Gravitational collapse of a charged fluid sphere

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A class of solutions of Einstein's gravitational field equations is discussed which describes the collapse or expansion of a charged, perfect-fluid spherical distribution of matter. These solutions reduce in the appropriate limits to certain charged Newtonian polytropes. A physical interpretation of the collapsing configurations is given, and it is shown that these solutions can describe the gravitational collapse of a bounded, charged fluid around a charged black hole. In all these configurations the singular region is either a spacelike or a null hypersurface. Therefore, the final state of collapse cannot be described by the complete analytic extension of the Reissner-Nordström spacetime. The special case of uniform density models is investigated in detail and it is shown that they describe the accretion of neutral matter by a charged black hole. On the basis of this analysis, it is suggested that for the realistic collapse of charged matter the singular region formed within the matter is either spacelike or null.

I. INTRODUCTION

Spacetime singularities generally arise in the relativistic theory of gravitation as "events" at which the physical laws break down. Spacetime regions of this nature occur in many solutions of the gravitational field equations. It is expected, nevertheless, that in the neighborhood of a singularity reasonable physical conditions hold since a "complete" gravitational collapse of matter is supposed to give rise to a "physical" singularity. Thus, following the argument of Laplace, the strong attractive field of such a singularity is assumed to prevent material particles and electromagnetic rays from leaving its vicinity. The existence of trapped surfaces of this nature in the relativistic theory of gravitation was first shown by Penrose.[1] There is as yet no general proof that all physical singularities allowed within Einstein's theory of gravitation are surrounded by an event horizon.

The physical nature of the singularity requires that all physical observers following (timelike) geodesics that cross (a horizon) into a trapped region should eventually end up at the singularity. This implies that the singularity should be a spacelike or a null hypersurface. For the case of matter-free black-hole solutions, this requirement is satisfied for the Schwarzschild solution since the singular hypersurface is spacelike. However, for the Reissner-Nordström and (charged or uncharged) Kerr solutions the singular surface is timelike.[2] The stability of this timelike singularity has been discussed by Simpson and Penrose,[3] and by McNamara.[4] Simpson and Penrose presented arguments for the instability of the inner horizon of the Reissner-Nordström black hole and conjectured that electromagnetic perturbations of a Reissner-Nordström black hole will become singular on the inner horizon. McNamara has considered linear perturbations of the Reissner-Nordström field by a massless scalar field and of the Kerr field by gravitational, electromagnetic, and massless scalar fields with the conclusion that these perturbing fields can indeed become singular on the inner horizon. Perturbations of the Reissner-Nordström black hole have also been the subject of more recent investigations by several authors,[5] some of whom[6] have speculated on the possibility that the interior geometry of the Reissner-Nordström and Kerr black holes could be significantly altered by quantum processes in vacuum. These results generally indicate that the structure of the interior Reissner-Nordström and (charged or uncharged) Kerr spacetimes is not representative of the final state of realistic collapsing configurations, since these are expected to result in significantly different spacetime structures interior to the event horizon. However, the nature of the spacetime structure and the structure of the singularity in a realistic collapse are not examined in this approach.

It is the purpose of the present paper to present a general class of solutions of Einstein's gravitational field equations for the collapse of a sphere of charged perfect fluid. It is possible to study explicitly the nature of the singularity in these solutions and to determine that the singular region is either spacelike or null. The inherent simplicity of these solutions precludes any claim that they may correspond to realistic situations. Rather, they are considered to be simple, physically reasonable models representing some of the complex phenomena associated with the gravitational collapse of matter. The case of the gravitational collapse of rotating matter will not be considered in the present work since, among other things, no satisfactory interior Kerr solution is known.[7]

Following the pioneering work of Oppenheimer and Snyder the gravitational collapse of matter has been considered by many authors. Despite the progress achieved, many physical questions of interest remain; for instance, the nature of the final state of an electrically neutral, isolated, massive body endowed with a magnetic field.[8] A frequently encountered problem is the nature of the state of matter in various collapsing configurations. If the configuration is static or changing slowly, then the assumption of local thermodynamic equilibrium is valid and one may employ the usual description in terms of local thermodynamic variables. Near a physical singularity, however, the validity of such an assumption is far from obvious. In the present work it is assumed that the matter may be described by a perfect fluid at essentially the absolute zero of temperature, so that no heat is exchanged and such that the mass-energy density μ, the pressure p, and the charge density ζ are the only quantities that characterize the state of matter. The pressure and density depend on position and time, and there is no equation of state. For the physical interpretation of the solutions, however, certain physical requirements are imposed on μ and p (e.g., $\mu \geq 0$, $p \geq 0$ everywhere, and $\mu - 3p \geq 0$ outside the trapped region).

The plan of the paper is as follows: In Sec. II general properties of spherically symmetric spacetimes (in the presence of a charged perfect fluid) in the comoving frame are described. An explicit class of solutions is presented in Sec. III. It is demonstrated in Sec. IV that in the static limit some of these solutions have simple Newtonian analogs. The physical interpretation of the general solutions is given in Sec. V and uniform density models for gravitational collapse are treated in Sec. VI. Some details of the calculations and certain developments outside the main line of the paper are relegated to the Appendixes.

II. A SPHERICALLY SYMMETRIC SPACETIME WITH MATTER

The metric form for a spherically symmetric spacetime region depends in general on two arbitrary functions (of time and the radial coordinate). In such a coordinate system the motion of matter is arbitrary except for the constraint imposed by spherical symmetry. If one chooses to use coordinates in which the matter is stationary, then the general metric form depends on three arbitrary functions. Let the metric form in such a comoving coordinate system be written as[9]

$$\mathfrak{F} = -a^2 dt^2 + b^2 dr^2 + R^2 d\Omega^2 , \tag{1}$$

where a, b, and R are arbitrary, non-negative functions of t and r, $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$, and (r, θ, φ) denote "spherical" coordinates. The spacetime is assumed to be occupied by a perfect fluid which may be charged. Thus the energy-momentum of the matter and the electromagnetic field is

$$T_{\mu\nu} = (\mu + p) u_\mu u_\nu + p g_{\mu\nu} + \frac{1}{4\pi} \left(g^{\alpha\beta} F_{\alpha\mu} F_{\beta\nu} - \tfrac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}\right), \tag{2}$$

where u_μ is the four-velocity of matter and $F_{\mu\nu}$ is the electromagnetic field tensor. Let ϕ_μ be the vector potential, $F_{\mu\nu} = \phi_{\nu,\mu} - \phi_{\mu,\nu}$. Then spherical symmetry ensures that only a radial electric field can in general exist. A suitable choice of gauge then renders $\phi_i = 0$ and

$$\phi_0 = \phi_0(t, r) , \tag{3}$$

without any loss in generality. A similar simplification imposed by the comoving coordinate condition implies that $u^i = 0$ and

$$u^0 = a^{-1} . \tag{4}$$

The Riemann and Einstein tensors for the metric form (1) are given in Appendix A. Let ξ be defined as

$$\xi = \frac{1}{8\pi} \left(\frac{1}{ab} \frac{\partial \phi_0}{\partial r}\right)^2 , \tag{5}$$

so that the energy-momentum tensor (2) is diagonal and completely determined by $T_t{}^t = -\mu - \xi$, $T_r{}^r = p - \xi$, and $\mathrm{tr}(T_{\mu\nu}) = 3p - \mu$. The gravitational field equations are then given by

$$G_{\mu\nu} = 8\pi T_{\mu\nu} , \tag{6}$$

and the electromagnetic field equations by

$$\frac{\partial}{\partial x^\nu} [(-g)^{1/2} F^{\mu\nu}] = 4\pi (-g)^{1/2} J^\mu , \tag{7}$$

where $J^\mu = \zeta u^\mu$ is the electric current vector.

To express the field equations in a simple form, it is useful to introduce the functions $\tilde{m}(t, r)$ and $q(r)$ which have the interpretations of mass-energy and total charge within a sphere of "radius" r, respectively. The mass-energy function $\tilde{m}(t, r)$ is defined[10] by

$$1 - 2\tilde{m}/R = -\Phi\Psi , \tag{8}$$

where Ψ and Φ are given by

$$\Psi = \frac{1}{a}\frac{\partial R}{\partial t} + \frac{1}{b}\frac{\partial R}{\partial r}, \qquad (9)$$

$$\Phi = \frac{1}{a}\frac{\partial R}{\partial t} - \frac{1}{b}\frac{\partial R}{\partial r}. \qquad (10)$$

The quantity $q(r)$ is defined by

$$\frac{1}{ab}\frac{\partial \phi_0}{\partial r} = \frac{q(r)}{R^2}, \qquad (11)$$

so that ξ can be expressed as

$$\xi = \frac{1}{8\pi}\frac{q^2}{R^4}. \qquad (12)$$

The fact that q is independent of time follows directly from the Maxwell equations. These equations also imply the natural connection between q and the charge density ζ:

$$\frac{dq}{dr} = 4\pi b R^2 \zeta. \qquad (13)$$

The flux of matter energy vanishes in a comoving frame, therefore Einstein's equations imply

$$\frac{\partial^2 R}{\partial t \partial r} = \frac{1}{a}\frac{\partial a}{\partial r}\frac{\partial R}{\partial t} + \frac{1}{b}\frac{\partial b}{\partial t}\frac{\partial R}{\partial r}. \qquad (14)$$

The conservation of energy, $u_\mu T^{\mu\nu}{}_{;\nu} = 0$, implies that

$$-(\mu + p)^{-1}\frac{\partial \mu}{\partial t} = \frac{1}{b}\frac{\partial b}{\partial t} + \frac{2}{R}\frac{\partial R}{\partial t}. \qquad (15)$$

Equations (14) and (15) together with the following two equations yield the full content of the gravitational field equations (cf. Appendix A)

$$\frac{\partial \tilde{m}}{\partial t} = -4\pi(p - \xi)R^2\frac{\partial R}{\partial t}, \qquad (16)$$

$$\frac{\partial \tilde{m}}{\partial t} = 4\pi(\mu + \xi)R^2\frac{\partial R}{\partial r}. \qquad (17)$$

It proves convenient to introduce a "total" energy function $m(t,r)$, which represents the sum of the mass-energy function $\tilde{m}$ and the "electric energy":

$$m = \tilde{m} + \frac{q^2}{2R}. \qquad (18)$$

In terms of this total energy function, Eqs. (16) and (17) may be written as

$$\frac{\partial m}{\partial t} = -4\pi p R^2\frac{\partial R}{\partial t}, \qquad (19)$$

$$\frac{\partial m}{\partial r} = 4\pi\mu R^2\frac{\partial R}{\partial r} + \frac{q}{R}\frac{dq}{dr}, \qquad (20)$$

which are intuitively appealing.

It should be noted that there are only five equations [(13)–(17)] among the seven unknown functions a, b, R, μ, p, ζ, and q. However, if the charge distribution and an equation of state are *a priori* specified, then the problem will be fully determined. Several authors[11] have performed such calculations (numerically) for uncharged matter.

It proves useful to derive certain consequences of Eqs. (13)–(17). By demanding integrability for the pair (16) and (17), or rather (19) and (20), one can derive the Euler equation

$$\frac{\partial p}{\partial r} + (\mu + p)\frac{1}{a}\frac{\partial a}{\partial r} = \frac{b}{R^2}q\zeta. \qquad (21)$$

Furthermore, from the decomposition of the Riemann tensor into its Ricci and Weyl parts,[12] it follows that $\tilde{m}$ may be written as

$$\tilde{m} = \tfrac{4}{3}\pi\mu R^3 + \tfrac{1}{2}\frac{q^2}{R} + R^3\psi_2. \qquad (22)$$

Let Γ be defined by

$$\Gamma = R^3\psi_2 + \frac{q^2}{R}. \qquad (23)$$

Then m can be expressed as

$$m = \tfrac{4}{3}\pi\mu R^3 + \Gamma. \qquad (24)$$

It follows from (15), (19), and (24) that

$$\frac{\partial \Gamma}{\partial t} = \tfrac{4}{3}\pi(\mu + p)R^3\left(\frac{1}{b}\frac{\partial b}{\partial t} - \frac{1}{R}\frac{\partial R}{\partial t}\right). \qquad (25)$$

The motion of matter is characterized by the four-velocity u^μ. However, to study its qualitative behavior, it is of interest to construct the projection operator $h_{\mu\alpha} = g_{\mu\alpha} + u_\mu u_\alpha$ and consider the behavior of the expansion and vorticity tensors. The spherical symmetry immediately rules out any vorticity and thus

$$\omega_{\alpha\beta} = h_\alpha{}^\mu u_{[\mu;\nu]}h^\nu{}_\beta = 0. \qquad (26)$$

The expansion tensor is

$$\theta_{\alpha\beta} = h_\alpha{}^\mu u_{(\mu;\nu)}h^\nu{}_\beta = \sigma_{\alpha\beta} + \tfrac{1}{3}h_{\alpha\beta}\theta, \qquad (27)$$

where $\sigma_{\alpha\beta}$ is the shear tensor and θ is the relative rate of (volume) expansion. An explicit calculation shows that

$$\theta = \frac{1}{a}\left(\frac{1}{b}\frac{\partial b}{\partial t} + \frac{2}{R}\frac{\partial R}{\partial t}\right), \qquad (28)$$

$$\sigma_i{}^j = \frac{1}{a}\left(\frac{1}{b}\frac{\partial b}{\partial t} - \frac{1}{R}\frac{\partial R}{\partial t}\right)(S_i{}^j - \tfrac{1}{3}\delta_i{}^j), \qquad (29)$$

where $(S_i{}^j)$ is a matrix whose only nonzero element is $S_r{}^r = 1$. It is simple to see that if the shear tensor vanishes, then by Eq. (14) the rate of (volume) expansion θ is independent of position and depends only on time; the converse of this statement is not true in general.

III. A GENERAL CLASS OF SOLUTIONS

If no *a priori* restrictions are placed on the functional form of the matter variables, it is necessary to restrict the motion of matter in order to obtain solutions of the field equations systematically. Thus, let the matter be comoving in an isotropic coordinate system, which implies that the motion is shear-free. Let the metric form be

$$\mathfrak{F} = -A^2(t,\rho)dt^2 + B^2(t,\rho)(d\rho^2 + \rho^2 d\Omega^2), \qquad (30)$$

where A and B are arbitrary non-negative functions of time and the (new) radial coordinate ρ. Maxwell's equations imply that

$$\frac{B}{A}\frac{\partial\phi_0}{\partial\rho} = \psi(\rho), \qquad (31)$$

$$\frac{d}{d\rho}(\rho^2\psi) = 4\pi\rho^2 B^3\zeta, \qquad (32)$$

where ψ is a smooth but otherwise arbitrary function of ρ and $\rho^2\psi$ is simply the total electric charge within a sphere of radius ρ. Einstein's equations then reduce to the following two equations for A and B (cf. Appendix B)

$$\dot{B} = ABH(t), \qquad (33)$$

$$\left(\frac{A''}{A} + \frac{B''}{B}\right) - \left(\frac{1}{\rho} + 2\frac{B'}{B}\right)\left(\frac{A'}{A} + \frac{B'}{B}\right) = \frac{2}{B^2}\psi^2, \qquad (34)$$

together with the analogs of (19) and (20) from which μ and p can be determined. Here $H(t)$ is an arbitrary function of time. To proceed further, the functions A and B will be assumed to have a special form, which is suggested by the form of the Reissner-Nordström solution in isotropic coordinates. Let U and V be smooth functions of ρ, $U>0$, and let $f(t)>0$ be defined by

$$\dot{f} = Hf. \qquad (35)$$

Define the functions ϕ and χ as

$$\phi = \frac{1}{2}(\lambda_0 + \eta_0)\frac{U(\rho)}{f(t)}, \qquad (36)$$

$$\chi = \frac{1}{2}(\lambda_0 - \eta_0)\frac{U(\rho)}{f(t)}, \qquad (37)$$

where $\lambda_0 > 0$ and η_0 are constants. Then A and B are assumed to be of the form

$$A = \frac{1-\phi\chi}{(1+\phi)(1+\chi)}, \qquad (38)$$

$$B = (1+\phi)(1+\chi)\frac{f(t)}{V(\rho)}. \qquad (39)$$

It is simple to verify that Eq. (33) is satisfied. When expressions (38) and (39) for A and B are substituted into Eq. (34), a sixth-order polynomial in $f(t)$ results with coefficients that are functions of ρ only. This equation, which may be written as

$$\sum_{n=0}^{6}(\lambda_0 U)^{-n}\mathfrak{C}_n(\rho)f^n = 0, \qquad (40)$$

must hold for all time, implying that $\mathfrak{C}_n(\rho)=0$ for $n=0,1,\ldots,6$. In order to express $\mathfrak{C}_n$ explicitly, it is convenient to introduce the functions $\mathfrak{z}$, $\mathfrak{M}$, $\mathfrak{N}$, and $\mathfrak{P}$ as follows:

$$\mathfrak{z} = 2\frac{U'}{U} - \frac{V'}{V}, \qquad (41)$$

$$\mathfrak{M} = V^{-1}\left(V'' - \frac{1}{\rho}V'\right), \qquad (42)$$

$$\mathfrak{N} = \lambda_0^{-2}U^{-2}(\eta_0^2 U'^2 - \psi^2 V^2), \qquad (43)$$

$$\mathfrak{P} = \mathfrak{z}' - \mathfrak{z}^2 - \frac{1}{\rho}\mathfrak{z}. \qquad (44)$$

The coefficients $\mathfrak{C}_n$ may now be expressed as

$$\mathfrak{C}_0 = \nu^3\mathfrak{P}, \qquad (45)$$

$$\mathfrak{C}_1 = 2\nu^2\mathfrak{P}, \qquad (46)$$

$$\mathfrak{C}_2 = \nu[\nu\mathfrak{M} + 2\mathfrak{N} + (2\nu+1)\mathfrak{P}], \qquad (47)$$

$$\mathfrak{C}_3 = 2\nu(\mathfrak{M} + \mathfrak{P}), \qquad (48)$$

$$\mathfrak{C}_4 = (2\nu+1)\mathfrak{M} - 2\mathfrak{N} + \nu\mathfrak{P}, \qquad (49)$$

$$\mathfrak{C}_5 = 2\mathfrak{M} = 2\mathfrak{C}_6, \qquad (50)$$

where ν is a parameter given by

$$\nu = \tfrac{1}{4}(1-\sigma^2), \qquad (51)$$

and

$$\sigma = \eta_0/\lambda_0. \qquad (52)$$

If $\nu \neq 0$, it follows from Eqs. (45)–(50) that $\mathfrak{M} = \mathfrak{N} = \mathfrak{P} = 0$. The three unknown functions U, V, and ψ are thus determined. However if $\nu = 0$, then only $\mathfrak{M} = \mathfrak{N} = 0$ can be concluded, with U remaining completely arbitrary. In either case, it follows from $\mathfrak{M} = 0$ that

$$V = \gamma\rho^2 + \delta, \qquad (53)$$

where γ and δ are constants. Similarly, $\mathfrak{N} = 0$ implies that

$$\psi = -\eta_0\frac{U'}{V}, \qquad (54)$$

where the sign of η_0 has been left undetermined. If $\nu \neq 0$, then $\mathfrak{P} = 0$ may be solved to give

$$\mathfrak{z} = -\frac{2\alpha\rho}{\alpha\rho^2 + \beta}, \qquad (55)$$

where α and β are constants. It follows from Eqs. (41) and (53) that (for $\nu \neq 0$)

$$U = \left(\frac{\gamma\rho^2 + \delta}{\alpha\rho^2 + \beta}\right)^{1/2}. \qquad (56)$$

Some cases of the solutions obtained above are of special interest. For $f=1$, $\alpha=\delta=1$, and $\beta=\gamma=0$, one obtains the Reissner-Nordström solution[13] in isotropic coordinates with λ_0 and η_0 as the mass and the charge, respectively. When $\alpha>0$, $\beta=0$, and $\delta>0$, the charged version of the McVittie solution[14] is obtained. Finally, in the absence of charge, $\sigma=0$, the general solution reduces to the class of solutions considered by Glass and Mashhoon[15] and interpreted (for $\alpha>0$, $\beta\geq 0$, $\gamma<0$, $\delta>0$, and $\dot{f}<0$) as describing the last stages in the gravitational collapse of a spherical star system with a central collapsed core. These solutions may serve as highly simplified models for the (possible) collapse of the core of a globular star cluster.

IV. A NEWTONIAN ANALOG

Consider the general static solution for $\nu\neq 0$, and let $f=1$, $\alpha>0$, $\beta>0$, $\gamma=0$, and $\delta=1$. In the absence of charge, $\sigma=0$, this solution describes a regular distribution of matter for $0\leq\rho<\infty$ and it is the relativistic generalization of the classical Emden polytrope[16] of index $n=5$. In this section this correspondence is generalized in the presence of electric charge.

Using the results of Appendix B one may show that the physical parameters of the system, μ, p, and ζ, are given by

$$\mu=\frac{3}{4\pi}\lambda_0\alpha\beta B^{-3}U^5(1+2\lambda_0\nu U), \tag{57}$$

$$p=\frac{1}{2\pi}\nu\lambda_0{}^2\alpha\beta A^{-1}B^{-3}U^6, \tag{58}$$

$$\zeta=\frac{3}{4\pi}\eta_0\alpha\beta B^{-3}U^5. \tag{59}$$

The pressure should be positive (or zero), therefore $\nu\geq 0$ (and hence $\sigma^2\leq 1$). It follows that $\mu-3p\geq 0$ everywhere if $\phi_N\equiv\lambda_0 U\leq 1$ holds. ϕ_N has the interpretation of the Newtonian gravitational potential. In the Newtonian limit, where $\phi_N\ll 1$, one may write μ, p, and ζ as

$$\mu\simeq\frac{3}{4\pi}\left(\frac{\alpha\beta}{\lambda_0{}^4}\right)\phi_N{}^5, \tag{60}$$

$$p\simeq\frac{\nu}{2\pi}\left(\frac{\alpha\beta}{\lambda_0{}^4}\right)\phi_N{}^6, \tag{61}$$

$$\zeta\simeq\sigma\mu, \tag{62}$$

$$\phi_N=\phi_N^0(1+\rho^2/\rho_0{}^2)^{-1/2}. \tag{63}$$

Here $\phi_N^0=\lambda_0/\beta^{1/2}$ is the potential at the center and $\rho_0=(\beta/\alpha)^{1/2}$ is a characteristic radius of the system. Thus a regular distribution of matter is obtained for $0\leq\rho<\infty$ which in the Newtonian limit has a constant charge to mass ratio and a polytropic equation of state ($p\propto\mu^{1+1/n}$), with $n=5$. It is of interest to investigate the behavior of the quantities q and $\tilde{m}$ in this solution. One finds that

$$q=\eta_0\alpha\rho^3U^3, \tag{64}$$

$$\tilde{m}=q\frac{\mu}{\zeta}\mathcal{G}, \tag{65}$$

where $\mathcal{G}$ is a function of ϕ_N given by

$$\mathcal{G}=(1+\phi_N+\nu\phi_N{}^2)^{-1}\times[1+\tfrac{1}{2}\phi_N+\tfrac{1}{2}(\phi_N^0)^{-2}\phi_N{}^3(1+2\nu\phi_N)]. \tag{66}$$

It follows that as $\rho\to\infty$, $\phi_N\to 0$, the total charge is given by $Q=\eta_0/\alpha^{1/2}$, and the total mass by $M=\lambda_0/\alpha^{1/2}=\rho_0\phi_N^0$, implying that $Q=\sigma M$ in this solution. Moreover, the system under consideration obeys a scaling law. To see this, let $\omega=\alpha^{-1/2}$ be the scaling parameter and consider a solution with a given λ_0, β, and $\omega=1$. Then for any other $\omega\neq 0$, a solution is obtained with t, ρ, $\tilde{m}$, q, and ρ_0 scaled by ω, and p, μ, and ζ scaled by ω^{-2}.

Consider now a spherical distribution of charged fluid in equilibrium within the Newtonian theory. Let the fluid have a polytropic equation of state[17] (with index n) and a constant charge to mass ratio $\sigma=\zeta/\mu=q(\rho)/m(\rho)$. Thus

$$\frac{dm}{d\rho}=4\pi\rho^2\mu, \tag{67}$$

$$\frac{dp}{d\rho}=-4\frac{\nu}{\rho^2}\mu m, \tag{68}$$

$$p=4\nu\kappa\mu^{1+1/n}. \tag{69}$$

It follows from Eq. (68) that $\nu\geq 0$ in order for the pressure to decrease outward. For $\nu=0$ the pressure is constant according to (68), and since it should decrease to zero at the boundary of the system, one concludes that $p=0$ everywhere. This fact is incorporated in Eq. (69) in which κ is a constant. Let $\mu=\mu_c\Theta^n$, where $\Theta=1$ at the origin $\rho=0$ and μ_c is the central density, and $z=3^{1/2}\rho/\rho_0$. Then Θ satisfies the (Lane-Emden) equation

$$z^{-2}\frac{d}{dz}\left(z^2\frac{d}{dz}\Theta\right)=-\Theta^n. \tag{70}$$

Here $\rho_0>0$ is given by

$$\rho_0{}^2=\frac{3}{4\pi}\kappa(n+1)\mu_c{}^{-1+1/n}. \tag{71}$$

It is simple to see by inspection of Eqs. (67)–(69) that if Θ is finite at the origin, then $d\Theta/dt=0$ there. With these boundary conditions Eq. (70) has a unique solution which has been given in closed form only for $n=0$, 1, 5. For $n=5$ the solution is $\Theta=(1+\rho^2/\rho_0{}^2)^{-1/2}$. The exact correspondence with Eqs. (60)–(63) is established if we let

$$\kappa = \frac{1}{6}\left(\frac{4\pi}{3}\,\frac{\lambda_0^{\,4}}{\alpha\beta}\right)^{1/5}, \tag{72}$$

$$\mu_c = \frac{3}{4\pi}\,\alpha\lambda_0\beta^{-3/2}. \tag{73}$$

These results indicate that the general solutions under consideration in this paper for certain values of the parameters may be regarded as the time-dependent, relativistic generalizations of a charged fluid sphere with a polytropic equation of state with index $n=5$. Note that for $\nu=0$ the Newtonian system is in neutral equilibrium and that the potential function $\phi_N(\rho)$ can be completely arbitrary. The choice (63) is thus a particular solution in this case. This situation is completely analogous to the discussion of the $\nu=0$ solutions in the time-dependent relativistic case (cf. Appendix D).

V. PHYSICAL INTERPRETATION

For the physical interpretation of the general solution obtained in Sec. III, it is convenient to introduce a new radial coordinate

$$r = \left(\frac{\alpha\rho^2+\beta}{\gamma\rho^2+\delta}\right)^{1/2}, \tag{74}$$

which leaves the comoving character of the coordinate system unchanged but casts Eq. (30) into the form of Eq. (1) with

$$a = (1-\nu\lambda^2/r^2)(1+\lambda/r+\nu\lambda^2/r^2)^{-1}, \tag{75}$$

$$b = \frac{\lambda_0}{\lambda}\, r(1+\lambda/r+\nu\lambda^2/r^2)W^{-1/2}, \tag{76}$$

$$R = \frac{\lambda_0}{\lambda}\,\Delta^{-1}(1+\lambda/r+\nu\lambda^2/r^2)W^{1/2}, \tag{77}$$

where[18] $\lambda(t)=\lambda_0/f$, $\Delta=\alpha\delta-\beta\gamma>0$, and

$$W=(\alpha-\gamma r^2)(\delta r^2-\beta). \tag{78}$$

Since the metric can be put in the isotropic form, the shear tensor vanishes and thus

$$\frac{\dot R}{R} = \frac{\dot b}{b}, \tag{79}$$

as may be simply checked from Eqs. (76) and (77). The expansion rate is then given by Eq. (28)

$$\theta = 3a^{-1}\,\frac{\dot R}{R} = -3\,\frac{\dot\lambda}{\lambda}, \tag{80}$$

so that $\dot\lambda>0$ corresponds to a collapsing configuration.

The amount of charge within a radius r is given by

$$q(r) = \eta_0\Delta^{-2}r^{-3}W^{3/2}, \tag{81}$$

and the total energy function m at time t within a radius r is given by Eq. (24) with Γ independent of time. It turns out that for the particular solution under consideration,

$$\Gamma = \lambda_0\Delta^{-3}r^{-5}W^{5/2}.$$

The mass-energy density can be calculated from Eqs. (8), (18), and (20),

$$\frac{4\pi}{3}\,\mu = \tfrac{1}{2}(\dot\lambda/\lambda)^2 + (\lambda/\lambda_0)^2(1+\lambda/r+\nu\lambda^2/r^2)^{-3} \times[\gamma\delta(2+\lambda/r)+\alpha\beta\lambda/r^5+2\nu\alpha\beta\lambda^2/r^6]. \tag{82}$$

The charge density is simply calculated from Eq. (13) with the result

$$\frac{4\pi}{3}\,\zeta = \sigma\left(\frac{\lambda}{r}\right)\left(\frac{\lambda}{\lambda_0}\right)^2(1+\lambda/r+\nu\lambda^2/r^2)^{-3} \times(\alpha\beta r^{-4}-\gamma\delta). \tag{83}$$

The pressure can similarly be obtained from Eq. (15),

$$p = \frac{1}{3}\left(\frac{\dot\lambda}{\lambda}\,a\right)^{-1}\dot\mu - \mu. \tag{84}$$

It will now be assumed that the matter is confined to the region $r\leqslant r_B$. The metric of the space-time region $r>r_B$ is then given by the (exterior) Reissner-Nordström solution. The two solutions can be joined smoothly at the boundary surface $r=r_B$ provided that

$$p(t,r_B)=0. \tag{85}$$

The exterior solution is completely characterized by the total mass and charge of the matter M and Q, given by

$$M = m(t,r_B), \tag{86}$$

$$Q = q(r_B) = \eta_0\Delta^{-2}r_B^{\,-3}W_B^{\,3/2}. \tag{87}$$

A differential equation for $\lambda(t)$ can then be obtained from Eqs. (24), (82), (83), and (86), which describes the dynamical evolution of the system. A detailed account of the join between the interior ($r\leqslant r_B$) and the exterior ($r\geqslant r_B$) solutions can be found in Appendix C. A physical parameter of interest for describing and classifying the collapsed configuration is the red-shift z_B associated with radial null rays emitted from the boundary surface of the system ($r=r_B$) and received by stationary observers at infinity. It follows from the discussion in Appendix C that

$$1+z_B = \Psi^{-1}(t,r_B). \tag{88}$$

The red-shift z_B is a combination of gravitational and Doppler shifts.

Let us now consider the nature of the singular region. It is necessary to assume that $\nu\geqslant 0$ at this point in order to establish a connection with the physically reasonable static solution discussed in the previous section. It follows that for $r>0$ the charge and matter densities are regular but the pressure diverges at $(\nu>0)$

$$r_S=\lambda\nu^{1/2}\,, \tag{89}$$

where $a(t,r_S)=0$. The physical region then consists of $r\geqslant r_S$ where a is non-negative. For $\nu=0$, the general solution is discussed in Appendix D and the following remarks in reference to the $\nu=0$ case apply to the $\nu=0$ limit of the particular solution under consideration in this section. Hence in that limit, the singularity occurs at the origin $(r_S=0)$, and in order to have the charge and mass-energy finite as $r\to 0$, Eqs. (81) and (82) imply that $\alpha\beta=0$. Thus, as before, the pressure diverges at the singularity since $a=0$ there, but the charge and matter densities remain finite.

Let N^μ be the vector normal to the hypersurface $r=\lambda{\nu_0}^{1/2}$, where ν_0 is a positive constant. Then it follows that

$$N^\mu N_\mu=-\nu_0\dot{\lambda}^2/a^2+1/b^2\,. \tag{90}$$

As $\nu_0\to\nu$, $a\to 0$, and thus the normal vector is timelike as the singular surface is approached. Hence if $\nu\neq 0$ and the system is time-dependent, then the singularity is spacelike. On the other hand, if $\nu=0$, then at the singularity $N^\mu N_\mu=0$, hence the singular surface is lightlike. It is interesting to note that for $\nu\neq 0$, if the system is collapsing then the singular region expands if the charge density in the immediate vicinity of the singularity has the same sign as the total charge enclosed within the singular surface, and contracts if it has the opposite sign. To see this, note that the sign of the charge within the singular surface (and more generally, within any given radius) is characterized by η_0 according to Eq. (81). The charge density at the singularity has the same (opposite) sign as η_0 if $\alpha\beta-\nu^2\gamma\delta\lambda^4$ is positive (negative). On the other hand, the physical radius of the singularity $R_S=R(t,r_S)$ expands or contracts depending on whether $\alpha\beta-\nu^2\gamma\delta\lambda^4$ is positive or negative since

$$\frac{1}{R_S}\,\frac{dR_S}{dt}=(\dot{\lambda}/\lambda)\,{W_S}^{-1}(\alpha\beta-\nu^2\gamma\delta\lambda^4)\,. \tag{91}$$

The solutions under consideration are physically reasonable if the singularity is enclosed within a horizon. The apparent horizon is the time development of a two-dimensional surface, which at each instant of time is the outer boundary of all trapped surfaces at that time. Penrose[1] has given the condition for the existence of a trapped surface. A two-dimensional, closed, spacelike surface is trapped if the null geodesic rays that emanate normally from this surface converge in both the outward and inward directions. Let $l^\mu=(a^{-1},b^{-1},0,0)$ and $n^\mu=(a^{-1},-b^{-1},0,0)$ be the two null rays in the (t,r,θ,φ) coordinates that are tangent to null geodesics and are orthogonal to the constant (t,r) surfaces. Then $\Psi=l^\mu R_{,\mu}$ and $\Phi=n^\mu R_{,\mu}$ are equal, up to positive-definite proportionality factors, to the expansion parameters for the radially outgoing and ingoing null geodesic congruences, respectively. A spacelike surface of constant t and r is trapped if both Ψ and Φ are negative. The marginally trapped surface which is the *outer* boundary of all trapped surfaces at a given time is then given by $\Psi(t,r)=0$. For all time t, this is the equation for the apparent horizon which may also be written as $2\tilde{m}/R=1$ or, alternatively, as

$$1-2m/R+q^2/R^2=0\,. \tag{92}$$

This equation has the solutions $R_\pm=m\pm(m^2-q^2)^{1/2}$; R_- is the inner boundary and R_+ the outer boundary of the trapped surfaces. The equation for the time development of the apparent horizon can be found from Eqs. (16) and (17) together with $2\tilde{m}/R=1$ and $\Psi(t,r)=0$. One finds that

$$\frac{dr}{dt}=\frac{a}{b}\;\frac{1-q^2/R^2+8\pi pR^2}{1-q^2/R^2-8\pi\mu R^2} \tag{93}$$

at the apparent horizon.[19]

The general solutions under consideration allow a variety of physical configurations for a collapsing or an expanding matter distribution. In the case of collapse, and for a physically reasonable distribution of matter, it is expected that the marginally trapped surface, $R_+=m+(m^2-q^2)^{1/2}$, would expand as the boundary radius contracts until the two coincide, i.e., $R_B=M+(M^2-Q^2)^{1/2}$, at which time this surface also constitutes the event horizon for the exterior observers. Subsequently, the collapsing matter will not be able to communicate with the external observers. The event horizon is a null hypersurface which may be continued back in time into the region occupied by matter so that at a given time the event horizon will be a spherical surface outside the marginally trapped surface since orthogonally emitted null rays will be bent into parallel beams (by the attraction of matter) upon emerging into the matter-free region.

The boundary surface continues to contract until all the matter has reached the singular region. This may be seen from $\dot{R}_B=-R_B(\dot{\lambda}/\lambda)a_B$, so that as long as $r_B\geqslant r_S$ and $\dot{\lambda}>0$ the boundary contracts, and reaches the singular surface with zero "speed." The pressure is indeterminate at $r_B=r_S$, and the time development of the system can no longer be determined by means of Einstein's equations. The spacetime region outside the boundary surface is, however, part of the analytically extended Reissner-Nordström spacetime.[20]

Among the general solutions under discussion, three categories [to be referred to as (i), (ii), and (iii)] may be distinguished depending on the ratio of the charge density (in the region $r \geq r_S$) to the total charge of the system. Cases (i) and (ii) correspond to configurations in which this ratio is positive and negative, respectively. In case (iii) this ratio is zero, so that the infalling matter is neutral. Some properties of cases (i) and (ii) will be described here and in Appendix E. A detailed discussion of case (iii) is contained in the next section.

It follows from Eq. (83) that cases (i) and (ii) correspond to the conditions $\alpha\beta r^{-4} - \gamma\delta > 0$ and $\alpha\beta r^{-4} - \gamma\delta < 0$ in the physical region, respectively. An explicit calculation shows that so long as $\nu \geq 0$, the sign of $\partial\mu/\partial r$ is opposite to that of $(\alpha\beta r^{-4} - \gamma\delta)$. Thus in case (i) the density decreases outward whereas in (ii) it increases outward. Case (iii) corresponds to uniform density models.

Case (i) is realized, for example, if $\alpha > 0$, $\beta \geq 0$, $\gamma < 0$, and $\delta > 0$. It may be shown that the curvature coordinate R satisfies $\partial R/\partial r > 0$ in this case. It is therefore natural to characterize R as the "physical radius" of the system.

This intuitive characterization generally fails, however, when case (ii) is considered. The behavior of R as a function of r is more complex; nevertheless, a consistent physical interpretation is possible. The remarks below and in Appendix E are restricted to case (ii) only, which besides its intrinsic significance may perhaps be of some astrophysical interest as well. The charge density ζ and the charge q have opposite signs, therefore Eq. (21) implies that the pressure monotonically decreases outward. Thus the pressure is positive everywhere in the physical region and is zero at the boundary. On the other hand, the density has in general a finite value on the boundary μ_B, $\partial\mu_B/\partial t \geq 0$, and decreases monotonically inward at any given time. It may be assumed that $\alpha > 0$, $\beta \geq 0$, $\gamma > 0$, and $\delta > 0$. The physical range of the radial coordinate is then given by $r_{\min} = (\beta/\delta)^{1/2}$ and $r_{\max} = (\alpha/\gamma)^{1/2}$, with $r_{\min} < r_{\max}$, since $\Delta > 0$. Furthermore, at $t = 0$ a singular region is assumed to exist with $r_S{}^2 > r_{\min} r_{\max}$. The total charge of the infalling matter is always insufficient to completely neutralize the charge at the initial singularity. The curvature coordinate of the singular region decreases with time as may be seen from Eq. (91). An interesting feature of these solutions is that the system may start to collapse from rest (i.e., $d\lambda/dt = 0$ at $t = 0$) under certain circumstances. Extensive numerical work is required for a complete description of these solutions. However, if it is assumed that the boundary surface is sufficiently close to the apparent horizon initially, then certain analytic results may be derived which are discussed in Appendix E.

VI. UNIFORM-DENSITY MODELS

The discussion of the collapse of a configuration in the previous section assumed a physically reasonable distribution of matter. For the case of a perfect fluid one may assume that the conditions $\mu \geq 0$ and $p \geq 0$ should hold everywhere and $\mu - 3p \geq 0$ just outside the apparent horizon. An adequate treatment of the question of propagation of disturbances in the medium necessitates an analysis of the perturbation of the spacetime under consideration, which is beyond the scope of the present work.[21]

To impose these conditions on the density and pressure, even in the simple solutions under study, requires recourse to numerical work. Therefore, to study the collapse analytically, it proves expedient to consider the special case of the solutions with uniform density, i.e., $\beta = \gamma = 0$. Let $\nu \geq 0$, $\alpha\delta > 0$, and $\omega = (\alpha\delta)^{-1/2}$ be the scaling parameter. Then the metric form (1) is isotropic, with $R = br$, and

$$a = (1 - r_S{}^2/r^2)(1 + \lambda/r + r_S{}^2/r^2)^{-1}, \tag{94}$$

$$b = \omega(\lambda_0/\lambda)(1 + \lambda/r + r_S{}^2/r^2). \tag{95}$$

The charge density ζ vanishes, so that the charge is a constant given by

$$q(r) = \omega\eta_0 . \tag{96}$$

Hence these models represent the accretion of neutral matter by a charged system. The total energy function is given by

$$m(t, r) = \omega\lambda_0 + \tfrac{4}{3}\pi\mu R^3 , \tag{97}$$

where μ, given by $\mu = 3(8\pi)^{-1}(\dot\lambda/\lambda)^2 \geq 0$, is a function of time only, and may be alternatively expressed in terms of $\epsilon \geq 0$, $M = \omega\lambda_0(1 + \epsilon)$, as

$$\mu = \frac{3}{4\pi}\,\epsilon\omega\lambda_0 R_B{}^{-3} . \tag{98}$$

The pressure is then expressed by

$$p = \mu\left(\frac{a_B}{a} - 1\right) , \tag{99}$$

so that for $r_S \leq r \leq r_B$, the pressure diverges at the singularity and decreases monotonically to zero at the boundary. The time development of the system is given by the differential equation

$$(\dot\lambda/\lambda)^2 = 2\epsilon\omega\lambda_0 R_B{}^{-3} , \tag{100}$$

which may be solved by means of the Legendre elliptic integrals as shown in Appendix F. The singularity occurs at $r = r_S$, where $a = 0$. Let $\tilde{F} \equiv R/(\omega\lambda_0)$ be the "reduced physical radius" of the system so that at the singularity

$$\tilde{F}_S = 1 + 2\nu^{1/2}. \qquad (101)$$

The marginally trapped surface at a given time is at $R = m + (m^2 - q^2)^{1/2}$, so that from Eqs. (96)–(98) one finds that

$$\tilde{F}^2 - 2\tilde{F} + \sigma^2 = 2\epsilon\tilde{F}^4/\tilde{F}_B{}^3, \qquad (102)$$

at the apparent horizon. It is interesting to note that although the coordinate radius of the singular surface increases in time, yet the physical radius of this surface, $\omega\lambda_0\tilde{F}_S$, is a constant. Moreover, at a given time, the reduced physical radius ranges monotonically from $\tilde{F}_S$ to $\tilde{F}_B$ (cf. $\partial R/\partial r = ab \geq 0$). The behavior of the apparent horizon can be simply described in terms of a graphic solution of Eq. (102). In the Cartesian $(\tilde{F}, \tilde{G})$ plane, consider the two curves

$$\tilde{G}_1 = (\tilde{F} - \tilde{F}_S)(\tilde{F} - 1 + 2\nu^{1/2})$$

and

$$\tilde{G}_2 = 2\epsilon\tilde{F}^4/\tilde{F}_B{}^3 \quad \text{for} \quad \tilde{F} \geq \tilde{F}_S.$$

There can be at most one point of crossing $\tilde{G}_1 = \tilde{G}_2$ where the apparent horizon occurs and $\tilde{F} = \tilde{F}_{AH}$. Let there be such a point at some given time. At any later time $\tilde{F}_B$ has decreased and hence the graph of $\tilde{G}_2$ is more highly curved resulting in a larger value of $\tilde{F}_{AH}$. A time may come, however, at which the two curves are tangent, i.e., $\tilde{G}_1 = \tilde{G}_2$ and $d\tilde{G}_1/d\tilde{F} = d\tilde{G}_2/d\tilde{F}$ at the maximum value of $\tilde{F}_{AH}$, namely, $\tilde{F}_{AH}^M = \frac{1}{2}[3 + (9 - 8\sigma^2)^{1/2}]$. From this time on no solution exists. Therefore the corresponding reduced boundary radius $\tilde{F}_B$ can range from ∞ down to $[4\epsilon/(\tilde{F}_{AH}{}^M - 1)]^{1/3}\tilde{F}_{AH}{}^M$.

It is simple to show that

$$a = \left(1 - \frac{2}{\tilde{F}} + \frac{\sigma^2}{\tilde{F}^2}\right)^{1/2}, \qquad (103)$$

and hence $da/d\tilde{F} > 0$, so that a is a monotonic function of $\tilde{F}$. Therefore, if at the initial instant $t = 0$, $(p/\mu)_{AH} = -1 + a_B/a_{AH}$ is so specified that the physical requirement $(\mu - 3p)_{AH} \geq 0$ is satisfied, then at any later time a_B decreases and a_{AH} increases, forcing $(p/\mu)_{AH}$ to decrease with time. Thus $\mu - 3p \geq 0$ is always true outside the apparent horizon once it is satisfied at the marginally trapped surface initially. The singular surface $r_S = \lambda\nu^{1/2}$ at which the pressure diverges is spacelike for $\nu > 0$ and null for $\nu = 0$.

The uniform density models obey a scaling law. All physical length parameters scale as ω and the density and pressure scale as ω^{-2}. The total charge to mass ratio for these models is given by $\sigma/(1+\epsilon)$ which, together with $|\sigma| \leq 1$ and $\epsilon > 0$, implies that $|Q| < M$ if the system is time dependent. Thus an extreme Reissner-Nordström black hole cannot result from the collapse of such a configuration of matter. It now remains to give explicit examples of such configurations. Let $\omega = 1$ and $\lambda(t=0) = \lambda_0$ in what follows. If at $t = 0$ one chooses $F_{AH} = 2$ and $F_B = \frac{7}{3}$, then a one-parameter class of models is obtained with $32\epsilon = (\frac{7}{3})^3\sigma^2$ and $\frac{63}{115} \leq \sigma^2 < \frac{1008}{1105}$. For the upper limit of $\sigma^2 \sim 0.9$, the boundary surface is initially almost trapped and from Eq. (88), $z_B \sim \infty$. However, for the minimum allowed value of $\sigma^2 \sim 0.5$, one has the minimum value of the initial redshift given by $z_B \sim 15$. It is simple to show that in each configuration z_B increases monotonically until it diverges at the instant that all matter has just passed through the horizon. To see an example of a $\nu = 0$ configuration, let $F_{AH} = \frac{3}{2}$ and $F_B = \frac{8}{5}$ at $t = 0$. Then ϵ is given by Eq. (102) as $81\epsilon = 2(\frac{8}{5})^3$.

It is a general property of the uniform density models that the inner horizon does not exist in the physical region since $R_- < R_S$. The spacetime region exterior to the matter is part of the analytically-extended Reissner-Nordström solution. In the present solutions the nature of the (spacelike or null) singular region that consists of charged (completely) collapsed matter is not altered by the accretion of neutral perfect fluid matter. A characteristic feature of the uniform density models is that the solutions possess a singularity at $t = 0$. However, this need not be the case for the general class of solutions discussed in the previous section. It has been shown[15] that (in the absence of electric charge) solutions exist in that class which are regular at $t = 0$ but develop a (spacelike) singularity in the process of gravitational collapse, the singular region always being enclosed within a horizon.

Note added in proof. In connection with the question raised at the end of Appendix B, we have undertaken a comprehensive study (to be published elsewhere) of the spherically symmetric, shear-free motions of charged or uncharged perfect fluids obeying an equation of state $p = p(\mu)$, including the special case of $p = 0$ everywhere. Among other things, we have answered the above question in the negative, that is, we have shown that there does not exist an interior Reissner-Nordström solution whose fluid is subject to the conditions specified above.

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APPENDIX A

In this appendix the Riemann and Einstein tensors for the metric form (1) are given. The nonzero components of the Riemann tensor may be obtained from the following:

$$\mathcal{R}_{0101} = aa_{,rr} - \frac{a}{b}\, a_{,r}\, b_{,r} - bb_{,tt} + \frac{b}{a}\, a_{,t}\, b_{,t}\,, \tag{A1}$$

$$\mathcal{R}_{0202} = \frac{aR}{b^2}\, a_{,r}R_{,r} - RR_{,tt} + \frac{R}{a}\, R_{,t}\, a_{,t}\,, \tag{A2}$$

$$\mathcal{R}_{0303} = \sin^2\theta\, \mathcal{R}_{0202}\,, \tag{A3}$$

$$\mathcal{R}_{0221} = R^2 \mathcal{S}\,, \tag{A4}$$

$$\mathcal{R}_{0331} = \sin^2\theta\, \mathcal{R}_{0221}\,, \tag{A5}$$

$$\mathcal{R}_{1212} = \frac{bR}{a^2}\, b_{,t}R_{,t} - RR_{,rr} + \frac{R}{b}\, b_{,r}R_{,r}\,, \tag{A6}$$

$$\mathcal{R}_{1313} = \sin^2\theta\, \mathcal{R}_{1212}\,, \tag{A7}$$

$$\mathcal{R}_{2323} = 2\tilde{m}R\sin^2\theta\,, \tag{A8}$$

where $(0,1,2,3) = (t, r, \theta, \varphi)$ and $\mathcal{S}$ is given by

$$\mathcal{S} = \frac{1}{R}\left(R_{,tr} - \frac{1}{a}\, a_{,r}R_{,t} - \frac{1}{b}\, b_{,t}R_{,r}\right). \tag{A9}$$

Similarly, the nonzero components of the Einstein tensor are given by the following relations:

$$G_1{}^0 = \frac{2}{a^2}\,\mathcal{S}\,, \tag{A10}$$

$$a^2 G_1{}^0 + b^2 G_0{}^1 = 0\,, \tag{A11}$$

$$R_{,r}G_0{}^0 - R_{,t}G_1{}^0 = -2\,\frac{\tilde{m}_{,r}}{R^2}\,, \tag{A12}$$

$$R_{,t}G_1{}^1 - R_{,r}G_0{}^1 = -2\,\frac{\tilde{m}_{,t}}{R^2}\,, \tag{A13}$$

and

$$\begin{aligned} G_2{}^2 = G_3{}^3 &= b^{-2}\left(\frac{R_{,rr}}{R} + \frac{a_{,rr}}{a} + \frac{R_{,r}a_{,r}}{Ra} - \frac{R_{,r}b_{,r}}{Rb} - \frac{a_{,r}b_{,r}}{ab}\right) \\ &\quad - a^{-2}\left(\frac{R_{,tt}}{R} + \frac{b_{,tt}}{b} + \frac{R_{,t}b_{,t}}{Rb} - \frac{R_{,t}a_{,t}}{Ra} - \frac{a_{,t}b_{,t}}{ab}\right). \end{aligned} \tag{A14}$$

It is interesting to note that

$$\begin{aligned} \tfrac{1}{4}G_{\mu\nu}G^{\mu\nu} &= \tfrac{1}{4}R_{\mu\nu}R^{\mu\nu} \\ &= 16\pi^2(\mu^2 + 3p^2) + 4\pi\,\frac{q^2}{R^4}\,(\mu + p) + \frac{q^4}{R^8}\,. \end{aligned} \tag{A15}$$

APPENDIX B

The purpose of this appendix is to express the gravitational field equations for the metric form (30). The only nonzero components of the Einstein tensor are given by

$$G_{00} = 3\,\frac{\dot{B}^2}{B^2} - \frac{A^2}{B^2}\left(2\,\frac{B''}{B} - \frac{B'^2}{B^2} + \frac{4}{\rho}\,\frac{B'}{B}\right), \tag{B1}$$

$$G_{01} = G_{10} = -2A\left(\frac{\dot{B}}{AB}\right)'\,, \tag{B2}$$

$$\begin{aligned} G_{11} &= \frac{B'^2}{B} + 2\,\frac{A'}{A}\,\frac{B'}{B} + \frac{2}{\rho}\left(\frac{A'}{A} + \frac{B'}{B}\right) \\ &\quad - \frac{B^2}{A^2}\left(2\,\frac{\ddot{B}}{B} + \frac{\dot{B}^2}{B^2} - 2\,\frac{\dot{A}}{A}\,\frac{\dot{B}}{B}\right), \end{aligned} \tag{B3}$$

$$\begin{aligned} \rho^{-2}G_{22} &= \frac{A''}{A} + \frac{B''}{B} - \frac{B'^2}{B^2} + \frac{1}{\rho}\left(\frac{A'}{A} + \frac{B'}{B}\right) \\ &\quad - \frac{B^2}{A^2}\left(2\,\frac{\ddot{B}}{B} + \frac{\dot{B}^2}{B^2} - 2\,\frac{\dot{A}}{A}\,\frac{\dot{B}}{B}\right), \end{aligned} \tag{B4}$$

$$G_{33} = \sin^2\theta\, G_{22}\,, \tag{B5}$$

where $(0,1,2,3) = (t, \rho, \theta, \varphi)$. The energy-momentum tensor is diagonal and the gravitational field equations are given by

$$G_{01} = 0\,, \tag{B6}$$

$$G_{11} - \rho^{-2}G_{22} = -2B^{-2}\psi^2\,, \tag{B7}$$

$$A^{-2}G_{00} - B^{-4}\psi^2 = 8\pi\mu\,, \tag{B8}$$

$$B^{-2}G_{11} + B^{-4}\psi^2 = 8\pi p\,. \tag{B9}$$

Equations (33) and (34) follow directly from (B6) and (B7), respectively, and the density and pressure can be evaluated from (B8) and (B9).

It has been shown[22] that Eqs. (B6) and (B7) combine to give an interesting nonlinear partial differential equation for B^{-1}. Let $x = \rho^2$, $E(t, x) = 1/B(t, \rho)$, $D(x) = \psi^2(\rho)$, and eliminate A between Eqs. (B6) and (B7). It follows that

$$2x(\dot{E}_{,xx}/\dot{E} - 2E_{,xx}/E) = DE^2\,. \tag{B10}$$

This equation may be integrated to give

$$2xE_{,xx} = C(x)E^2 + D(x)E^3\,, \tag{B11}$$

where C is an arbitrary function of x. With a particular choice of C and D, and the appropriate boundary conditions, Eq. (B11) may in principle be solved to find B. It can also be shown that C and D are related to the conformal curvature of spacetime by

$$\psi_2 = -\tfrac{2}{3}CE^3 - DE^4\,. \tag{B12}$$

In connection with Eqs. (B8) and (B9), an interesting result has been obtained by Mansouri.[23] He has shown that if the state of a bounded, perfect fluid is time dependent, then, in the absence of a net electric charge, no equation of state of the form $p = p(\mu)$ can exist, except for the trivial case $p = 0$. This result is contingent on the assumption of spherically-symmetric shear-free motion. It is important to note that no restrictions are placed on the local laws of physics within the framework of the relativistic theory of gravitation. It follows that if an equation of state of the form $p = p(\mu)$ is imposed, then, in time, the motion will develop shear. Alternatively, if the motion is to be shear-free, then a more complicated equation of state, capable of taking due account of hear conduction, etc., must be assumed.

In the light of Mansouri's result, the question naturally arises whether a charged perfect-fluid sphere undergoing shear-free collapse (or expansion) admits an equation of state of the form $p = p(\mu)$. This appears to be an open question at the present time.

APPENDIX C

In this appendix the conditions under which the solutions of the gravitational field equations interior and exterior to the matter region can be joined smoothly will be discussed. Consider a matter distribution such that there is a hypersurface of matter discontinuity in spacetime. The spacetime may be covered by different coordinate patches, but the metric tensor should be such that the proper distance between any two events (even across the surface of discontinuity) may be properly defined. Thus the metric tensor should be a continuous function of position and time. However, the partial derivatives of the metric tensor may change discontinuously across the boundary surface of matter as shown by O'Brien and Synge.[24] The O'Brien-Synge conditions, which are the same as the requirement that the first and second fundamental forms be continuous at any non-null hypersurface, have been shown to be equivalent to the Lichnerowicz conditions by Israel[25] and Robson.[26]

Let t^*, r^*, θ^*, and φ^* be the Reissner-Nordström coordinates for the matter-free region. The metric form is then given by

$$\mathfrak{F}^* = -\Lambda dt^{*2} + \Lambda^{-1} dr^{*2} + r^{*2}(d\theta^{*2} + \sin^2\theta^* d\varphi^{*2}), \tag{C1}$$

where $\Lambda = 1 - 2M/r^* + Q^2/r^{*2}$. Consider a coordinate transformation of the form

$$t^* = F(t, r), \tag{C2}$$

$$r^* = G(t, r), \tag{C3}$$

$\theta^* = \theta$ and $\varphi^* = \varphi$. Under this transformation the Reissner-Nordström metric takes the form

$$\begin{aligned}\mathfrak{F}^* = &-(\Lambda F_{,t}{}^2 - \Lambda^{-1} G_{,t}{}^2) dt^2 \\ &-2(\Lambda F_{,t} F_{,r} - \Lambda^{-1} G_{,t} G_{,r}) dt\, dr \\ &+ (-\Lambda F_{,r}{}^2 + \Lambda^{-1} G_{,r}{}^2) dr^2 + G^2 d\Omega^2 .\end{aligned} \tag{C4}$$

The boundary of the matter region is given by the hypersurface

$$r - r_B = 0, \tag{C5}$$

which is always timelike. Thus the O'Brien-Synge conditions require that $g_{\mu\nu}$ and $g_{\mu\nu,\rho}$ be continuous at $r = r_B$ except perhaps for $g_{rr,r}$ and $g_{tr,r}$. Thus at $r = r_B$, and for any (allowed) value of t, the following equalities hold:

$$\Lambda F_{,t}{}^2 - \Lambda^{-1} G_{,t}{}^2 = a^2, \tag{C6}$$

$$-\Lambda F_{,r}{}^2 + \Lambda^{-1} G_{,r}{}^2 = b^2, \tag{C7}$$

$$\Lambda F_{,t} F_{,r} - \Lambda^{-1} G_{,t} G_{,r} = 0, \tag{C8}$$

$$G = R, \tag{C9}$$

$$G_{,r} = R_{,r}. \tag{C10}$$

It follows from these relations that

$$M = \tilde{m}(t, r_B) + \tfrac{1}{2} Q^2/R_B. \tag{C11}$$

The O'Brien-Synge conditions also require in this case that $T_r{}^r$ be continuous at $r = r_B$. For the interior solution $T_r{}^r = p - \xi$, whereas for the exterior solution $T_r{}^r = -Q^2/(8\pi G^4)$, so that

$$p(t, r_B) = \frac{q^2(r_B) - Q^2}{8\pi R_B{}^4}. \tag{C12}$$

This equation simply follows from (C11) by differentiation with respect to time. It must be remarked that the conditions to be imposed on the continuity of $g_{\mu\nu,\rho}$, other than Eq. (C10), are all already contained in (C6)–(C10) so that no new relation can be obtained.

In a physical problem fields other than the gravitational may be present in which case certain continuity conditions should be imposed on these fields in conformity with the corresponding field equations. In the case under consideration, it is clear that the absence of a surface concentration of charge at $r = r_B$ requires the continuity of the radial (electric) field. This is equivalent to the requirement that

$$q(r_B) = \eta_0 \Delta^{-2} r_B{}^{-3} W_B{}^{3/2} = Q. \tag{C13}$$

It follows from (C11) and (C12) that $p(t, r_B) = 0$ and $m(t, r_B) = M$, as expected.

The functions F and G may now be chosen such that $\mathfrak{F}^*$ is a proper metric form for the analytic extension of the Reissner-Nordström spacetime beyond the outer horizon $r^* = R_+$. Thus throughout the collapse of the charged fluid sphere the exterior spacetime region is part of the complete analytic continuation of the Reissner-Nordström spacetime.[20] The junction conditions discussed here, therefore, apply throughout the gravitational collapse of the fluid.

From the point of view of the stationary (Reissner-Nordström) observers at the asymptotically flat region of spacetime, the equation of motion of the boundary surface is given by $t^* = F(t, r_B)$ and $r^* = R(t, r_B)$, where F may be determined from

$$a \frac{\partial R}{\partial r} - b\left(1 - \frac{2M}{R} + \frac{Q^2}{R^2}\right)\frac{\partial F}{\partial t} = 0 \quad \text{at } r = r_B . \tag{C14}$$

This relation follows simply from (C6) and (C11). If a radial electromagnetic ray is emitted from the surface of the collapsing system and reaches the stationary observers at infinity, then the wave vector is given by

$$k^\mu = \omega_\infty(\Lambda^{-1}, 1, 0, 0,), \tag{C15}$$

where ω_∞ is the observed frequency at $r^* \to \infty$. The emitted frequency is given by $\omega_e = -k_\mu u_{(e)}{}^\mu$, where

$$u_{(e)}{}^\mu = a^{-1}(\dot F, \dot G, 0, 0) \quad \text{at } r = r_B . \tag{C16}$$

The redshift z_B observed at infinity owing to the motion of the boundary surface is defined by $1 + z_B = \omega_e/\omega_\infty$. Equation (88) then follows from (C11) and (C14)–(C16).

Once the surface of collapsing matter can no longer communicate with the external observers, a test electromagnetic ray emitted radially outward from the boundary surface is trapped by the gravitational field in the matter-free region and eventually reaches the timelike (Reissner-Nordström) singularity at $r^* = 0$. Thus this timelike singularity still exists in the matter-free region in the complete extension of the solutions under consideration in this paper. However, the existence of a singularity in a manifestly matter-free region is not in accord with the physical nature of a spacetime singularity which results from the gravitational collapse of matter (i.e., a *matter* singularity). It appears that this unphysical feature is perhaps due to the spherical symmetry of the spacetimes under discussion and will disappear in a generic spacetime manifold.

APPENDIX D

In this appendix the general solution for $\nu = 0$ is considered. This solution can be given by the metric form (30) with

$$A = (1 + \lambda U)^{-1} , \tag{D1}$$

$$B = \frac{\lambda_0}{\lambda} V^{-1}(1 + \lambda U) , \tag{D2}$$

where $V = \gamma\rho^2 + \delta$ and $U(\rho) > 0$ is arbitrary. The density and pressure are given by

$$8\pi\mu = 3(\dot\lambda/\lambda)^2 - 4(\lambda/\lambda_0)^2 V^2 A'^2 + 12\gamma\delta(\lambda/\lambda_0)^2 A^2 + 2AV(\lambda/\lambda_0)^2(VA'' + 2\delta\rho^{-1}A') , \tag{D3}$$

$$8\pi p = -3(\dot\lambda/\lambda)^2 + \frac{2}{A}\frac{d}{dt}(\dot\lambda/\lambda) - 4\gamma\delta(\lambda/\lambda_0)^2 A^2 . \tag{D4}$$

Furthermore, the mass and charge within a spherical region of "radius" ρ can be expressed as

$$m = \frac{1}{2}B\rho^3\left[\left(\frac{\dot\lambda}{\lambda}\right)^2 B^2 + \left(\frac{A'}{A}\right)^2 - \left(\frac{B'}{B}\right)^2 - \frac{2}{\rho}\frac{B'}{B}\right], \tag{D5}$$

$$q = \sigma\rho^2 B \frac{A'}{A} . \tag{D6}$$

It is interesting to note that in the static case $\lambda = \lambda_0$ the pressure is simply given by $p = -(2\pi)^{-1}\gamma\delta A^2$ so that a reasonable solution may be obtained if $\gamma\delta \leq 0$. In the special case where $V = 1$ (i.e., $\gamma = 0$ and $\delta = 1$), the pressure vanishes and $\zeta = \sigma\mu$ with $\sigma^2 = 1$ and

$$4\pi\mu = \rho^{-2}A^3 \frac{d}{d\rho}\left(\rho^2 \frac{A'}{A^2}\right) . \tag{D7}$$

It also follows from Eqs. (D5) and (D6) that in this case $q = \sigma m$ and $\phi_0 = \sigma A$. This special class of solutions has been discussed by Bonner.[27]

For many choices of the function U a regular solution may be obtained for $\nu = 0$. However, a singularity may appear if $U \to \infty$ (or $A \to 0$) as $\rho \to 0$. This may be seen by an examination of (D3) and (D4), and by stipulating that m and q be finite in the region under consideration even in the presence of a singularity. In particular, the pressure diverges at $A = 0$ if $\lambda(t)$ differs from a simple exponential, i.e., $d(\dot\lambda/\lambda)/dt \neq 0$. It may be simply shown that if a singularity of this nature does appear, then the singular hypersurface is null.

APPENDIX E

The aim of this appendix is to consider a situation where the ratio of the charge density to the total charge, ζ/Q, is negative. Assume that a collapsing configuration of this type exists which starts from rest with $\alpha > 0$, $\beta = 0$, $\gamma > 0$, and $\delta > 0$. The total charge Q and the total mass M are then given by

$$Q = \omega\eta_0\left(1 - \frac{r_B{}^2}{r_{\max}{}^2}\right)^{3/2} , \tag{E1}$$

$$Q/M = \sigma\left(1 + \frac{r_B{}^2}{r_{\max}{}^2} + 4\frac{r_B{}^3}{\lambda r_{\max}{}^2}\right)^{-1} , \tag{E2}$$

where $\omega = (\alpha\delta)^{-1/2}$ is a scaling parameter. One may set $\lambda = \lambda_0$ at $t = 0$ without any loss in generality. It follows that the time evolution of the system is given by

$$\int_1^{\lambda/\lambda_0} \frac{(1 + hx + \nu h^2 x^2)^{3/2}}{x^2(x-1)}\, dx = 2t/(\omega r_{\max}) , \tag{E3}$$

where $h \equiv \lambda_0/r_B$.

The graph of R versus r for the general situation in which α, β, γ, and δ are all positive can have at most three local extrema (one minimum and two maxima) for positive r. This can be simply seen by the application of Descartes' rule to the sixth-order polynomial that is obtained from $\partial R/\partial r=0$. It may also be shown that the r coordinate of the minimum increases with time, whereas those of the maxima decrease. When $\beta=0$ this graph has at most two extrema (a minimum and a maximum). This is indeed the case if in addition the relation $r_{\max}\gg r_S$ holds for all time. Let r_1 and r_2 denote the r coordinates of the minimum and the maximum, respectively. Then $r_1<r_2$ and at $t=0$, the apparent horizon is at r_1, $r_S<r_1<r_B<r_2$. Assume that $\sigma^2\ll 1$ and that r_B is infinitesimally close to r_1. Then it is possible to show that the apparent horizon expands outward, i.e., $1-q^2R^{-2}-8\pi\mu R^2>0$ at the apparent horizon. It is also possible to satisfy the requirement that $\mu-3p>0$ outside the apparent horizon since μ is finite at the boundary. Thus with all physical requirements satisfied, the above furnishes the mathematical description of a charged, gravitationally-collapsed system which accretes perfect-fluid matter of opposite charge that is initially static, and for which the event horizon subsequently forms in an infinitesimally short period of time.

APPENDIX F

The solution of the differential equation governing the time development of the uniform density collapsing configurations under consideration in this paper, Eq. (100), is studied in this appendix. Let $\tau=t/(\omega\lambda_0)$ and define

$$I_\nu(x)=\int^x \frac{(1+y+\nu y^2)^{3/2}}{y^{5/2}}\,dy\,. \tag{F1}$$

Then it follows simply that

$$I_\nu(\lambda/r_B)-I_\nu(\lambda_0/r_B)=(2\epsilon)^{1/2}\tau\,, \tag{F2}$$

where it is assumed that $\lambda(t=0)=\lambda_0$. This assumption does not diminish the generality of the treatment given here since the configurations of interest are invariant under the transformation $r\to\tilde{\Omega}r$ and $\lambda\to\tilde{\Omega}\lambda$, where $\tilde{\Omega}\neq 0$ is a constant.

If $\nu=0$, then it is easy to see that

$$I_0(x)=2\sinh^{-1}(x^{1/2})-2\left(1+\frac{1}{x}\right)^{1/2}-\frac{2}{3}\left(1+\frac{1}{x}\right)^{3/2}+\text{constant}\,. \tag{F3}$$

Next, let $\nu\neq 0$ and define F and E to be Legendre's elliptic integrals of the first and second kind, respectively. Then

$$\mathsf{F}(u,v)=\int_0^u (1-v^2\sin^2u')^{-1/2}du'\,, \tag{F4}$$

$$\mathsf{E}(u,v)=\int_0^u (1-v^2\sin^2u')^{1/2}du'\,, \tag{F5}$$

for $v^2<1$. $I_\nu(x)$ may then be expressed as

$$I_\nu(x)=\frac{2}{3}\left(\nu x+\frac{1}{x}+1\right)^{1/2}\left(\nu x-\frac{1}{x}+4\,\frac{x-s^2}{x+s^2}\right)+2s(1+\tfrac{4}{3}\nu)\mathsf{F}(\tilde{\phi},s|\sigma|^{1/2})-\tfrac{16}{3}s^{-1}\mathsf{E}(\tilde{\phi},s|\sigma|^{1/2})+\text{constant}\,, \tag{F6}$$

where $s^{-2}=(1+|\sigma|)/2$, $\tilde{\phi}=\tan^{-1}(x^{1/2}/s)$ and the constant term depends only on ν. Equations (F2), (F3), and (F6) implicitly determine λ as a function of τ.

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[10]The mass-energy function $\tilde{m}$ may be invariantly defined as $2\tilde{m}/R^3=K$, where K is the sectional curvature defined by

$$K=\mathcal{R}_{\mu\nu\rho\sigma}\vartheta^\mu\varphi^\nu\vartheta^\rho\varphi^\sigma/\vartheta^\alpha\vartheta_\alpha\varphi^\beta\varphi_\beta.$$

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There is an extensive literature on spherically-symmetric solutions of the gravitational field equations, and no attempt at completeness has been made here. The pioneering work on the gravitational collapse of a dust sphere to form a black hole as a model for stellar collapse is found in J. R. Oppenheimer and H. Snyder, Phys. Rev. 56, 455 (1939). Models for the final stages of the gravitational collapse of a globular star cluster have been investigated by Glass and Mashhoon (Ref. 15).

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$$2\psi_2 = \mathcal{C}_{\mu\nu\rho\sigma}\vartheta^{\mu}\varphi^{\nu}\vartheta^{\rho}\varphi^{\sigma}/\vartheta^{\alpha}\vartheta_{\alpha}\varphi^{\beta}\varphi_{\beta},$$

where $\mathcal{C}_{\mu\nu\rho\sigma}$ is the Weyl tensor given by

$$\mathcal{C}_{\mu\nu\rho\sigma} = \mathcal{R}_{\mu\nu\rho\sigma} - \tfrac{1}{2}\left(g_{\mu\rho}\mathcal{R}_{\nu\sigma} + g_{\nu\sigma}\mathcal{R}_{\mu\rho} - g_{\mu\sigma}\mathcal{R}_{\nu\rho} - g_{\nu\rho}\mathcal{R}_{\mu\sigma}\right) + \tfrac{1}{6}\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right)\mathcal{R}.$$

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[18]The case $\Delta = 0$ is of interest since U is then a constant and the coordinate transformation (74) is not meaningful. However, the resulting solution in isotropic coordinates is simply the Friedmann solution. The net charge is, of course, zero in such a solution.

[19]Note that Eq. (18) may be written as $m = \tilde{m} + q^2/(4\tilde{m})$ at the apparent horizon. In a collapsing configuration $\tilde{m}$ increases with time if $1 - 8\pi\mu R^2 - q^2/R^2 > 0$ at the apparent horizon. It follows that the physical radius of the black hole increases until $r_{\mathrm{AH}} = r_B$ at which time $M = m_0 + Q^2/(4m_0)$, where $m_0 = \tilde{m}(t, r_{\mathrm{AH}})$ at $r_{\mathrm{AH}}(t) = r_B$. Thus, at the apparent horizon, $\tilde{m}$ may be thought of as the "irreducible" mass of the black hole. This constitutes a generalization of an idea first put forward by Christodoulou; D. Christodoulou, Ph.D. thesis, Princeton University, 1971 (unpublished); D. Christodoulou and R. Ruffini, Phys. Rev. D 4, 3552 (1971).

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