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Feynman propagator in curved spacetime: A momentum-space representation

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We obtain a momentum-space representation of the Feynman propagator $G(x,x')$ for scalar and spin-1/2 fields propagating in arbitrary curved spacetimes. The construction uses Riemann normal coordinates with origin at the point x' and is therefore only valid for points x lying in a normal neighborhood of x'. We show that the resulting momentum-space representation is equivalent to the DeWitt-Schwinger proper-time representation. Our momentum-space representation permits one to apply momentum-space techniques used in Minkowski space to arbitrary curved spacetimes. The usefulness of this representation in discussing the renormalizability of interacting field theories in curved spacetime is illustrated by an explicit renormalization, to second order in the coupling constant, of a quartically self-interacting scalar field theory in an arbitrary spacetime.

I. INTRODUCTION

In Minkowski space there are well-developed momentum-space methods for dealing with ultraviolet divergences which arise in the theory of interaction quantized fields. In a general curved spacetime, the homogeneity required for the existence of a global momentum-space representation is lacking. Consequently, it would appear that one must forego the convenience of momentum-space techniques and work directly in configuration space, except perhaps for spacetimes which are sufficiently homogeneous, or which can be treated as weak perturbations of homogeneous geometries. Nevertheless, one feels intuitively that because ultraviolet divergences involve only the short-wavelength behavior of the theory, and because, according to the principle of equivalence, curved spacetime can be viewed as approximately flat in a sufficiently small region, one ought to be able to apply Minkowski-space techniques to ultraviolet divergences of interacting fields in curved spacetime. On the other hand, one cannot expect the problem of renormalizing such divergences to be reducible to the corresponding problem in Minkowski space, as additional divergent terms involving the Riemann tensor are present.

In this paper, we introduce a local momentum-space representation near any given point in a general curved spacetime, and show how this enables one to apply standard techniques to the renormalization of ultraviolet divergences of interacting quantized fields. The transition to local momentum space is carried out by using Riemann normal coordinates[1,2] with origin at the point under consideration. An event is specified by the normal coordinates $y^\mu = \tau\xi^\mu$, where ξ^μ is the unit tangent vector (at the origin) to the geodesic which joins the origin to that event and τ is the arc length along that geodesic. Riemann normal coordinates are valid in normal neighborhoods of the origin in which the geodesics from the origin do not intersect. Because the divergences under consideration involve arbitrarily short wavelengths, they should be adequately described in terms of local normal coordinates. At the origin of those coordinates, the metric is Minkowskian and the first derivatives of the metric vanish, so that the description of the local dynamics is as nearly like that of special relativity as is possible in a general curved spacetime.

In Sec. II, we consider a scalar field in an arbitrary curved spacetime and give the local momentum-space representation of the Feynman propagator. In Sec. III, we derive the well-known proper-time representation of the propagator from the above momentum-space representation. In Sec. IV, we use the momentum-space representation of the propagator to show that the theory of a quartically self-interacting scalar field in an arbitrary curved spacetime is renormalizable to second order in the coupling constant. This generalizes earlier results obtained using momentum-space representations in conformally flat spacetimes.[3-5] In Sec. V, we give the local momentum-space representation of the Feynman propagator for a spin-$\frac{1}{2}$ field in curved spacetime. A momentum-space representation based on the normal modes used in adiabatic regularization is discussed in the Appendix.

II. MOMENTUM-SPACE REPRESENTATION OF THE FEYNMAN PROPAGATOR OF A SCALAR FIELD

Consider a scalar field ϕ satisfying the field equation

$$-\nabla^\mu \nabla_\mu \phi + (m^2 + \xi R)\phi = 0 , \tag{2.1}$$

where ∇_μ denotes the covariant derivative, R is the scalar curvature of the spacetime, ξ is an arbitrary real number, and m is the mass. [We use units with $\hbar = c = 1$, metric signature $(-+++)$, and the conventions of Ref. 6.] The Feynman Green's function is a solution of

$$(-\nabla^\mu \nabla_\mu + m^2 + \xi R) G(x,x') = g^{-1/2}(x) \delta(x - x') , \tag{2.2}$$

where $g(x) = |\det g_{\mu\nu}(x)|$. Introducing Riemann normal coordinates y^μ with origin at the point x', one has[2,7]

$$\begin{aligned} g_{\mu\nu} = \eta_{\mu\nu} &- \tfrac{1}{3} R_{\mu\alpha\nu\beta} y^\alpha y^\beta - \tfrac{1}{6} R_{\mu\alpha\nu\beta;\gamma} y^\alpha y^\beta y^\gamma \\ &+ (-\tfrac{1}{20} R_{\mu\alpha\nu\beta;\gamma\delta} + \tfrac{2}{45} R_{\alpha\mu\beta\lambda} R^\lambda{}_{\gamma\nu\delta}) y^\alpha y^\beta y^\gamma y^\delta \\ &+ \cdots , \end{aligned} \tag{2.3}$$

$$\begin{aligned} g = 1 &- \tfrac{1}{3} R_{\alpha\beta} y^\alpha y^\beta - \tfrac{1}{6} R_{\alpha\beta;\gamma} y^\alpha y^\beta y^\gamma \\ &+ (\tfrac{1}{18} R_{\alpha\beta} R_{\gamma\delta} - \tfrac{1}{90} R_{\lambda\alpha\beta}{}^\kappa R_{\lambda\gamma\delta\kappa} - \tfrac{1}{20} R_{\alpha\beta;\gamma\delta}) \\ &\times y^\alpha y^\beta y^\gamma y^\delta + \cdots , \end{aligned} \tag{2.4}$$

where the coefficients are evaluated at $y = 0$ and $\eta_{\mu\nu}$ denotes the Minkowski metric. All indices on the right-hand side of (2.3) and (2.4) are raised and lowered with the Minkowski metric $\eta_{\mu\nu}$. Working in normal coordinates about x' and defining $\overline{G}(x,x')$ by

$$\begin{aligned} G(x,x') &= g^{-1/4}(x) \overline{G}(x,x') g^{-1/4}(x') \\ &= g^{-1/4}(x) \overline{G}(x,x') \end{aligned} \tag{2.5}$$

one finds after some calculation that $\overline{G}(x,x')$ satisfies the equation

$$\begin{aligned} &\eta^{\mu\nu} \partial_\mu \partial_\nu \overline{G} - [m^2 + (\xi - \tfrac{1}{6}) R] \overline{G} - \tfrac{1}{3} R_\alpha{}^\nu y^\alpha \partial_\nu \overline{G} + \tfrac{1}{3} R^\mu{}_\alpha{}^\nu{}_\beta y^\alpha y^\beta \partial_\mu \partial_\nu \overline{G} \\ &\quad - (\xi - \tfrac{1}{6}) R_{;\alpha} y^\alpha \overline{G} + (-\tfrac{1}{3} R_\alpha{}^\nu{}_{;\beta} + \tfrac{1}{6} R_{\alpha\beta}{}^{;\nu}) y^\alpha y^\beta \partial_\nu \overline{G} + \tfrac{1}{6} R^\mu{}_\alpha{}^\nu{}_{\beta;\gamma} y^\alpha y^\beta y^\gamma \partial_\mu \partial_\nu \overline{G} \\ &\quad - \tfrac{1}{2} (\xi - \tfrac{1}{6}) R_{;\alpha\beta} y^\alpha y^\beta \overline{G} + (-\tfrac{1}{30} R_\alpha{}^\lambda R_{\lambda\beta} + \tfrac{1}{60} R^\kappa{}_\alpha{}^\lambda{}_\beta R_{\kappa\lambda} + \tfrac{1}{60} R^{\lambda\mu\kappa}{}_\alpha R_{\lambda\mu\kappa\beta} - \tfrac{1}{120} R_{;\alpha\beta} + \tfrac{1}{40} \Box R_{\alpha\beta}) y^\alpha y^\beta \overline{G} \\ &\quad + (-\tfrac{3}{20} R^\nu{}_{\alpha;\beta\gamma} + \tfrac{1}{10} R_{\alpha\beta}{}^{;\nu}{}_\gamma - \tfrac{1}{60} R^\kappa{}_\alpha{}^\nu{}_\beta R_{\kappa\gamma} + \tfrac{1}{15} R^\kappa{}_{\alpha\lambda\beta} R_\kappa{}^\nu{}_\gamma{}^\lambda) y^\alpha y^\beta y^\gamma \partial_\nu \overline{G} \\ &\quad + (\tfrac{1}{20} R^\mu{}_\alpha{}^\nu{}_{\beta;\gamma\delta} + \tfrac{1}{15} R^\mu{}_{\alpha\lambda\beta} R^\lambda{}_\gamma{}^\nu{}_\delta) y^\alpha y^\beta y^\gamma y^\delta \partial_\mu \partial_\nu \overline{G} = -\delta(y) , \end{aligned} \tag{2.6}$$

where y^α are the coordinates of the point x and $\partial_\mu = \partial / \partial y^\mu$. We have retained only terms with coefficients involving four derivatives of the metric or fewer. This will prove sufficient for dealing with all ultraviolet divergences that arise in the course of renormalization. The above equation is valid in n dimensions.

In normal coordinates with origin at x', $\overline{G}(x,x')$ is a function of y (and x'). We introduce the momentum space associated with the point x' $(y = 0)$ by making the n-dimensional Fourier transformation

$$\overline{G}(x,x') = \int \frac{d^n k}{(2\pi)^n} e^{iky} \overline{G}(k) , \tag{2.7}$$

where $ky \equiv k_\alpha y^\alpha \equiv \eta^{\alpha\beta} k_\alpha y_\beta$. Note that $\overline{G}(k)$ is a function of x': $\overline{G}(k) = \overline{G}(k; x')$. Strictly, $\overline{G}(k)$ is defined by the inverse of Eq. (2.7), which requires discussion of the behavior of $\overline{G}(x,x') \equiv \overline{G}(y; x')$ for all y. As we are only interested here in the singularity structure of $G(x,x')$ as x approaches x', we may take $\overline{G}(k)$ to be defined as the Fourier transform of a function which coincides with a solution of Eq. (2.6) in an open set containing x' and which also has compact support in a normal neighborhood of x'. This procedure does not affect the singularity structure as x approaches x', so that the Fourier transform $\overline{G}(k)$ so defined will be sufficient for finding that singularity structure, although not for finding the global behavior of $G(x,x')$.

Before writing down the equation satisfied by $\overline{G}(k)$ [the Fourier transform of Eq. (2.6)] it will be convenient to indicate how the solution $\overline{G}(k)$ is to be obtained. An iterative procedure will be used which is obtained by writing

$$\overline{G}(k) = \overline{G}_0(k) + \overline{G}_1(k) + \overline{G}_2(k) + \cdots \tag{2.8}$$

and

$$\overline{G}_i(x,x') = \int \frac{d^n k}{(2\pi)^n} e^{iky} \overline{G}_i(k) \quad (i = 0, 1, 2, \ldots) , \tag{2.9}$$

where $\overline{G}_i(k)$ has a geometrical coefficient involving i derivatives of the metric. On dimensional grounds, it follows that $\overline{G}_i(k)$ is of order $k^{-(2+i)}$ so that (2.8) is an asymptotic expansion of $\overline{G}(k)$ in large k. It is not difficult to see that the lowest-order solution is the Minkowski-space solution

$$\overline{G}_0(k) = (k^2 + m^2)^{-1} \tag{2.10}$$

and that

$$\overline{G}_1(k) = 0 . \tag{2.11}$$

Then $\overline{G}_2(x,x')$ satisfies

$$\begin{aligned} \eta^{\mu\nu} \partial_\mu \partial_\nu \overline{G}_2 - m^2 \overline{G}_2 - (\xi - \tfrac{1}{6}) R \overline{G}_0 - \tfrac{1}{3} R_\alpha{}^\nu y^\alpha \partial_\nu \overline{G}_0 \\ + \tfrac{1}{3} R^\mu{}_\alpha{}^\nu{}_\beta y^\alpha y^\beta \partial_\mu \partial_\nu \overline{G}_0 = 0 . \end{aligned} \tag{2.12}$$

But now $\overline{G}_0(x,x')$ is Lorentz invariant so that it is a function only of $\eta_{\alpha\beta} y^\alpha y^\beta \equiv y^\alpha y_\alpha$. For such a function,

$$-\tfrac{1}{3}R_{\alpha}{}^{\nu}y^{\alpha}\partial_{\nu}\overline{G}_0+\tfrac{1}{3}R^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta}y^{\alpha}y^{\beta}\partial_{\mu}\partial_{\nu}\overline{G}_0\equiv 0 \qquad (2.13)$$

so that (2.12) becomes

$$\eta^{\mu\nu}\partial_{\mu}\partial_{\nu}\overline{G}_2-m^2\overline{G}_2-(\xi-\tfrac{1}{6})R\overline{G}_0=0\,. \qquad (2.14)$$

In momentum space this is

$$(k^2+m^2)\overline{G}_2(k)+(\xi-\tfrac{1}{6})R\overline{G}_0(k)=0\,. \qquad (2.15)$$

Hence

$$\overline{G}_2(k)=(\tfrac{1}{6}-\xi)R/(k^2+m^2)^2\,. \qquad (2.16)$$

The Lorentz invariance of $\overline{G}_0(x,x')$ leads to further simplifications of (2.6) when $\overline{G}_3$ and $\overline{G}_4$ are calculated, namely

$$(-\tfrac{1}{3}R_{\alpha}{}^{\nu}{}_{;\beta}+\tfrac{1}{6}R_{\alpha\beta}{}^{;\nu})y^{\alpha}y^{\beta}\partial_{\nu}\overline{G}_0+\tfrac{1}{6}R^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta;\gamma}y^{\alpha}y^{\beta}y^{\gamma}\partial_{\mu}\partial_{\nu}\overline{G}_0\equiv 0\,, \qquad (2.17)$$

$$(-\tfrac{3}{20}R^{\nu}{}_{\alpha;\beta\gamma}+\tfrac{1}{10}R_{\alpha\beta}{}^{;\nu}{}_{\gamma}-\tfrac{1}{60}R^{\kappa}{}_{\alpha}{}^{\nu}{}_{\beta}R_{\kappa\gamma}+\tfrac{1}{15}R^{\kappa}{}_{\alpha\lambda\beta}R_{\kappa}{}^{\nu}{}_{\gamma}{}^{\lambda})y^{\alpha}y^{\beta}y^{\gamma}\partial_{\nu}\overline{G}_0+(\tfrac{1}{20}R^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta;\gamma\delta}+\tfrac{1}{15}R^{\mu}{}_{\alpha\lambda\beta}R^{\lambda}{}_{\gamma}{}^{\nu}{}_{\delta})y^{\alpha}y^{\beta}y^{\gamma}y^{\delta}\partial_{\mu}\partial_{\nu}\overline{G}_0\equiv 0\,. \qquad (2.18)$$

In addition, (2.13) continues to hold with $\overline{G}_0$ replaced by $\overline{G}_2$ since $\overline{G}_2(x,x')$ is also Lorentz invariant and hence a function only of $y_{\alpha}y^{\alpha}$. Thus (2.6) simplifies to the following equation for $\overline{G}$ to fourth order in derivatives of the metric:

$$\eta^{\mu\nu}\partial_{\mu}\partial_{\nu}\overline{G}-[m^2+(\xi-\tfrac{1}{6})R]\overline{G}-(\xi-\tfrac{1}{6})R_{;\alpha}y^{\alpha}\overline{G}-\tfrac{1}{2}(\xi-\tfrac{1}{6})R_{;\alpha\beta}y^{\alpha}y^{\beta}\overline{G}$$
$$+(-\tfrac{1}{30}R_{\alpha}{}^{\lambda}R_{\lambda\beta}+\tfrac{1}{60}R^{\kappa}{}_{\alpha}{}^{\lambda}{}_{\beta}R_{\kappa\lambda}+\tfrac{1}{60}R^{\lambda\mu\kappa}{}_{\alpha}R_{\lambda\mu\kappa\beta}-\tfrac{1}{120}R_{;\alpha\beta}+\tfrac{1}{40}\Box R_{\alpha\beta})y^{\alpha}y^{\beta}\overline{G}=-\delta(y)\,. \qquad (2.19)$$

Converting to momentum space using (2.7),

$$[k^2+m^2+(\xi-\tfrac{1}{6})R]\overline{G}(k)+i(\xi-\tfrac{1}{6})R_{;\alpha}\partial^{\alpha}\overline{G}(k)$$
$$+[-\tfrac{1}{2}(\xi-\tfrac{1}{6})R_{;\alpha\beta}-\tfrac{1}{120}R_{;\alpha\beta}+\tfrac{1}{40}\Box R_{\alpha\beta}-\tfrac{1}{30}R_{\alpha}{}^{\lambda}R_{\lambda\beta}+\tfrac{1}{60}R^{\kappa}{}_{\alpha}{}^{\lambda}{}_{\beta}R_{\kappa\lambda}+\tfrac{1}{60}R^{\lambda\mu\kappa}{}_{\alpha}R_{\lambda\mu\kappa\beta}]\partial^{\alpha}\partial^{\beta}\overline{G}(k)=1\,, \qquad (2.20)$$

where

$$\partial^{\alpha}\overline{G}(k)=\partial\overline{G}/\partial k_{\alpha}\,.$$

The complete solution up to fourth order in derivatives of the metric is

$$\begin{aligned}\overline{G}(k)=&(k^2+m^2)^{-1}+(\tfrac{1}{6}-\xi)R(k^2+m^2)^{-2}\\&+i(\tfrac{1}{6}-\xi)R_{;\alpha}(k^2+m^2)^{-1}\partial^{\alpha}(k^2+m^2)^{-1}\\&+(\tfrac{1}{6}-\xi)^2R^2(k^2+m^2)^{-3}\\&+a_{\alpha\beta}(k^2+m^2)^{-1}\partial^{\alpha}\partial^{\beta}(k^2+m^2)^{-1}\,,\end{aligned} \qquad (2.21)$$

where

$$\begin{aligned}a_{\alpha\beta}=&\tfrac{1}{2}(\xi-\tfrac{1}{6})R_{;\alpha\beta}+\tfrac{1}{120}R_{;\alpha\beta}-\tfrac{1}{40}\Box R_{\alpha\beta}\\&+\tfrac{1}{30}R_{\alpha}{}^{\lambda}R_{\lambda\beta}-\tfrac{1}{60}R^{\kappa}{}_{\alpha}{}^{\lambda}{}_{\beta}R_{\kappa\lambda}-\tfrac{1}{60}R^{\lambda\mu\kappa}{}_{\alpha}R_{\lambda\mu\kappa\beta}\,.\end{aligned} \qquad (2.22)$$

The Feynman propagator is obtained by replacing m^2 by $m^2-i\epsilon$, with ϵ an arbitrary small positive real quantity to be taken to zero at the end of any calculation, and carrying out the momentum integrations along the real axis. Alternatively, the propagator can be evaluated in a Euclidean spacetime for which $\eta_{\alpha\beta}=\delta_{\alpha\beta}$: The derivation given here is equally valid for this case.

Equations (2.7) and (2.21) give the momentum-space expression for $\overline{G}(x,x')$ which is related to $G(x,x')$ by (2.5). The Fourier transform of $G(x,x')$ itself can readily be obtained by performing integrations by parts to absorb the factor $g^{-1/4}(x)$ expressed as a polynomial in y^{α} [see Eq. (4.20) below]. The Fourier transform of $\overline{G}(x,x')$, Eq. (2.21), is sufficient for treating divergences.

III. RELATION TO PROPER-TIME FORMALISM

The well-known proper-time representation of the Feynman propagator in curved spacetime[8,9] can be derived from the momentum-space representation obtained in Sec. III. This approach bypasses the introduction of the formal mathematical objects used in the conventional derivation. Using

$$(k^2+m^2)^{-1}\partial^{\alpha}(k^2+m^2)^{-1}\equiv\tfrac{1}{2}\partial^{\alpha}(k^2+m^2)^{-2}\,, \qquad (3.1)$$

$$(k^2+m^2)^{-1}\partial^{\alpha}\partial^{\beta}(k^2+m^2)^{-1}\equiv\tfrac{1}{3}\partial^{\alpha}\partial^{\beta}(k^2+m^2)^{-2}-\tfrac{2}{3}\eta^{\alpha\beta}(k^2+m^2)^{-3}\,, \qquad (3.2)$$

(2.21) may be rewritten

$$\begin{aligned}\overline{G}(k)=&(k^2+m^2)^{-1}+(\tfrac{1}{6}-\xi)R(k^2+m^2)^{-2}\\&+\tfrac{1}{2}i(\tfrac{1}{6}-\xi)R_{;\alpha}\partial^{\alpha}(k^2+m^2)^{-2}\\&+\tfrac{1}{3}a_{\alpha\beta}\partial^{\alpha}\partial^{\beta}(k^2+m^2)^{-2}\\&+[(\tfrac{1}{6}-\xi)^2R^2-\tfrac{2}{3}a^{\lambda}{}_{\lambda}](k^2+m^2)^{-3}\,.\end{aligned} \qquad (3.3)$$

Substituting (3.3) in (2.7) and integrating by parts leads to

$$\overline{G}(x,x')=\int\frac{d^nk}{(2\pi)^n}e^{iky}\left[1+f_1(x,x')\left(-\frac{\partial}{\partial m^2}\right)+f_2(x,x')\left(\frac{\partial}{\partial m^2}\right)^2\right]\frac{1}{k^2+m^2}\,, \qquad (3.4)$$

where, to fourth order in derivatives of the metric,

$$f_1(x,x') = (\tfrac{1}{6} - \xi)R + \tfrac{1}{2}(\tfrac{1}{6} - \xi)R_{;\alpha}y^\alpha - \tfrac{1}{3}a_{\alpha\beta}y^\alpha y^\beta \,, \tag{3.5}$$

$$f_2(x,x') = \tfrac{1}{2}(\tfrac{1}{6} - \xi)^2 R^2 - \tfrac{1}{3}a^\lambda{}_\lambda \,. \tag{3.6}$$

Now put

$$(k^2+m^2)^{-1} = \int_0^\infty i ds \exp[-is(k^2+m^2)] \,, \tag{3.7}$$

where the usual Feynman boundary conditions are obtained on making the replacement $m^2 \to m^2 - i\epsilon$, and define

$$F(x,x';is) = 1 + f_1(x,x')is + f_2(x,x')(is)^2 \,. \tag{3.8}$$

Making use of

$$\int \frac{d^n k}{(2\pi)^n} \exp[-is(k^2+m^2) + iky] = \frac{i}{(4\pi)^{n/2}}(is)^{-n/2}\exp\left(-im^2 s - \frac{\sigma}{2is}\right), \tag{3.9}$$

where $\sigma(x,x') = \frac{1}{2}\tau^2 = \frac{1}{2}y_\alpha y^\alpha$ is half the square of the geodesic distance between x and x', we obtain

$$\overline{G}(x,x') = \frac{i}{(4\pi)^{n/2}} \times \int_0^\infty \frac{ids}{(is)^{n/2}} \exp\left(-im^2 s - \frac{\sigma}{2is}\right) F(x,x';is) \,, \tag{3.10}$$

where (3.4) has been used. The propagator $G(x,x')$ is related to $\overline{G}(x,x')$ by (2.5). The usual expression for the proper-time representation of $G(x,x')$ in n dimensions[10] is now obtained by noticing that the van Vleck determinant

$$\Delta(x,x') = -g^{-1/2}(x)\det[-\partial_\mu \partial_{\nu'}\sigma(x,x')]g^{-1/2}(x') \tag{3.11}$$

reduces in normal coordinates about x' to $g^{-1/2}(x)$.[7] Thus we obtain

$$G(x,x') = \frac{i\Delta^{1/2}(x,x')}{(4\pi)^{n/2}} \times \int_0^\infty \frac{ids}{(is)^{n/2}} \exp\left(-im^2 s - \frac{\sigma}{2is}\right) F(x,x';is) \tag{3.12}$$

with $F(x,x';is)$ given by Eqs. (3.4)–(3.6). This expression is in generally covariant form if, in Eq. (3.5), y^α is written as $\sigma^{;\alpha}$. It agrees with the work of Ref. 9. The equivalence of the momentum-space and proper-time representations means that they both give the same renormalization of the free-field stress tensor and both lead to the same conformal trace anomaly.

IV. APPLICATION TO SELF-INTERACTING SCALAR FIELD THEORY

This section will demonstrate the usefulness of the momentum-space representation in studying the renormalizability of $\lambda\phi^4$ field theory in curved spacetime. It was shown in Refs. 3 and 4 that all physical processes which are first or second order in λ (including vacuum-to-vacuum processes, which are nontrivial in curved spacetime) can be made finite by renormalization of the physical parameters of the theory appearing in the scalar field action and the Einstein gravitational action. A different treatment has also recently been given by Birrell.[5] The derivation of these results made use of momentum-space representations valid only in conformally flat spacetimes. However, the results are valid for spacetimes having arbitrary metric since no new divergences appear if the spacetime ceases to be conformally flat. This will be demonstrated explicitly in this section using the general momentum-space representation of Sec. III.

The Lagrangian density in the interaction picture is

$$\mathcal{L} = \tfrac{1}{2}\sqrt{g}\,[g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - (Z_2 m^2 + Z_3\xi R)\phi^2] - \tfrac{1}{4}\sqrt{g}\,\lambda\mu^{4-n}Z_4\phi^4 \,, \tag{4.1}$$

where μ is the unit of mass required to keep the dimensions of $\mathcal{L}$ consistent for all n, m is the renormalized mass, ξ and λ are dimensionless renormalized coupling constants, and ϕ is the bare field related to the renormalized field ϕ_R by

$$\phi = Z_1^{1/2}\phi_R \,. \tag{4.2}$$

The renormalization constants Z_i are power series in λ (and hence dimensionless):

$$Z_i = 1 + \sum_{r=1}^\infty Z_i^{(r)}\lambda^r \,. \tag{4.3}$$

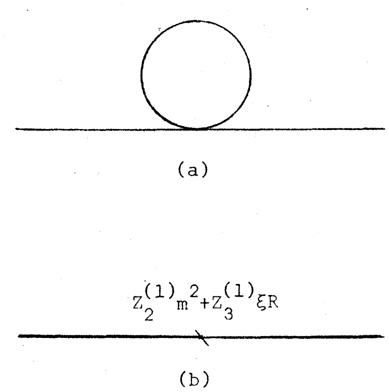

FIG. 1. First-order corrections to the two-point function.

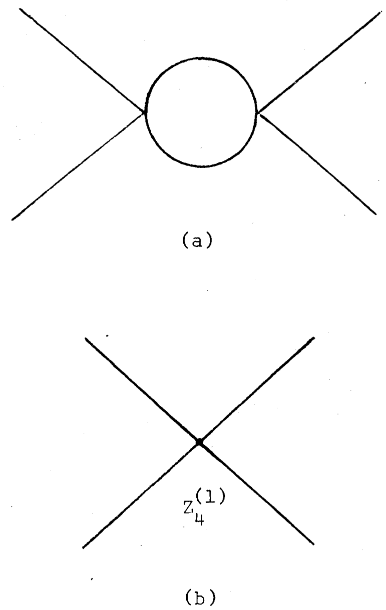

FIG. 2. Second-order corrections to the four-point function.

To second order in λ, the interaction Hamiltonian density in the interaction picture is

$$\begin{aligned}\mathcal{H}' = &\tfrac{1}{2}\lambda\sqrt{g}\,[Z_2^{(1)}m^2+Z_3^{(1)}\xi R]\phi^2+\tfrac{1}{4}\lambda\sqrt{g}\,\mu^{4-n}\phi^4\\ &+\tfrac{1}{2}\lambda^2\sqrt{g}\,[Z_2^{(2)}m^2+Z_3^{(2)}\xi R]\phi^2\\ &+\tfrac{1}{4}\lambda^2\sqrt{g}\,\mu^{4-n}Z_4^{(1)}\phi^4 .\end{aligned} \tag{4.4}$$

In Refs. 3 and 4, the interaction Hamiltonian was normal ordered and hence had a slightly different form from (4.4). Whether the Hamiltonian is normal ordered or not does not affect the final renormalized theory. In Refs. 3 and 11, it was shown that the renormalizability of a field theory in curved spacetime can be investigated by looking at S-matrix elements between interaction picture states which are conveniently chosen to be physical particle states at early times (before the self-interaction is switched on). Once these S-matrix elements are renormalized, finite particle creation amplitudes are obtained by performing Bogolubov transformations to late-time physical particle states. Power-counting arguments show that the divergent Feynman diagrams are those which involve zero, two, or four external lines. To second order in λ, the diagrams having two or four external lines which need to be considered are shown in Figures 1–3. It does not make much difference to the discussion of renormalization whether the external lines are wave functions (as in S-matrix elements) or free-field Feynman propagators

$$G(x,y)\equiv i\langle 0|T(\phi(x)\phi(y))|0\rangle ,$$

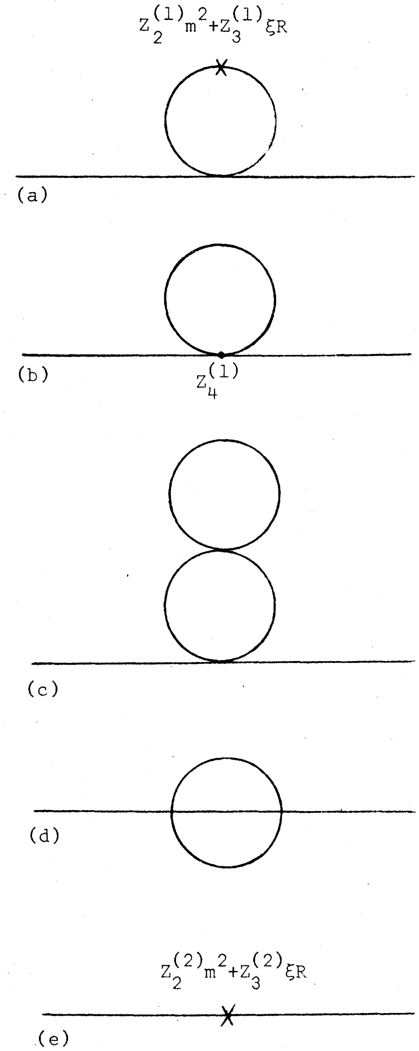

FIG. 3. Second-order corrections to the two-point function.

where $|0\rangle$ denotes the interaction-picture vacuum state (as in corrections to two- or four-point functions). Taking the external lines to be propagators gives the following expressions for the mathematical equivalent of Figs. 1(a)–3(e), respectively:

$$3i\lambda\mu^{4-n}\int G(y,x')G(x',x')G(x',z)[g(x')]^{1/2}d^nx' , \tag{4.5}$$

$$-\lambda\int [Z_2^{(1)}m^2+Z_3^{(1)}\xi R(x')]G(y,x')G(x',z)\times[g(x')]^{1/2}d^nx' , \tag{4.6}$$

$$54i\lambda^2\mu^{8-2n}\int G(y,x)G(y',x)G^2(x,x')G(x',z)G(x',z')\times[g(x)g(x')]^{1/2}d^nx\,d^nx' , \tag{4.7}$$

$$6\lambda^2\mu^{4-n}Z_4^{(1)}\int G(y,x')G(y',x')G(x',z)G(x',z')\times[g(x')]^{1/2}d^nx' , \quad (4.8)$$

$$-3i\lambda^2\mu^{4-n}\int[Z_2^{(1)}m^2+Z_3^{(1)}\xi R(x')]G(y,x)G^2(x,x')G(x,z)\times[g(x)g(x')]^{1/2}d^nxd^nx' , \quad (4.9)$$

$$3i\lambda^2\mu^{4-n}Z_4^{(1)}\int G(y,x')G(x',x')G(x',z)[g(x')]^{1/2}d^nx' , \quad (4.10)$$

$$-9\lambda^2\mu^{8-2n}\int G(y,x)G^2(x,x')G(x',x')G(x,z)\times[g(x)g(x')]^{1/2}d^nxd^nx' , \quad (4.11)$$

$$-6\lambda^2\mu^{8-2n}\int G(y,x)G^3(x,x')G(x',z)\times[g(x)g(x')]^{1/2}d^nxd^nx' , \quad (4.12)$$

$$-\lambda^2\int[Z_2^{(2)}m^2+Z_3^{(2)}\xi R(x')]G(y,x')G(x',z)\times[g(x')]^{1/2}d^nx' . \quad (4.13)$$

Using an approach similar to the treatments given in Refs. 4 and 5, we will show that the contributions of Eqs. (4.5)–(4.13) to physical amplitudes can be made finite by suitable choices of the renormalization constants. Consider first (4.5) and (4.6). The divergence in (4.5) arises from the factor $G(x',x')$. Define the finite part of the propagator, $G_R(x,x')$, by

$$G(x,x')=g^{-1/4}(x)\int\frac{d^nk}{(2\pi)^n}e^{iky}\left[\frac{1}{k^2+m^2}+\frac{(\frac{1}{6}-\xi)R}{(k^2+m^2)^2}\right]+G_R(x,x') . \quad (4.14)$$

This definition of $G_R(x,x')$ is not quite the same as the definition used in Ref. 4. In Eq. (4.14),

$$G_R(x,x')=g^{-1/4}(x)\overline{G}_R(x,x') ,$$

where $\overline{G}_R(x,x')$ has a Fourier transform $\overline{G}_R(k)$ which is of order k^{-5}. The leading terms in an asymptotic expansion of $\overline{G}_R(k)$ in large k were calculated in Sec. II. Setting $x=x'$ (or $y=0$) in (4.14) and evaluating the integral using Eq. (A1) of Ref. 12 gives

$$G(x',x')=\left(\frac{m^2}{4\pi}\right)^{n/2-2}\left[\frac{i[m^2+(\xi-\frac{1}{6})R]}{8\pi^2(n-4)}+\frac{im^2(\gamma-1)+i(\xi-\frac{1}{6})R\gamma}{16\pi^2}\right]+G_R(x',x') , \quad (4.15)$$

where terms of order $(n-4)$ have been omitted. Define $G_R(x')$ by

$$G_R(x')=G_R(x',x')+\frac{i\mu^{n-4}[m^2+(\xi-\frac{1}{6})R]}{16\pi^2}\left[\ln\left(\frac{m^2}{4\pi\mu^2}\right)+\gamma\right]-\frac{im^2\mu^{n-4}}{16\pi^2} . \quad (4.16)$$

Then (4.15) may be written more simply as

$$G(x',x')=\frac{i\mu^{n-4}[m^2+(\xi-\frac{1}{6})R]}{8\pi^2(n-4)}+G_R(x')+o(n-4) . \quad (4.17)$$

It now follows that the sum of (4.5) and (4.6) is finite provided that

$$Z_2^{(1)}=-\frac{3}{8\pi^2(n-4)} \quad (4.18)$$

and

$$\xi Z_3^{(1)}=(\xi-\tfrac{1}{6})Z_2^{(1)} . \quad (4.19)$$

Consider now (4.7) and (4.8). The divergence in (4.7) arises when $G^2(x,x')$ is integrated over x in a neighborhood of x'. Thus we can find the divergences in $G^2(x,x')$ by using the momentum-space representation (4.14). Using (2.4) and integrating by parts, (4.14) may be rewritten

$$G(x,x')=\int\frac{d^nk}{(2\pi)^n}e^{iky}\left[\frac{1}{k^2+m^2}+\frac{(\frac{1}{3}-\xi)R}{(k^2+m^2)^2}-\frac{2R_{\alpha\beta}k^\alpha k^\beta}{3(k^2+m^2)^3}+o(k^{-5})\right] . \quad (4.20)$$

It is not difficult to see (for example, by power counting) that the only divergent contribution to $G^2(x,x')$ is

$$G^2(x,x')\approx\int\int\frac{d^nkd^np}{(2\pi)^{2n}}\frac{e^{i(k+p)y}}{(k^2+m^2)(p^2+m^2)} . \quad (4.21)$$

Setting $q=k+p$ leads to

$$G^2(x,x') \approx \int \frac{d^nq}{(2\pi)^{2n}} e^{iqy} \int \frac{d^np}{[(p-q)^2+m^2](p^2+m^2)} \tag{4.22}$$

$$= \int \frac{d^nq}{(2\pi)^{2n}} e^{iqy} \int_0^1 d\alpha \int \frac{d^np}{[p^2-2pq\alpha+q^2\alpha+m^2]^2} \tag{4.23}$$

$$= i\pi^{n/2}\Gamma(2-n/2) \int \frac{d^nq}{(2\pi)^{2n}} e^{iqy} \int_0^1 \frac{d\alpha}{[m^2+q^2\alpha(1+\alpha)]^{2-n/2}}, \tag{4.24}$$

where Eq. (A1) of Ref. 12 has been used. Thus $G^2(x,x')$ has a simple pole at $n=4$ which may be obtained by expanding about $n=4$. The result is

$$G^2(x,x') = -\frac{i\mu^{n-4}}{8\pi^2(n-4)}\delta(x-x'), \tag{4.25}$$

where we have written $\delta(x-x')$ instead of $\delta(y)$. The factor μ^{n-4} arises when the α integral is expanded about $n=4$. Notice that this δ function can be taken to be the covariant δ function $g^{-1/2}(x')\delta(x-x')$ since $g(x')\equiv 1$ in normal coordinates at x'. Thus Eq. (4.25) is a covariant statement. Substituting (4.25) in (4.7) leads to the requirement

$$Z_4^{(1)} = -\frac{9}{8\pi^2(n-4)}, \tag{4.26}$$

which ensures that the sum of (4.7) and (4.8) is finite. Using Eqs. (4.17)–(4.19), (4.25), and (4.26) we find that the sum of Eqs. (4.9)–(4.11) contains the following divergent contribution:

$$\frac{27\lambda^2}{64\pi^4(n-4)^2}\int [m^2+(\xi-\tfrac{1}{6})R(x')]G(y,x')G(x',z)[g(x')]^{1/2}d^nx' - \frac{9i\lambda^2\mu^{4-n}}{4\pi^2(n-4)}\int G(y,x')G_R(x')G(x',z)[g(x')]^{1/2}d^nx'. \tag{4.27}$$

Using (4.16) this may be written in the alternative form

$$\left(\frac{27\lambda^2}{64\pi^4(n-4)^2}+\frac{9\lambda^2}{64\pi^4(n-4)}\left[\ln\left(\frac{m^2}{4\pi\mu^2}\right)+\gamma-1\right]\right)\int [m^2+(\xi-\tfrac{1}{6})R(x')]G(y,x')G(x',z)[g(x')]^{1/2}d^nx'$$
$$+\frac{9\lambda^2(\xi-\frac{1}{6})}{64\pi^4(n-4)}\int G(y,x')R(x')G(x',z)[g(x')]^{1/2}d^nx' - \frac{9i\lambda^2\mu^{4-n}}{4\pi^2(n-4)}\int G(y,x')G_R(x',x')G(x',z)[g(x')]^{1/2}d^nx'. \tag{4.28}$$

In flat spacetime, in which $R(x')=0$ and $G_R(x',x')=0$, (4.28) can be readily compared with the calculations performed by Collins.[12] The main point of Collins's paper is to show that dimensional regularization provides a mass-independent renormalization of $\lambda\phi^4$ field theory, and to demonstrate this he has to show that all terms involving $\ln(m^2/4\pi\mu^2)$ cancel. However, in making the transition from (4.27) to (4.28) it was clear that these terms come from $G_R(x')$ so that Collins's work implies that all terms involving $G_R(x')$ must cancel. It will be seen in what follows that this guarantees renormalizability of $\lambda\phi^4$ field theory in curved spacetime to second order in λ.

To complete our investigation of second-order corrections to the propagator we must evaluate the divergences in (4.12), which requires knowing the behavior of $G^3(x,x')$ for x in a neighborhood of x'. It will be more convenient to investigate $\overline{G}^3(x,x')=g^{3/4}(x)G^3(x,x')$. We will write

$$\overline{G}(x,x') = I_1(x,x')+I_2(x,x')+\overline{G}_R(x,x'), \tag{4.29}$$

where

$$I_1(x,x') = \int \frac{d^nk}{(2\pi)^n}\frac{e^{iky}}{k^2+m^2} \tag{4.30}$$

and

$$I_2(x,x') = (\xi-\tfrac{1}{6})R(x')\frac{\partial I_1}{\partial m^2}(x,x'). \tag{4.31}$$

Then

$$\overline{G}^3(x,x') = I_1^3(x,x')+(\xi-\tfrac{1}{6})R(x')\frac{\partial}{\partial m^2}I_1^3(x,x')+3I_1^2(x,x')\overline{G}_R(x,x')+\cdots. \tag{4.32}$$

The terms omitted from (4.32) such as $I_1(x,x')I_2^2(x,x')$ are all finite as can be seen by power counting. Notice that $I_1^2(x,x')$ is given by (4.25) so that the divergence in the third term of (4.32) is simply

$$-\frac{3i\mu^{n-4}}{8\pi^2(n-4)}G_R(x',x')\delta(x-x') . \tag{4.33}$$

The first term in (4.32) is

$$I_1^3(x,x') = \int \frac{d^nk}{(2\pi)^{3n}} e^{iky} \int\int \frac{d^np\,d^nq}{[(p+q-k)^2+m^2](p^2+m^2)(q^2+m^2)} . \tag{4.34}$$

The double integral in (4.34) has been evaluated by Collins.[12] He obtains

$$\int\int \frac{d^np\,d^nq}{[(p+q-k)^2+m^2](p^2+m^2)(q^2+m^2)} \approx \pi^n\Gamma(3-n)\left[\frac{6m^{2(n-3)}}{n-4} - 3\,m^{2(n-3)} - \frac{k^{2(n-3)}}{2}\right]. \tag{4.35}$$

Thus we find

$$\mu^{8-2n}I_1^3(x,x') \approx \frac{m^2}{256\pi^4}\left[\frac{6}{(n-4)^2} + \frac{6[\ln(m^2/4\pi\mu^2)+\gamma-1]-3}{(n-4)}\right]\delta(y) + \frac{1}{512\pi^4(n-4)}\eta^{\alpha\beta}\partial_\alpha\partial_\beta\delta(y) \tag{4.36}$$

and finally

$$\mu^{8-2n}\bar{G}^3(x,x') \approx \frac{m^2+(\xi-\frac{1}{6})R}{256\pi^4}\left[\frac{6}{(n-4)^2} + \frac{6[\ln(m^2/4\pi\mu^2)+\gamma-1]-3}{(n-4)}\right]\delta(y)$$
$$+\left[\frac{3(\xi-\frac{1}{6})R}{128\pi^4(n-4)} - \frac{3i\mu^{4-n}G_R(x',x')}{8\pi^2(n-4)}\right]\delta(y) + \frac{\eta^{\alpha\beta}\partial_\alpha\partial_\beta\delta(y)}{512\pi^4(n-4)} . \tag{4.37}$$

From this we can obtain a covariant expression for

$$G^3(x,x') = g^{-3/4}(x)\bar{G}^3(x,x')$$

by noticing that, in normal coordinates,

$$\eta^{\alpha\beta}\partial_\alpha\partial_\beta\delta(y) = g^{3/4}\Box\delta(y) + \tfrac{1}{6}R\delta(y) , \tag{4.38}$$

where $\Box \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$, and we can take $\delta(y)$ to be the covariant δ function $\tilde{\delta}(x-x') = g^{-1/2}(x')\delta(x-x')$. Thus we find

$$\mu^{8-2n}G^3(x,x') \approx \frac{m^2+(\xi-\frac{1}{6})R}{256\pi^4}\left[\frac{6}{(n-4)^2} - \frac{3}{(n-4)}\right]\tilde{\delta}(x-x') + \frac{R\tilde{\delta}(x-x')}{3072\pi^4(n-4)}$$
$$-\frac{3i\mu^{4-n}G_R(x')\tilde{\delta}(x-x')}{8\pi^2(n-4)} + \frac{\Box\tilde{\delta}(x-x')}{512\pi^4(n-4)} , \tag{4.39}$$

where (4.16) has been used. Thus the divergences in (4.12) are

$$\frac{3\lambda^2}{256\pi^4(n-4)}G(y,z) + \left(-\frac{9\lambda^2}{64\pi^4(n-4)^2} + \frac{15\lambda^2}{256\pi^4(n-4)}\right)\int [m^2+(\xi-\tfrac{1}{6})R(x')]G(y,x')G(x',z)[g(x')]^{1/2}d^nx'$$
$$-\frac{\lambda^2}{256\pi^4(n-4)}\int G(y,x')R(x')G(x',z)[g(x')]^{1/2}d^nx' + \frac{9i\lambda^2\mu^{4-n}}{4\pi^2(n-4)}\int G(y,x')G_R(x')G(x',z)[g(x')]^{1/2}d^nx' . \tag{4.40}$$

The first term in (4.40) is removed by performing a field renormalization of the form (4.2). This gives

$$Z_1^{(2)} = \frac{3}{256\pi^4(n-4)} . \tag{4.41}$$

When (4.40) is added to (4.27), the terms involving $G_R(x')$ cancel and the remaining terms, which involve only m^2 and R, are removed by making a suitable choice of $Z_2^{(2)}$ and $Z_3^{(2)}$ in (4.13). We find that

$$Z_2^{(2)} = \frac{9}{32\pi^4(n-4)^2} + \frac{15}{256\pi^4(n-4)} , \tag{4.42}$$

$$\xi Z_3^{(2)} = (\xi-\tfrac{1}{6})Z_2^{(2)} - \frac{1}{256\pi^4(n-4)} . \tag{4.43}$$

Expressions (4.42) and (4.43) differ slightly from the corresponding expressions in Ref. 4. This difference arises because the interaction Hamiltonian density of Ref. 4 was normal ordered.

To complete the proof that $\lambda\phi^4$ field theory is renormalizable to second order in λ requires demonstrating that the sum of all vacuum-to-vacuum diagrams is finite after renormalization of coupling constants in the gravitational action. This was done for conformally flat spacetimes in Ref. 4, the most important part being to show that all state-dependent divergences cancel when all second-order vacuum-to-vacuum diagrams are summed. It is not difficult to see that the arguments of Ref. 4 apply equally when the momentum-space representation of Sec. II is used, so that renormalizability holds to second order in λ in any spacetime.

V. MOMENTUM-SPACE REPRESENTATION OF THE FEYNMAN PROPAGATOR OF A SPINOR FIELD

The Dirac equation in curved spacetime is[13-15]

$$[\underline{\gamma}^{\mu}(x)\nabla_{\mu}+m]\psi=0\,, \tag{5.1}$$

where the $\underline{\gamma}^{\mu}(x)$ matrices satisfy

$$\underline{\gamma}^{\mu}(x)\underline{\gamma}^{\nu}(x)+\underline{\gamma}^{\nu}(x)\underline{\gamma}^{\mu}(x)=2g^{\mu\nu}(x)\,, \tag{5.2}$$

m is the mass, and ∇_{μ} is the covariant derivative acting on the four-component spinor field ψ:

$$\nabla_{\mu}\psi=(\partial_{\mu}-\Gamma_{\mu})\psi\,. \tag{5.3}$$

Here the spinorial affine connections $\Gamma_{\mu}(x)$ are matrices defined, to within an additive term proportional to the unit matrix, by the vanishing of the covariant derivative:

$$\nabla_{\mu}\underline{\gamma}_{\nu}=\partial_{\mu}\underline{\gamma}_{\nu}-\Gamma^{\lambda}_{\mu\nu}\underline{\gamma}_{\lambda}-\Gamma_{\mu}\underline{\gamma}_{\nu}+\underline{\gamma}_{\nu}\Gamma_{\mu}=0\,. \tag{5.4}$$

Introducing the vierbein field $b_{\alpha}{}^{\mu}(x)$ such that

$$\eta_{\alpha\beta}=b_{\alpha}{}^{\mu}(x)b_{\beta}{}^{\nu}(x)g_{\mu\nu}(x)\,, \tag{5.5}$$

one finds that the solution of Eq. (5.4) is

$$\Gamma_{\mu}(x)=\tfrac{1}{4}\gamma_{\alpha}\gamma_{\beta}b^{\beta}{}_{\lambda}(x)g^{\lambda\sigma}(x)\nabla_{\mu}b^{\alpha}{}_{\sigma}(x)\,, \tag{5.6}$$

where the additive term proportional to the unit matrix has been set equal to zero and

$$\nabla_{\mu}b^{\alpha}{}_{\sigma}=\partial_{\mu}b^{\alpha}{}_{\sigma}-\Gamma^{\lambda}{}_{\mu\sigma}b^{\alpha}{}_{\lambda}\,. \tag{5.7}$$

In Eq. (5.6) the matrices γ_{α} (without the underlining) refer to the special relativistic Dirac matrices satisfying

$$\gamma_{\alpha}\gamma_{\beta}+\gamma_{\beta}\gamma_{\alpha}=2\eta_{\alpha\beta}\,, \tag{5.8}$$

and related to the $\underline{\gamma}_{\mu}$ by

$$\underline{\gamma}_{\mu}(x)=b^{\alpha}{}_{\mu}(x)\gamma_{\alpha}\,. \tag{5.9}$$

It can be shown[14] that application of the operator $(\underline{\gamma}^{\mu}\nabla_{\nu}-m)$ to Eq. (5.1) yields

$$(g^{\mu\nu}\nabla_{\mu}\nabla_{\nu}+\tfrac{1}{4}R-m^{2})\psi=0\,, \tag{5.10}$$

where ∇_{λ} denotes a covariant derivative (including the spinorial affine connection). Let $\mathcal{G}(x,x')$ denote the 4×4 matrix Green's function of Eq. (5.10):

$$(\nabla^{\mu}\nabla_{\mu}+\tfrac{1}{4}R-m^{2})\mathcal{G}(x,x')=-g^{-1/2}\delta(x-x')1\,. \tag{5.11}$$

Under general coordinate transformations $\mathcal{G}$ is a biscalar, while under Lorentz transformations of the vierbein fields at x and x' it transforms like $\psi(x)\overline{\psi}(x')$,

$$\overline{\psi}=\psi^{\dagger}\gamma \tag{5.12}$$

is the Pauli adjoint of ψ and the matrix γ is defined by

$$\gamma\gamma_{\alpha}\gamma^{-1}=-\gamma_{\alpha}^{T}\,. \tag{5.13}$$

In Minkowski spacetime with $b^{\alpha}{}_{\sigma}$ constant, $\mathcal{G}(x,x')$ reduces to the scalar field Green's function multiplied by the unit matrix. The matrix $S(x,x')$ defined by

$$S(x,x')=-i(\underline{\gamma}^{\mu}\nabla_{\mu}-m)\mathcal{G}(x,x')\gamma^{-1} \tag{5.14}$$

is the Green's function satisfying[15]

$$i\gamma(\gamma^{\mu}\nabla_{\mu}+m)S(x,x')=-g^{-1/2}\delta(x-x')1\,. \tag{5.15}$$

In the proper-time representation one has (Ref. 15, pp. 154, 158)

$$\mathcal{G}(x,x')=\Delta^{1/2}(x,x')\sum_{j=0}^{\infty}A_{j}(x,x')\left(-\frac{\partial}{\partial m^{2}}\right)^{j}\times\int_{0}^{\infty}\frac{ids}{(4\pi is)^{n/2}}\times\exp\left[-i\left(m^{2}s-\frac{\sigma}{2s}\right)\right]\,, \tag{5.16}$$

where $\Delta(x,x')$ is defined in Eq. (3.11), the $A_{n}(x,x')$ are matrices transforming like $\psi(x)\overline{\psi}(x')$, and $\sigma=\frac{1}{2}\tau^{2}$.

Working in Riemann normal coordinates at x', it is straightforward to go from the proper-time representation to the momentum-space representation. As a consequence of Eqs. (2.8), (2.9), (3.8), (3.9), and $\Delta(x,x')=g^{-1/2}(x)$, one can write Eq. (5.16) as

$$\begin{aligned}\mathcal{G}(x,x')&=g^{-1/4}(x)\sum_{j=0}^{\infty}A_{j}(x,x')\left(-\frac{\partial}{\partial m^{2}}\right)\overline{G}_{0}(x,x')\\&=g^{-1/4}(x)\left[1+(A_{1}+A_{1\alpha}y^{\alpha}+A_{1\alpha\beta}y^{\alpha}y^{\beta})\left(-\frac{\partial}{\partial m^{2}}\right)+A_{2}\left(-\frac{\partial}{\partial m^{2}}\right)^{2}\right]\overline{G}_{0}(x,x')\,,\end{aligned} \tag{5.17}$$

where the $A_{j}(x,x')$ have been expanded about the point x' [the $A_{j\alpha\beta}\ldots$ are proportional to derivatives of the A_{j} evaluated at $x=x'$, and we have anticipated the result proved below that $A_{0}(x,x')=1$]. Only those coefficients involving up to four spacetime derivatives (i.e., which may contribute to ultraviolet divergences) have been retained in Eq.

(5.17). Because

$$\overline{G}_0(x,x') = \int \frac{d^nk}{(2\pi)^n} e^{iky}(k^2+m^2)^{-1}, \tag{5.18}$$

one can replace each y^α by $i\partial/\partial k_\alpha \equiv i\partial^\alpha$ applied to $\exp(iky)$ and then integrate by parts, to obtain the result

$$\mathcal{G}(x,x') = g(x)^{-1/4} \int \frac{d^nk}{(2\pi)^n} e^{iky}\mathcal{G}(k), \tag{5.19}$$

where

$$\mathcal{G}(k) = \left[1 + (A_1 + iA_{1\alpha}\partial^\alpha - A_{1\alpha\beta}\partial^\alpha\partial^\beta)\left(-\frac{\partial}{\partial m^2}\right) + A_2\left(-\frac{\partial}{\partial m^2}\right)^2\right](k^2+m^2)^{-1}. \tag{5.20}$$

In the Feynman propagator it is understood that m^2 is replaced by $m^2 - i\epsilon$ as in the scalar case. [If desired, the factor of $g(x)^{-1/4}$ in Eq. (5.19) can also be expanded about x' with the result that the $A_{j\alpha\beta}\ldots$ coefficients will be somewhat altered.]

The quantity $A_0(x,x')$ satisfies the equation

$$\frac{dx^\sigma}{d\tau} A_0(x,x')_{;\sigma} = 0 \tag{5.21}$$

with the boundary condition

$$A_0(x',x') = 1. \tag{5.22}$$

Here $dx^\sigma/d\tau$ is the tangent at x to the geodesic from x' to x. [$A_0(x,x')$ is the bispinor of geodesic parallel transport.] In the normal neighborhood of x' we choose the vierbein field such that $b^\alpha{}_\sigma(x)$ is obtained from $b^\alpha{}_\sigma(x')$ by parallel transport along the geodesic from x' to x. Then

$$\frac{dx^\mu}{d\tau}\nabla_\mu b^\alpha{}_\sigma(x) = 0. \tag{5.23}$$

It follows from Eq. (5.6) that $(dx^\mu/d\tau)\Gamma_\mu(x) = 0$, and hence the covariant derivative in Eq. (5.21) can be replaced by the ordinary derivative. Integration along the geodesic from x' to x then yields

$$A_0(x,x') = 1. \tag{5.24}$$

The coefficients A_1 and A_2 are given in Ref. 15, while $A_{1\mu}$ can be obtained from Ref. 16 if one notes that Γ_μ vanishes at the origin of the normal coordinate system. The coefficient $A_{1\mu\nu} = \frac{1}{2}A_{1,\mu\nu}(x',x')$ requires additional calculation. The results are

$$A_1 = \tfrac{1}{12}R1, \tag{5.25}$$

$$A_2 = (-\tfrac{1}{120}R_{;\mu}{}^{\mu} + \tfrac{1}{288}R^2 - \tfrac{1}{180}R_{\mu\nu}R^{\mu\nu} + \tfrac{1}{180}R_{\mu\nu\sigma\tau}R^{\mu\nu\sigma\tau})1 + \tfrac{1}{48}G_{[\alpha\beta]}G_{[\gamma\delta]}R^{\alpha\beta\lambda\xi}R^{\gamma\delta}{}_{\lambda\xi}, \tag{5.26}$$

$$A_{1\mu} = -\tfrac{1}{24}R_{;\mu}1 + \tfrac{1}{12}G_{[\alpha\beta]}R^{\alpha\beta}{}_{\mu}{}^{\lambda}{}_{;\lambda}, \tag{5.27}$$

$$A_{1\mu\nu} = \tfrac{1}{30}(\tfrac{1}{4}R_{\mu\nu;\lambda}{}^{\lambda} - \tfrac{1}{2}R_{;\mu\nu} - \tfrac{1}{3}R_{\mu\lambda}R^{\lambda}{}_{\nu} + \tfrac{1}{6}R^{\lambda\xi}R_{\lambda\mu\xi\nu} + \tfrac{1}{6}R^{\lambda\xi\sigma}{}_{\mu}R_{\lambda\xi\sigma\nu})1 + \tfrac{1}{48}G_{[\alpha\beta]}(RR^{\alpha\beta}{}_{\mu\nu} - R^{\alpha\beta\lambda}{}_{\mu;\lambda\nu} - R^{\alpha\beta\lambda}{}_{\nu;\lambda\mu}) + \tfrac{1}{96}G_{[\alpha\beta]}G_{[\gamma\delta]}(R^{\alpha\beta\lambda}{}_{\mu}R^{\gamma\delta}{}_{\lambda\nu} + R^{\alpha\beta\lambda}{}_{\nu}R^{\gamma\delta}{}_{\lambda\mu}), \tag{5.28}$$

where

$$G_{[\alpha\beta]} = \tfrac{1}{4}(\gamma_\alpha\gamma_\beta - \gamma_\beta\gamma_\alpha). \tag{5.29}$$

The quantities on the right side of the above equations are evaluated at x'. We take $b^\alpha{}_\sigma(x') = \delta^\alpha{}_\sigma$ in these normal coordinates, so that vierbein indices on the Riemann tensor need not be distinguished from spacetime indices in the above equations. These results permit one to deal with divergences arising in quantization of the spin-$\frac{1}{2}$ field by means of momentum-space techniques analogous to those used in Minkowski space.

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APPENDIX: DERIVATION OF MOMENTUM-SPACE REPRESENTATION FROM WKB FORMALISM

In Refs. 17 and 18, a method of defining a renormalized quantum stress tensor for free fields in cosmological spacetimes, called adiabatic regularization, was introduced. The first step in this method is to define approximate physical quantum particle states, and this is done by solving the homogeneous wave equation by separation of variables and using a WKB approximation to obtain solutions of the time-dependent part of the wave equation. In this appendix we show that a momentum-space representation of the Feynman propagator can be constructed directly from these WKB solutions. The representation is identical to that obtained for conformally flat spacetimes in Ref. 4. For a spatially flat Robertson-Walker spacetime with metric

$$ds^2 = C(\eta)(-d\eta^2 + dx_1{}^2 + dx_2{}^2 + dx_3{}^2) \tag{A1}$$

the positive-frequency WKB solutions are

$$\phi_{\underline{k}}(x) = \frac{C^{-1/2}(\eta)}{(2\pi)^{3/2}} \frac{\exp[-i\int_{\eta'}^{\eta} W_k(t)dt + i\underline{k}\cdot(\underline{x}-\underline{x}')]}{[W_k(\eta)]^{1/2}}, \tag{A2}$$

where $k=|\underline{k}|$, $(\eta', \underline{x}')$ is some fixed point at which the phase of $\phi_{\underline{k}}(x)$ is chosen so that $\phi_{\underline{k}}(x)$ is real, and W_k is a real function of time which satisfies

$$W_k{}^2=\omega_k{}^2+(\xi-\tfrac{1}{6})CR-\tfrac{1}{2}\left[\frac{W_k''}{W_k}-\frac{3}{2}\left(\frac{W_k'}{W_k}\right)^2\right], \quad \text{(A3)}$$

$$\omega_k{}^2=k^2+C(\eta)m^2. \quad \text{(A4)}$$

R is the Ricci scalar which is a function only of time.

The Feynman propagator is

$$\begin{aligned} G(x,x') &= i\langle 0|T(\phi(x)\phi(x'))|0\rangle \\ &= i\theta(\eta-\eta')\int \phi_{\underline{k}}(x)\phi_{\underline{k}}^*(x')d^3\underline{k} \\ &\quad + i\theta(\eta'-\eta)\int \phi_{\underline{k}}(x')\phi_{\underline{k}}^*(x)d^3\underline{k}, \end{aligned} \quad \text{(A5)}$$

where

$$\theta(\eta-\eta')=1 \quad \text{if } \eta>\eta'$$
$$=0 \text{ otherwise.}$$

In evaluating the propagator, we will only be interested in terms which are divergent in the coincidence limit $x\to x'$, and for these purposes it is sufficient to take the lowest-order solution to (A3), namely,

$$W_k{}^2=\omega_k{}^2+(\xi-\tfrac{1}{6})CR \quad \text{(A6)}$$

or

$$W_k=\omega_k+\frac{(\xi-\frac{1}{6})CR}{2\omega_k}+o(k^{-3}). \quad \text{(A7)}$$

We can expand

$$\int_{\eta'}^{\eta} W_k(t)dt=(\eta-\eta')W_k(\eta')+o((\eta-\eta')^2), \quad \text{(A8)}$$

$$\exp\left[-i\int_{\eta'}^{\eta} W_k(t)dt\right]=\exp[-i(\eta-\eta')\omega_k]\left[1-\frac{i(\eta-\eta')(\xi-\frac{1}{6})CR}{2\omega_k}+\cdots\right], \quad \text{(A9)}$$

$$[2W_k(\eta)]^{-1/2}[2W_k(\eta')]^{-1/2}=\frac{1}{2\omega_k(\eta')}\left[1-\frac{(\xi-\frac{1}{6})CR}{2\omega_k{}^2}+o(k^{-4})\right]+o(\eta-\eta'). \quad \text{(A10)}$$

In each of Eqs. (A7)–(A10), terms omitted do not give rise to divergences in $\int \phi_{\underline{k}}(x)\phi_{\underline{k}}^*(x')d^3\underline{k}$ in the coincidence limit $x\to x'$. Using (A9) and (A10) we find

$$\int \phi_{\underline{k}}(x)\phi_{\underline{k}}^*(x')d^3\underline{k}=\frac{C^{-1/2}(\eta)C^{-1/2}(\eta')}{(2\pi)^3}\int\frac{d^3\underline{k}}{2\omega_k}\exp[-i(\eta-\eta')\omega_k+i\underline{k}\cdot(\underline{x}-\underline{x}')]$$
$$\times\left[1-\frac{(\xi-\frac{1}{6})CR}{2\omega_k{}^2}-\frac{i(\eta-\eta')(\xi-\frac{1}{6})CR}{2\omega_k}\right]. \quad \text{(A11)}$$

In (A11), all quantities ω_k, C, and R appearing under the integral sign are functions of the fixed time $\eta=\eta'$. We can write the propagator as

$$G(x,x')=C^{-1/2}(\eta)C^{-1/2}(\eta')[I_1+(\xi-\tfrac{1}{6})CRI_2], \quad \text{(A12)}$$

where

$$I_1=i\theta(\eta-\eta')\int\frac{d^3\underline{k}}{(2\pi)^3}\frac{e^{ik(x-x')}}{2\omega_k}+i\theta(\eta'-\eta)\int\frac{d^3\underline{k}}{(2\pi)^3}\frac{e^{-ik(x-x')}}{2\omega_k}, \quad \text{(A13)}$$

$$I_2=i\theta(\eta-\eta')\int\frac{d^3\underline{k}}{(2\pi)^3}\left[-\frac{1}{4\omega_k{}^3}-\frac{i(\eta-\eta')}{4\omega_k{}^2}\right]e^{ik(x-x')}+i\theta(\eta'-\eta)\int\frac{d^3\underline{k}}{(2\pi)^3}\left[-\frac{1}{4\omega_k{}^3}+\frac{i(\eta-\eta')}{4\omega_k{}^2}\right]e^{-ik(x-x')}, \quad \text{(A14)}$$

where

$$k(x-x')=-k_0(\eta-\eta')+\underline{k}\cdot(\underline{x}-\underline{x}') \quad \text{(A15)}$$

and

$$k_0=\omega_k. \quad \text{(A16)}$$

But I_1 and I_2 can be expressed as four-dimensional momentum integrals:

$$I_\alpha=\int\frac{d^4k}{(2\pi)^4}\frac{e^{ik(x-x')}}{(k^2+M^2-i\epsilon)^\alpha} \quad (\alpha=1,2), \quad \text{(A17)}$$

where the contour of integration in the k_0 plane is the positive real axis, $M^2=C(\eta')m^2$, $k^2=-k_0{}^2+\underline{k}^2$, and the limit $\epsilon\to 0$ is taken after the integration has been performed. The propagator is thus given in a momentum-space representation by (A12) and (A17).

If the derivation in this appendix were carried out in n dimensions, the only differences in the propagator would be the replacement of $C^{-1/2}(\eta)$ by $C^{(2-n)/4}(\eta)$ and $\xi-\frac{1}{6}$ by $\xi-(n-2)/4(n-1)$. This representation is thus identical to that obtained in Ref. 4 for conformally flat spacetimes. Because the coordinate system (A1) is not normal, the representation is not identical in form to that of Sec. II. However, there is a close similarity between the two, and either representation can be used to investigate renormalizability of $\lambda\phi^4$ theory in conformally flat spacetimes along the lines of Sec. IV. The divergent parts of the Feynman diagrams are found to be the same whichever representation is used.

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