytically except when the exchanged mass is zero. Section III is devoted to constructing the singular kernel with which we shall concern ourselves. In Sec. IV, we show that the solution must have a δ-function-like dependence on the relative energy variable, which allows reduction of the equation to an ordinary differential equation. This reduction involves a caveat which we discuss in Sec. V. Section VI is devoted to a discussion of the differential equation, the nonrelativistic limit, and numerical results.

II. THE "YUKAWA" PROBLEM

To begin, let us examine the familiar problem of two scalar particles of mass m interacting via the exchange of a scalar particle of mass λ. In the usual ladder approximation, the Bethe-Salpeter equation is[9]

$$[(\tfrac{1}{2}P+p)^2-m^2+i\eta][(\tfrac{1}{2}P-p)^2-m^2+i\eta]\chi(p) = -i\left(\frac{g}{\pi}\right)^2\int\frac{1}{(p'-p)^2-\lambda^2+i\eta}\chi(p')d^4p' . \quad (1)$$

P is defined to be the center-of-mass four-momentum and p is the relative four-momentum of the two particles of mass m.

In the center-of-mass coordinate system, we shall define

$$p\equiv(\omega,\vec{k}) ,\qquad P\equiv(M,\vec{0}) . \quad (2)$$

This will result in the equation

$$[(-\omega^2+k^2+m^2-\tfrac{1}{4}M^2-i\eta)^2-M^2\omega^2]\chi(p) = -i\left(\frac{g}{\pi}\right)^2\int\frac{1}{(p'-p)^2-\lambda^2+i\eta}\chi(p')d^4p' . \quad (3)$$

Finally, defining the partial-wave decomposition to be

$$\chi(p)=\sum_{lm}\frac{1}{k}\chi_l(k,\omega)Y_l^m(\hat{k}) , \quad (4)$$

we find

$$[(-\omega^2+k^2+m^2-\tfrac{1}{4}M^2-i\eta)^2-M^2\omega^2]\chi_l(k,\omega) = \frac{2g^2}{\pi i}\int Q_l\left(\frac{(k'-k)^2-(\omega'-\omega)^2+\lambda^2-i\eta}{2kk'}+1\right) \times\chi_l(k',\omega')d\omega'dk' . \quad (5)$$

The solution of this problem has been resistant to attack except for the case $\lambda=0$ because it cannot be separated in the k and ω variables. Even though one can determine certain of the analytic properties of χ well enough to close the contour of integration and do the so-called "Wick rotation,[10]" one is still left with an integration around the cuts in the ω' plane which is too complicated to do analytically.[11]

However, if one were to add to this interaction one that differed from it only by having the opposite sign in the coupling g^2 and a slightly larger mass λ', then the net contribution to the integral will come only from the ends of the cuts where the singularity will be nearly like a pole. As in the Yukawa problem, when λ and λ' are taken to zero, one can reduce the equation to a much simpler form. This indeed shall be the basis of our approach. We recognize that if we are to apply this idea to the interaction of a particle-antiparticle pair, then changing the sign of g^2 is a bit cavalier. However, such sign changes could arise from the inclusion of additive quantum numbers or spin which are not considered here.

III. CONSTRUCTION OF A CONFINING INTERACTION KERNEL

In order to produce an interaction that looks like a simple harmonic oscillator, we follow the above suggestion, but it is necessary to add eight terms together with appropriate magnitudes and signs so that the results are derivatives with respect to the exchanged mass. This procedure yields a family of interactions

$$V^{(n)}(p,p')\approx-i\left(\frac{g}{\pi}\right)^2\frac{\partial^n}{\partial\lambda^n}\left[\frac{1}{(p'-p)^2-\lambda^2+i\eta}\right] . \quad (6)$$

The static limit of these behaves as $g^2r^{n-1}e^{-\lambda r}$, and the $V^{(2n+1)}$ potentials have a δ-function-like structure that we can exploit to "separate" the equation ($V^{(2n)}$ yield inseparable equations). $V^{(3)}$ is therefore the lowest-order interaction of interest to us here. Specifically,

$$V^{(3)}\approx2ig^2\frac{12\lambda}{\pi^2}\frac{-(p'-p)^2-\lambda^2-i\eta}{[-(p'-p)^2+\lambda^2-i\eta]^4} . \quad (7)$$

Henceforth, it is to be assumed that we are considering this case and in the limit that λ is very small. Explicitly, the Bethe-Salpeter equation of interest here shall be

$$[(-\omega^2+k^2+m^2-\tfrac{1}{4}M^2-i\eta)^2-M^2\omega^2]\chi(p) = \lim_{\lambda\to0}\frac{12\lambda}{i}\frac{2g^2}{\pi^2}\int\frac{-(p'-p)^2-i\eta-\lambda^2}{[-(p'-p)^2+\lambda^2-i\eta]^4}\chi(p')d^4p' . \quad (8)$$

The first task, as usual, is to perform a partial-wave decomposition of this equation. This can be done by brute force, of course, but it is easiest to make the observation that

$$\lim_{\lambda\to0}\frac{12\lambda}{\pi^2}\frac{(\vec{x}-\vec{x}')^2-\lambda^2}{[(\vec{x}-\vec{x}')^2+\lambda^2]^4}=\Delta\delta^{(3)}(\vec{x}'-\vec{x}) , \quad (9)$$

where $\vec{x}$ and $\vec{x}'$ are three-vectors. From our experience with this object, we know already that

$$\Delta\delta^{(3)}(\vec{x}'-\vec{x})=\sum_l\frac{2l+1}{4\pi xx'}\left\{\left[\delta''(x'-x)-\frac{l(l+1)}{x^2}\delta(x'-x)\right]-(-1)^l\left[\delta''(x'+x)-\frac{l(l+1)}{x^2}\delta(x'+x)\right]\right\}P_l(\cos\alpha) . \quad (10)$$

Either way, the partial-wave equation in the limit of small λ is found to be

$$[(-\omega^2+k^2+m^2-\tfrac{1}{4}M^2-i\eta)^2-M^2\omega^2]\chi_l(k,\omega)$$

$$=\frac{\lambda g^2}{\pi i}\int \chi_l(k',\omega')\left\{\frac{6(k'-k)^2-6(\omega'-\omega)^2-6i\eta-2\lambda^2}{[(k'-k)^2-(\omega'-\omega)^2+\lambda^2-i\eta]^3}-\frac{l(l+1)}{k^2}\frac{1}{[(k'-k)^2-(\omega'-\omega)^2+\lambda^2-i\eta]}\right\}d\omega' dk'$$

$$-\frac{(-1)^l\lambda g^2}{\pi i}\int \chi_l(k',\omega')\left\{\frac{6(k'+k)^2-6(\omega'-\omega)^2-6i\eta-2\lambda^2}{[(k'+k)^2-(\omega'-\omega)^2+\lambda^2-i\eta]^3}-\frac{l(l+1)}{k^2}\frac{1}{[(k'+k)^2-(\omega'-\omega)^2+\lambda^2-i\eta]}\right\}d\omega' dk' . \tag{11}$$

We define the variables

$$p_1{}^2=(k'-k)^2-(\omega'-\omega)^2 ,$$
$$p_2{}^2=(k'+k)^2-(\omega'-\omega)^2 \tag{12}$$

and the functions

$$\delta_\lambda(p)=\frac{\lambda}{\pi}\frac{1}{p^2+\lambda^2} ,$$
$$\delta_\lambda''(p)=\frac{\partial^2}{\partial p^2}\delta_\lambda(p) , \tag{13}$$

so that (11) can be written

$$[(-\omega^2+k^2+m^2-\tfrac{1}{4}M^2-i\eta)^2-M^2\omega^2]\chi_l(k,\omega)=-ig^2\int\left(\delta_\lambda''(p_1)-\frac{l(l+1)}{k^2}\delta_\lambda(p_1)\ \ \chi_l(p_1)\right)\chi_l(k',\omega')d\omega' dk'$$
$$+i(-1)^l g^2\int\left(\delta_\lambda''(p_2)-\frac{l(l+1)}{k^2}\delta_\lambda(p_2)\right)\chi_l(k',\omega')d\omega' dk' . \tag{14}$$

It is easy to show that the integrands must be even in k' and the integrals in (14) are to be taken from $-\infty$ to ∞ in k' and ω'.

IV. SEPARATION OF THE EQUATION WHEN $\lambda\to 0$

We shall assume λ is fixed, but very small compared to m. If Eq. (11) is examined, we can see first of all that if $\chi_l(k,\omega)$ vanishes at large $|\omega|$ in the upper or lower half plane so that the contour can be closed and the integral evaluated, the right-hand side will vanish with λ unless $\chi_l(k,\omega)$ is singular. We confirm, therefore, that $\chi_l=0$ is a solution in the limit $\lambda\to 0$. Obviously, we must look for singular solutions. Since it is usually a property of integrals that they smooth singularities, we expect that the result of the right-hand side should be at least less singular than $\chi_l(k,\omega)$ itself, if not completely smooth. The only way that this can come about is for the singularity of χ_l to fall on the zeros of the quantity multiplying it on the left-hand side of the equation. This occurs at the values of ω

$$\Omega_0(k)\equiv\tfrac{1}{2}M-(k^2+m^2)^{1/2}+i\epsilon,\quad \epsilon=\frac{\eta}{2(k^2+m^2)^{1/2}} ,$$
$$\Omega_1(k)\equiv\tfrac{1}{2}M+(k^2+m^2)^{1/2}-i\epsilon . \tag{15}$$

We wish to define also

$$\omega_0(k)\equiv\tfrac{1}{2}M-(k^2+m^2)^{1/2} ,$$
$$\omega_1(k)\equiv\tfrac{1}{2}M+(k^2+m^2)^{1/2} , \tag{16}$$

so that

$$\Omega_{0,1}=\omega_{0,1}\pm i\epsilon .$$

We have the choice of placing simple poles at these points; however, if we do, it will be found that we have not exhausted the ω dependence of the wave function. The next obvious choice is to make $\chi_l(k,\omega)$ a δ function of ω. Thus, we shall take

$$\chi_l(k,\omega)=\lim_{\epsilon\to 0}\frac{\epsilon}{\pi}\frac{1}{[\omega-\frac{1}{2}M+(k'^2+m^2)^{1/2}]^2+\epsilon^2}W_l(k) . \tag{17}$$

This will result in the equation

$$[\omega^2-\Omega_1{}^2(k)][\omega+\omega_0(k)+i\epsilon]\frac{\epsilon W_l(k)}{\omega-\omega_0+i\epsilon}=+\frac{g^2}{i}\int\left[\delta_\lambda''((p_1{}^2-i\eta)^{1/2})-\frac{l(l+1)}{k^2}\delta_\lambda((p_1{}^2-i\eta)^{1/2})\right]$$
$$\times\,\delta_\epsilon(\omega'-\omega_0(k'))W(k')\,dk'$$
$$-\frac{(-1)^l g^2}{i}\int\left[\delta_\lambda''((p_2{}^2-i\eta)^{1/2})-\frac{l(l+1)}{k^2}\delta_\lambda((p_2{}^2-i\eta)^{1/2})\right]$$
$$\times\,\delta_\epsilon(\omega'-\omega_0(k'))\,W_l(k')d\omega'dk'\,. \qquad (18)$$

At this point, we shall define the variable

$$q\equiv\omega'-\tfrac{1}{2}M+(k'^2+m^2)^{1/2} \qquad (19)$$

and the Jacobians

$$J_1(k',\omega',k,\omega)=\frac{\partial(k',\omega')}{\partial(p_1,q)}=\frac{[(k'-k)^2-(\omega'-\omega)^2]^{1/2}}{[(\omega'-\omega)/(k'^2+m^2)^{1/2}+1]k'-k}\;,$$
$$J_2(k',\omega',k,\omega)=\frac{\partial(k',\omega')}{\partial(P_2,q)}=\frac{(k'+k)^2-(\omega'-\omega)^2]^{1/2}}{[(\omega'-\omega)/(k'^2+m^2)^{1/2}+1]k'+k}\,. \qquad (20)$$

If $W_l(k)$ is well behaved at infinity, these integrals can now be evaluated and the right-hand side (RHS) of Eq. (18) becomes

$$\mathrm{RHS}=-ig^2\left[\left(\frac{\partial^2}{\partial p_1{}^2}-\frac{l(l+1)}{k^2}\right)J_1(k,\omega,k',\omega')\,W_l(k')\right]_{\substack{q=i\epsilon\\ p_1=i(\lambda^2-i\eta)^{1/2}}}$$
$$+i(-1)^l g^2\left[\left(\frac{\partial^2}{\partial p_2{}^2}-\frac{l(l+1)}{k^2}\right)J_2(k',\omega',k,\omega)\,W_l(k')\right]_{\substack{q=i\epsilon\\ p_2=i(\lambda^2-i\eta)^{1/2}}}\,. \qquad (21)$$

It is easy to show that

$$\frac{\partial}{\partial p_1}=J_1\left(\frac{\partial}{\partial k'}-\frac{k'}{(k'^2+m^2)^{1/2}}\frac{\partial}{\partial\omega'}\right)\,. \qquad (22)$$

Because of the symmetry of W_l and the similarity between J_1 and J_2, we can multiply the first term by two and drop the second. Thus we have

$$\mathrm{RHS}=-2ig^2\left\{J_1{}^3W_l''+3J_1{}^2\left(\frac{\partial J_1}{\partial k'}-\frac{k'}{(k'^2+m^2)^{1/2}}\frac{\partial J_1}{\partial\omega'}\right)W_l'\right.$$
$$\left.+J_1\left[\left(\frac{\partial}{\partial k'}-\frac{k'}{(k'^2+m^2)^{1/2}}\frac{\partial}{\partial\omega'}\right)\left(J_1\frac{\partial J_1}{\partial k'}-\frac{k'}{(k'^2+m^2)^{1/2}}J_1\frac{\partial}{\partial\omega'}J_1\right)\right]W_l-\frac{l(l+1)}{k^2}J_1W_1\right\}_{\substack{q=i\epsilon\\ p_1=i(\lambda^2-i\eta)^{1/2}}} \qquad (23)$$

which can be put into the form

$$\mathrm{RHS}=+\frac{2g^2}{i}\left\{J_1{}^3W_l''+\frac{3J_1}{x}\left[1-\frac{m^2(\frac{1}{2}M-\omega+q)}{(k'^2+m^2)^{3/2}}J_1{}^2\right]W_l'\right.$$
$$\left.+\frac{3m^2(\frac{1}{2}M-\omega+q)}{(k'^2+m^2)^{3/2}x^2}J_1\left[\frac{p_1k'}{(k'^2+m^2)}J_1-\left(1-\frac{m^2(\frac{1}{2}M-\omega+q)}{(k'^2+m^2)^{3/2}}J_1{}^2\right)\right]W_l-J_1\frac{l(l+1)}{k^2}W_l\right\}_{\substack{q=i\epsilon\\ p_1=i(\lambda^2-i\eta)^{1/2}}}\,, \qquad (24)$$

where

$$x \equiv \left[\frac{\omega' - \omega}{(k'^2 + m^2)^{1/2}} + 1\right] k' - k \,. \qquad (25)$$

We will, of course, eventually wish to take the limit ϵ and λ go to zero in this expression, as well as on the other side of the equation. As will be seen, the left side of Eq. (18) is always finite, and we must therefore require all the terms on the right to be so also. In particular, we must have

$$\lim_{\substack{q\to 0\\ p_1\to 0}} \frac{3p_1}{x^2}\left[1 - \frac{m^2(\frac{1}{2}M - \omega + q)}{(k'^2+m^2)^{3/2}} J_1{}^2\right]$$

$$= \text{finite number for all } k,\omega \,. \qquad (26)$$

But from (12) and (25), it is clear that x and p_1 are zero at the same point, i.e., at $k' = k$, $\omega' = \omega$. Thus the limit at $p \to 0$ involves a function of the form 0/0 in two variables. Since there are two variables, a variety of answers is possible depending on the relationship between p and q. In this case, a finite result for the limit in (26) is possible at the special points

$$\omega = \tfrac{1}{2}M - (k^2+m^2)^{1/2} \text{ or } \omega = -\tfrac{1}{2}M + (k^2+m^2)^{1/2} \qquad (27)$$

(note the variables are unprimed), only if q is very much smaller than p. That is, $\epsilon \ll \lambda$.

If q is negligible, we have

$$J(k',\omega',k,\omega) = \frac{\{(k'-k)^2 - [\frac{1}{2}M - \omega - (k'^2+m^2)^{1/2}]^2\}^{1/2}}{[(\frac{1}{2}M - \omega)/(k'^2+m^2)^{1/2}]k' - k}, \qquad (28)$$

$$\lim_{q\to 0} J(k',\omega',k,\omega) = \frac{p_1}{[(\frac{1}{2}M - \omega)/(k'^2+m^2)^{1/2}]k' - k} \,. \qquad (29)$$

Clearly, this vanishes with p_1, unless the denominator vanishes for the value of k' that corresponds to $p_1 = 0$. This can occur only near the points $\omega = \frac{1}{2}M - (k^2+m^2)^{1/2}$ or $\omega = -\frac{1}{2}M + (k^2+m^2)^{1/2}$. At these points,

$$\lim_{p\to 0}\lim_{q\to 0} J(k',\omega',k,\omega_0) = (k^2+m^2)^{1/2}/m \,. \qquad (30)$$

We also obtain in this limit

$$\lim_{p\to 0}\lim_{q\to 0}\left(\frac{\partial}{\partial k'} - \frac{k'}{(k'^2+m^2)^{1/2}}\frac{\partial}{\partial\omega'}\right)J_1(k',\omega',k,\omega_0) = \frac{k}{m}\frac{1}{(k^2+m^2)^{1/2}} \,,$$

$$\lim_{p\to 0}\lim_{q\to 0}\left(\frac{\partial}{\partial k'} - \frac{k'}{(k'^2+m^2)^{1/2}}\frac{\partial}{\partial\omega'}\right)J_1(k',\omega',k,\omega_0)\left(\frac{\partial}{\partial k'} - \frac{k'}{(k'^2+m^2)^{1/2}}\frac{\partial}{\partial\omega'}\right)J_1(k',\omega',k,\omega_0) = \frac{3}{4m^2} \,. \qquad (31)$$

Thus, at the point $\omega = \omega_0$ we find

$$\mathrm{RHS}(k,\omega_0) = \frac{2g^2(k^2+m^2)^{1/2}}{im}\left[\frac{k^2+m^2}{m^2}W_l'' + 3\frac{k}{m^2}W_l' + \left(\frac{3}{4m^2} - \frac{l(l+1)}{k^2}\right)W_l\right]. \qquad (32)$$

Meanwhile, if we evaluate the left-hand side (LHS) at the point (k,ω_0), we obtain

$$\mathrm{LHS}(k,\omega_0) = 4iM(k^2+m^2)^{1/2}[\tfrac{1}{2}M - (k^2+m^2)^{1/2}]W_l(k) \,. \qquad (33)$$

The left and right sides both vanish at points away from (k,ω_0).

V. EQUALITY OF THE LEFT AND RIGHT SIDES OF THE SEPARATED EQUATION

In the limit that all the infinitesimal parameters vanish, we are left with both sides of the equation vanishing except at the special values of $\omega = \omega_0(k)$. This is a set of measure zero and while we were correct in our surmise that the result of integration would be a function less singular than the δ function with which we started, it is still singular and definitely not smooth. It is therefore necessary to decide exactly what is meant by equality of the two sides of the equation.

As functions of ω, both sides of the equation converge to a function of the form

$$f(\omega) = \begin{cases} a, & \omega = \omega_0 \\ 0, & \text{otherwise}\,. \end{cases} \qquad (34)$$

However, the two sides do not converge at the same rate. The structure of the left-hand side is determined by the parameter ϵ in the function

$$g(\omega) = \frac{1}{\pi}\frac{\epsilon}{\omega - \omega + i\epsilon} \,. \qquad (35)$$

The structure of the right-hand side is determined by the function $+iJ_1$. If k_0' and ω_0' are the values of k' and ω' for which $p=q=0$, then

$$k_0' = \tfrac{1}{2}(k+\omega-\tfrac{1}{2}M) - \frac{m^2}{2(k+\omega-\tfrac{1}{2}M)},$$
$$\omega_0' = \tfrac{1}{2}(k+\omega+\tfrac{1}{2}M) + \frac{m^2}{2(k+\omega-\tfrac{1}{2}M)}. \tag{36}$$

Defining

$$r \equiv (\omega - \omega_0), \tag{37}$$

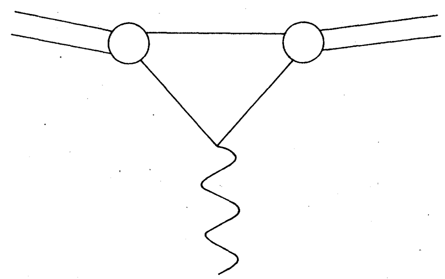

FIG. 1. Feynman diagram for the electromagnetic form factor.

we find

$$+iJ_1 \approx \frac{+ip_1}{\dfrac{(\tfrac{1}{2}M-\omega)k_0'}{(k_0'^2+m^2)^{1/2}} - k + (\tfrac{1}{2}M-\omega)p\dfrac{\partial}{\partial p}\dfrac{k_0'}{(k_0'^2+m^2)^{1/2}} + (\tfrac{1}{2}M-\omega)r\dfrac{\partial}{\partial r}\dfrac{k_0'}{(k_0'^2+m^2)^{1/2}}},$$

which reduces, in the limit, to the simple form

$$+iJ_1 \approx \frac{+ip_1}{(\omega-\omega_0) + [m/(k^2+m^2)^{1/2}]p_1}. \tag{38}$$

But $p_1 \sim i\lambda$, so the right-hand side structure is like that of the function

$$h(\omega) = -\frac{\lambda}{\omega - \omega_0 + i\lambda}. \tag{39}$$

We conclude that the left-hand side converges to its limit much faster than the right-hand side since we were forced to take $\epsilon \ll \lambda$ in order to keep the right-hand side finite. The process of integration has therefore not only pushed the peak of the δ function down to a finite height, but in a sense it has squashed it out to the side too. This will make a difference only in the case we wish to integrate our wave amplitude over a singular function, which is exactly the case if we wish to apply this formalism to the calculation of such practical objects as form factors and cross sections. For example in the diagram for the electromagnetic form (cf. Fig. 1), it will be necessary to evaluate an integral of the form

$$\int \chi^\dagger(\vec{k},\omega)[(\tfrac{1}{2}P+p)^2 - m^2 + i\eta]\chi(\vec{k},\omega)\cdots d\omega\, d^3k$$
$$= \int W^\dagger(k)(\omega-\omega_0-i\epsilon)(\omega-\omega_1-i\epsilon)W(k)\delta_\epsilon(\omega-\omega_0)\delta_\epsilon(\omega-\omega_0)\cdots d\omega\, dk. \tag{40}$$

That is, we must evaluate an integral of the form

$$I = \int (\omega-\omega_0-i\epsilon)\delta_\epsilon(\omega-\omega_0)\delta_\epsilon(\omega-\omega_0)d\omega. \tag{41}$$

Our choice of the parametric representation for $\delta_\epsilon(\omega-\omega_0)$ was chosen in the first place so that $(\omega-\omega_0-i\epsilon)\delta_\epsilon(\omega-\omega_0)$ has a definite meaning on the left-hand side of the Bethe-Salpeter equation. It is therefore possible to write down the value of this integral in a perfectly consistent way using the same construction, i.e.,

$$I = \frac{1}{\pi^2}\int \frac{\epsilon}{\omega-\omega_0+i\epsilon}\,\frac{\epsilon}{(\omega-\omega_0)^2+\epsilon^2}d\omega = \frac{1}{2\pi i}. \tag{42}$$

This is one-half the result one would get by simply using the second δ function in the conventional way; i.e., by evaluating the rest of the integrand at the point ω_0. This is because the expression $(\omega-\omega_0-i\epsilon)\delta_\epsilon(\omega-\omega_0)$ is singular with a characteristic scale the same as the δ function we use to evaluate the integral.

The Bethe-Salpeter equation contains the factor of $(\omega-\omega_0-i\epsilon)\delta_\epsilon(\omega-\omega_0)$ on the left side. Therefore, we can substitute the expression on the right-hand side of the Bethe-Salpeter equation for it where it occurs in other calculations. But, we have shown that the right-hand side is singular on the scale of λ which is much larger than ϵ. This means that in such integrals we can use the remaining δ function to evaluate the integral in the usual way and the factor of $\frac{1}{2}$ will not occur:

$$\frac{-i}{\pi^2}\int J_1{}^3 \frac{\epsilon}{(\omega-\omega_0)^2+\epsilon^2}\,d\omega \approx \frac{-i}{\pi}\int \frac{\lambda^3}{(\omega-\omega_0+i\epsilon)^3} \times \frac{\epsilon}{(\omega-\omega_0)^2+\epsilon^2}\,d\omega \approx 1\,. \quad (43)$$

We conclude that a pointwise interpretation of equality in the Bethe-Salpeter equation is inadequate in this case. We should instead adopt a more generalized integral definition of equality in much the same way we say distributions, such as δ functions, are equal. This means that the right-hand side of Eq. (18) has double the weight of the left-hand side in this "distribution" sense. Thus, the proper equation for $W_l(k)$ implied by (32) and (33) is

$$M[\tfrac{1}{2}M-(k^2+m^2)^{1/2}]W_l(k) = \frac{-g^2}{m^3}\left[(k^2+m^2)\frac{\partial^2}{\partial k^2}+3k\frac{\partial}{\partial k} +\left(\frac{3}{4}-\frac{m^2}{k^2}l(l+1)\right)\right]W_l(k)\,. \quad (44)$$

VI. SOLUTION OF THE DIFFERENTIAL EQUATION

Equation (44) can be written in the dimensionless form

$$\left[-(1+\xi^2)\frac{\partial^2}{\partial\xi^2}-3\xi\frac{\partial}{\partial\xi}+\left(b(1+\xi^2)^{1/2}+\frac{l(l+1)}{\xi^2}\right)\right]W_l(\xi) = a_n W_l(\xi)\,, \quad (45)$$

where

$$\xi=\frac{k}{m}\,, \qquad a_n=\frac{m^3M^2}{2g^2}+\frac{3}{4}\,, \qquad b=\frac{m^4M}{g^2}\,, \quad (46)$$

so that

$$M=\frac{4a_n-3}{2b}m\,, \qquad g^2=\frac{4a_n-3}{2b^2}m^5\,. \quad (47)$$

We see immediately that the weak-coupling limit is large b. The low-lying states in this case have wave functions that are significant in magnitude only where $\xi \ll 1$. Thus, the purely nonrelativistic limit of (45) is

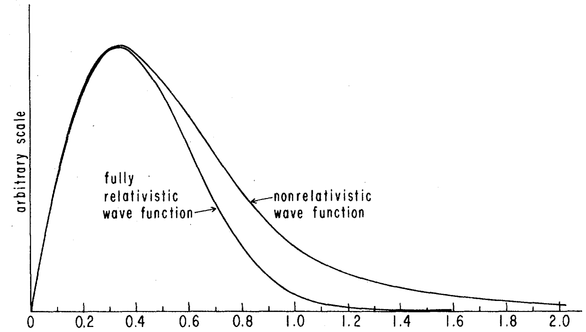

FIG. 2. Comparison of the nonrelativistic to fully relativistic ground-state wave function for $\kappa=0.25$ ($b=81$). The wave functions are not normalized in order to emphasize their difference.

$$\left[-\frac{\partial^2}{\partial\xi^2}+\tfrac{1}{2}b\xi^2+\frac{l(l+1)}{\xi^2}\right]W_l=(a_n-b)W_l\,. \quad (48)$$

We must keep the $\frac{1}{2}\xi^2$ term in the expansion of the square root because it is multiplied by the large number b; in addition, since a is large when b is large, it nearly cancels the constant from that expansion. These features are not present in the second derivative term so we are justified in dropping ξ^2 compared to 1 there. [The ultimate test, of course, is that using (48) gives a result in close agreement to a numerical integration of (45) with b taken very large, which was indeed the result found.] Equation (48) is the well-known simple-harmonic-oscillator equation and its solutions are familiar. The eigenvalues a_n-b are given by[12]

$$a_n-b=\sqrt{2b}\,(n+\tfrac{3}{2}),\quad n=0,1,2,\ldots. \quad (49)$$

We note that the lowest-lying state is odd in k here because of the definition of the partial-wave decomposition we used in Eq. (4). The mass spec-

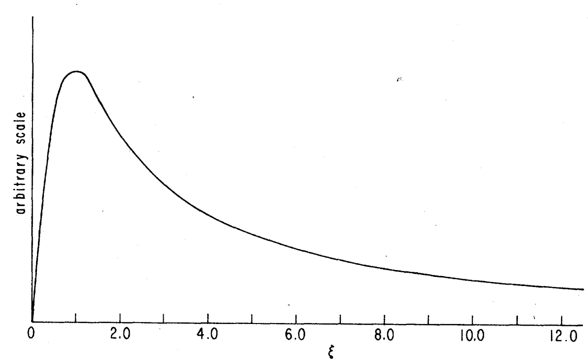

FIG. 3. Fully relativistic ground-state wave function for $\kappa=115$ ($b=0.031\,25$). The much more slowly declining wave function in the large-ξ region is apparent in this highly relativistic case.

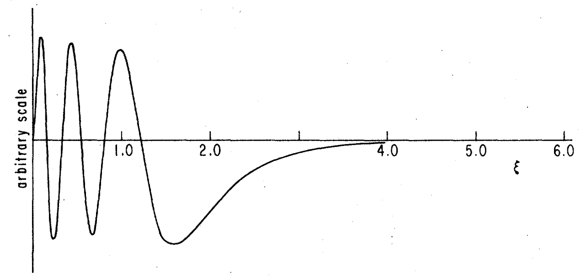

FIG. 4. The 6S-state wave function for $\kappa = 0.189$ ($b = 256$). This wave function is only slightly relativistic.

trum implied by (48) is

$$M_n \approx 2m + \left(\frac{2g^2}{m^3}\right)^{1/2} (n + \tfrac{3}{2}) . \qquad (50)$$

This implies that the classical spring constant κ is

$$\kappa = g^2/m^2 , \qquad (51)$$

so that M_n rises as the square root of κ when κ is small.

As can be seen from the numerical work, when κ grows to a value greater than $m^3/10$, the spring becomes so strong that even in the lowest-lying states, the motion is relativistic. In this case, we see that b is small and that if a approaches definite limits as $b \to 0$, then M_n must continue to rise as the square root of the classical spring constant κ.

While most of the qualitative features of the wave function follow those of the nonrelativistic simple harmonic oscillator, the asymptotic behavior is changed significantly. For very large ξ, Eq. (45) becomes

$$\left(\frac{\partial^2}{\partial \xi^2} + \frac{3}{\xi}\frac{\partial}{\partial \xi} - \frac{b}{\xi}\right) W = 0, \quad \xi \gg 1 \qquad (52)$$

which can easily be converted into Bessel's equation. Therefore, the asymptotic behavior of W_l

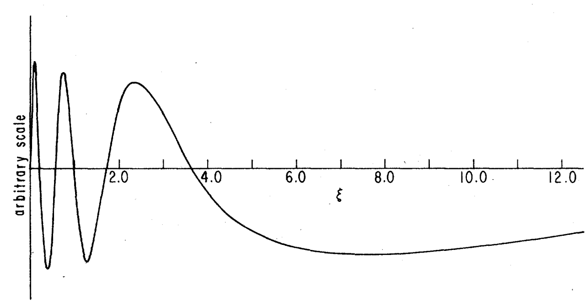

FIG. 5. Highly relativistic 6S-state wave function with $\kappa = 190$ ($b = 0.125$).

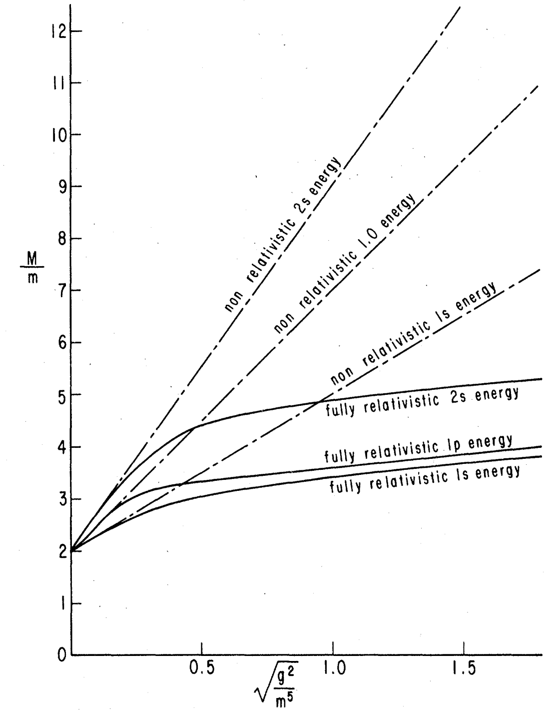

FIG. 6. Depression of the lowest states' energies from the nonrelativistic prediction as a function of coupling strength.

is given by

$$W_l(\xi) \sim \frac{1}{(4b\xi)^{3/2}} e^{-(4b\xi)^{1/2}}, \quad \xi \gg 1 . \qquad (53)$$

This, of course, differs from the ordinary simple harmonic oscillator which falls off like $e^{-\xi^2}$.

Because of the square root in (44) and (45), the equation for $W_l(\xi)$ is not that for a hypergeometric function. One is forced to integrate this problem numerically to find its eigenvalues and the wave amplitudes themselves.

The figures show the results of this numerical work. For the ground state, shown in Figs. 2 and 3, the relativistic corrections are small up to classical spring constants of the order of 0.25 or so. In Fig. 3, the difference in the large-k dependence of the wave function is very apparent for the very large coupling $\kappa = 115$. The comparison between weak and strong couplings for the higher-lying 6S state is shown in Figs. 4 and 5. Finally, in Figs. 6 and 7, the behavior of the energy levels as a function of the coupling constant is shown. It is clear that the effect of relativity upon the energy eigenvalues is to depress them relative to the expectation of the nonrelativistic approximation. Thus, in a loose sense the potential seems weaker than quadratic. Also, apparent in Fig. 7

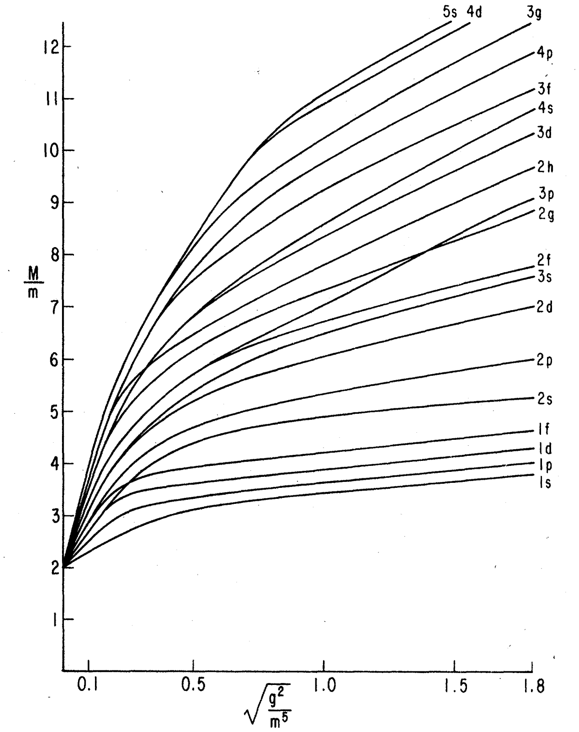

FIG. 7. The lowest several energy levels as a function of the spring constant. Note that the degeneracy of different angular momentum states is broken, and that the energy eigenvalues are no longer evenly spaced in the relativistic domain.

is the fact that the classical degeneracy of the different n and l levels is broken by the relativistic effects, and the well-known even spacing of the levels is no longer observed.

VII. CONCLUSIONS

The extension of well-understood classical potential problems into the realm of relativistic quantum field theory is by no means a completely well-defined process. In the case of the simple harmonic oscillator, we have proposed a method that preserves the δ-function-like infrared behavior of the classical problem without extending too much symmetry into the timelike dimension. The result is an equation that can be separated and thus reduced to a one-dimensional equation. At any point of our calculation, it is possible to take the weak coupling limit to recover the simple harmonic oscillator. Doing this gives a simple and well-defined meaning to the coupling constant in the fully relativistic equation by relating it to the classical spring constant.

While it was not possible to solve the separated Bethe-Salpeter equation analytically, certain limits are tractable, and in any case the simple differential equation is easily handled numerically. From such considerations, it is possible to conclude that there are some large differences in the strongly coupled oscillator and the weakly coupled one. In both cases, the asymptotic behavior is different from the result of the Schrödinger equation for the simple harmonic oscillator. When the coupling is strong, most of the nice symmetry of the simple oscillator is lost and the energy levels are no longer evenly spaced.

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