Manifestly conformally covariant description of spinning and charged particles

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It is shown that relativistic particles may be described in terms of conformal O(2,4)-symmetric actions and that this is the maximal $O(m,n)$ symmetry. The position is given as a six-component object where two are made fictitious by the introduction of two additional local invariances to the conventional reparametrization invariance. Spinning particles are obtained through the superspace technique, and interaction with an external electromagnetic field is obtained through minimal coupling in the case of massless particles.

I. INTRODUCTION

In a previous paper,[1] hereafter referred to as paper I, it was shown that a free massive relativistic particle may be embedded in a reparametrization and scale-invariant description with a global de Sitter symmetry. The particle was described by five coordinates $y^a(\tau)$, $a=0,1,2,3,5$, where τ and y^5 are unobservable, owing to the reparametrization and scale invariance, respectively. It was described by the Lagrangian

$$L(\tau)=\frac{1}{2V}(\dot{y}+Wy)^2, \tag{1.1}$$

where the indices are contracted with the diagonal metric tensor with signature $(+----)$. V is the einbein variable and W the one-dimensional vector "field" which is responsible for the local scale invariance.

The equations of a massive relativistic particle follow from (1.1) after the gauge choices $V=1/m$, $W=0$, and $y^5=1$. An interesting feature of (1.1), which was explored in paper I, is that another gauge choice yields a massless particle in a de Sitter space, namely $V=1/m$, $W=0$, and $Y^2=-L^2$ where the constant L is the radius of the de Sitter space.

In paper I it was also shown that the supersymmetrized version of (1.1) obtained by means of the superspace technique[2] yielded the same results for the classical spinning particle. Upon quantization one either gets the ordinary massive Dirac equation or the generalized massless one in the de Sitter space.

The purpose of the present paper is threefold: First, to show that the results of paper I follow more neatly from a theory with a further local invariance together with a global O(2,4) conformal symmetry (Secs. II and III). In particular in the spinning case we get a manifestly conformally invariant constraint algebra; second, to explore further equivalences (Sec. IV); third, to show that interaction with an external field can be introduced by means of a generalized minimal coupling (Sec. V).

II. THE MASSIVE RELATIVISTIC PARTICLE IN TERMS OF A MANIFESTLY CONFORMALLY INVARIANT ACTION

We shall show in this section that a free massive relativistic particle may be described by the Lagrangian $(c=1)$

$$L_0(\tau)=\frac{1}{2V}(\dot{y}+Wy)^2+\tfrac{1}{2}VDy^2, \tag{2.1}$$

where y^A, $A=0,1,2,3,5,6$ is a six-component object with dimensions of length, and where the indices are contracted by means of the diagonal metric tensor g_{AB} with signature $(+----+)$. V, W, and D are to be treated as dynamical variables. The action $S_0=\int d\tau\, L_0(\tau)$ is invariant under the parameter transformation

$$\begin{aligned}&\delta y^A=e\dot{y}^A, \quad \delta V=\dot{e}V+e\dot{V},\\ &\delta W=\dot{e}W+e\dot{W}, \quad \delta D=e\dot{D},\end{aligned} \tag{2.2}$$

and under the scale transformation

$$\begin{aligned}&\delta y^A=fy^A, \quad \delta V=2fV,\\ &\delta W=-\dot{f}, \quad \delta D=-4fD,\end{aligned} \tag{2.3}$$

and under the additional transformation

$$\begin{aligned}&\delta y^A=\delta V=0,\\ &\delta W=-hV, \quad \delta D=2h\frac{W}{V}-\frac{\dot{h}}{V},\end{aligned} \tag{2.4}$$

where $e(\tau)$, $f(\tau)$, and $h(\tau)$ are infinitesimal functions. As we shall see, these invariances will make τ, y^5, and y^6 unobservable.

The Lagrangian (2.1) yields the equations of motion [cf. (2.7) in paper I]

$$\ddot{y}^A=0, \quad \dot{y}^2=0, \quad y^2=0, \quad y\cdot\dot{y}=0 \tag{2.5}$$

after the gauge choices $V=V_0\neq 0$, $W=D=0$. The further choices $\dot{y}^5=1$ and $\dot{y}^6=0$ yield then the equations for a free massive particle, while $y^6=L$ yields the equations for a massless particle in a de Sitter space.

These properties can also be explained in terms of the Hamiltonian formulation of (2.1) (cf. paper I). We have the constraints

$$P_V=P_W=P_D=0\,, \tag{2.6a}$$

$$P^2=P\cdot y=y^2=0\,. \tag{2.6b}$$

These constraints satisfy a closed Poisson algebra and they generate gauge transformations. They are so-called first-class constraints.[3] The Hamiltonian theory is determined by these constraints together with the total Hamiltonian (cf. paper I)

$$\begin{aligned} H_{\text{tot}} &= H+\dot{V}P_V+\dot{W}P_W+\dot{D}P_D\,,\\ H &= \tfrac{1}{2}VP^2-WP\cdot y-\tfrac{1}{2}VDy^2\,, \end{aligned} \tag{2.7}$$

where $\dot{V}$, $\dot{W}$, and $\dot{D}$ are to be treated as parameter functions of the phase-space variables.[3]

The local gauge symmetry of (2.1) is determined by the constraints (2.6b). The gauge group is easily seen to be SO(2,1) by setting $J_{12}=\frac{1}{2}(P^2+y^2)$, $J_{13}=\frac{1}{2}(P^2-y^2)$, and $J_{23}=\frac{1}{2}P\cdot y$. The Lagrangian (2.1) is also invariant under an infinitesimal global conformal O(2,4) rotation

$$\delta y^A=\epsilon^A{}_B y^B\,, \tag{2.8}$$

where $\epsilon^{AB}=-\epsilon^{BA}$ are infinitesimal constants. The generator is $F=\frac{1}{2}\epsilon_{AB}J^{AB}$ where

$$J^{AB}=P^Ay^B-P^By^A\,, \tag{2.9}$$

which is conserved and gauge invariant.

When we now eliminate $P_D=0$ and $y^2=0$ by means of the gauge choice

$$D=0,\quad P_6=0\,, \tag{2.10}$$

we get exactly the theory (1.1) which was considered in paper I. Notice that $P_6=0$ is a good gauge choice provided $y^6\neq 0$, since

$$\{y^2,P_6\}=2y_6\,. \tag{2.11}$$

We have

$$y^6=\pm(-y^ay_a)^{1/2}\,,$$

where y^a, $a=0,1,2,3,5$ are the first five components of y^A. Hence $y^6\neq 0$ is equivalent to $y^ay_a\neq 0$ which we had to impose in the spinning case in paper I. Since $y^2=0$ is a quadratic constraint we get two possible solutions for y^6 which here are completely equivalent, however. From here on the analysis of paper I applies.

III. MANIFESTLY CONFORMALLY INVARIANT DESCRIPTION OF THE SPINNING PARTICLE

The generalization to D space-time dimensions of the Lagrangian (2.1) is given by

$$\begin{aligned} \mathcal{L}(x)=\tfrac{1}{2}\det(V^m_\mu)[\,&V^\mu_m(\partial_\mu+W_\mu)\phi\\ &\times V^\nu_n(\partial_\nu+W_\nu)\phi\eta^{nm}+D\phi^2]\,, \end{aligned} \tag{3.1}$$

where the notation is in accordance with Eq. (2.1) in paper I. D is a scalar field and the indices of the field object ϕ are suppressed.

The supersymmetric generalization of the particle theory (2.1) follows from the general representation (3.1) and the superspace technique.[2] We get the action

$$\begin{aligned} S=-\tfrac{1}{2}i\int\!\!\int d\tau\, d\theta\, \det(h^A_M)[(h^{-1})^M_A(\partial_M+W_M)z\\ \times(h^{-1})^N_B(\partial_N+W_N)z\eta^{AB}+Dz^2]\,, \end{aligned} \tag{3.2}$$

where θ is an anticommuting space coordinate, an odd element of a Grassmann algebra. h^A_M are zweibein fields, $M=1,2$ is the index of the curvilinear coordinates, and $A=1,2$ that of the flat tangent space, where 1 corresponds to τ and 2 to θ. h^A_M, W_M, D, and z depend on both τ and θ. η^{AB} is chosen such that S is scalar in the flat tangent space. Notice that the indices of z are suppressed [cf. Eq. (3.1) in paper I].

Evaluating the determinant of the zweibein field, the action (3.2) simplifies to [$d\equiv(h^{-1})^2_2$, $\gamma\equiv(h^{-1})^1_2$, $b\equiv h^1_1$, $\dot{z}\equiv\partial z/\partial\tau$, and $z'=\partial z/\partial\theta$]

$$\begin{aligned} S&=\int\!\!\int d\tau\, d\theta\, \mathcal{L}(\tau,\theta)\,,\\ \mathcal{L}(\tau,\theta)&=-\tfrac{1}{2}i[d\gamma(\dot{z}+W_1z)^2\\ &\quad+d^2(\dot{z}+W_1z)\cdot(z'+W_2z)+bdDz^2]\,, \end{aligned} \tag{3.3}$$

from which we obtain the equations of motion

$$\begin{aligned} &\ddot{z}=\dot{z}'=0\,,\\ &\dot{z}^2=\dot{z}\cdot z=z^2=\dot{z}\cdot z'=0\,, \end{aligned} \tag{3.4}$$

where we have imposed the gauge choice

$$\begin{aligned} &d=R^{-1/2},\quad \gamma=i\theta R^{-1/2},\quad b=R\,,\\ &D=W_1=W_2=0\,. \end{aligned} \tag{3.5}$$

R is a constant with dimension length (corresponding to V_0 in the previous section). Notice that we measure all dimensions in length, and with the usual choice Dim(τ) = 1, Dim(z) = 1, Dim(θ) = $\frac{1}{2}$, we get Dim(γ) = 0, Dim(d) = $-\frac{1}{2}$, Dim(W_1) = -1, Dim $(W_2)=-\frac{1}{2}$, Dim$(b)=1$, and Dim$(D)=-3$. The last equations in (3.4) are constraint equations coming from the variations of d, γ, b, W_1, W_2, and D.

The Hamiltonian formulation of (3.2) is given in terms of the constraints

$$P_d=P_\gamma=P_b=P_{W_1}=P_{W_2}=P_D=0\,, \tag{3.6}$$

$$P'^2=P'\cdot z(\theta)=z^2(\theta)=0\,, \tag{3.7}$$

and the Hamiltonian is a generalization of (2.7). When we eliminate the constraints (3.6) by means of the gauge choice (3.5), and when we expand $z(\tau,\theta)$ in θ,

$$z^A(\tau,\theta)=y^A(\tau)+i\theta\psi^A(\tau)R^{1/2}\,, \tag{3.8}$$

and set $P'_A=P_A$, we are left with the reduced theory

$$\begin{aligned}&H=\tfrac{1}{2}RP^2\,,\\ &P^2=P\cdot y=y^2=P\cdot\psi=y\cdot\psi=0\,.\end{aligned} \tag{3.9}$$

One may easily check that these constraints satisfy a closed Poisson algebra [a graded SO(2,1) algebra] in terms of the following Poisson brackets[4]:

$$\begin{aligned}&\{P_A,P_B\}=\{y^A,y^B\}=0\,,\\ &\{y^A,P_B\}=\delta^A_B\,,\\ &\{P_A,\psi^B\}=\{y^A,\psi^B\}=0\,,\\ &\{\psi^A,\psi^B\}=ig^{AB}\,.\end{aligned} \tag{3.10}$$

Now we break the manifestly conformally covariant description and eliminate the constraint $y^2=0$ by means of the gauge choice

$$P_6=0\,. \tag{3.11}$$

Again there are two possible solutions for y^6, namely

$$y^6=\pm(-y^a_a y_a)^{1/2}\,.$$

When we choose

$$y^6=-(-y^a y_a)^{1/2}\,,$$

we obtain exactly the theory given in paper I, e.g., the remaining constraints satisfy the closed algebra (3.35) of paper I. There we showed that the massive gauge choice $P_5=-m$ upon quantization with the operators $\hat{\psi}^A$ represented by 8×8 matrices yields the ordinary Dirac equation, and that the de Sitter gauge $y^a y_a=-L^2$ yields the generalized massless one in de Sitter space.

IV. FURTHER EQUIVALENCES

There is a choice of coordinates other than that of Sec. II which seems to be more relevant here, in particular when interaction is included as will be seen in the next section. Consider therefore again the Hamiltonian formulation of the system (2.1). After the gauge choice $V=1/m$, $W=D=0$, it reduces to (the canonical momenta to y^A are here denoted by Π^A)

$$H=\frac{1}{2m}\Pi_A\Pi^A=0\,, \tag{4.1}$$

$$\Pi_A y^A=y^2=0\,. \tag{4.2}$$

We shall here show that one may perform a point transformation which linearizes the constraints (4.2). The new set of coordinates x^A obtained in this way is related to y^A in the following fashion:

$$x^\mu=\frac{y^\mu}{y^6-y^5}R,\quad x^5=y^6-y^5,\quad x^6=\frac{y^2}{R}, \tag{4.3}$$

or equivalently

$$\begin{aligned}&y^\mu=\frac{x^5}{R}x^\mu,\quad y^5=-\tfrac{1}{2}x^5\left(1+\eta_{\mu\nu}\frac{x^\mu x^\nu}{R^2}\right)+\frac{x^6}{2x^5}R\,,\\ &y^6=\tfrac{1}{2}x^5\left(1-\eta_{\mu\nu}\frac{x^\mu x^\nu}{R^2}\right)+\frac{x^6}{2x^5}R\,,\end{aligned} \tag{4.4}$$

where R is a constant with dimensions of length and where $x^5\neq 0$ is required. The new metric tensor $g_{AB}(x)$ is given by the relation

$$g_{AB}(x)=\frac{\partial}{\partial x^A}y^C(x)\frac{\partial}{\partial x^B}y_C(x)\,, \tag{4.5}$$

and becomes explicitly

$$\begin{aligned}&g_{\mu\nu}(x)=\left(\frac{x^5}{R}\right)^2\eta_{\mu\nu}\,,\\ &g_{55}(x)=-\frac{x^6}{(x^5)^2}R,\ g_{56}(x)=\frac{R}{2x^5}\,,\\ &g_{\mu5}(x)=g_{\mu6}(x)=g_{66}(x)=0\,.\end{aligned} \tag{4.6}$$

The inverse $g^{AB}(x)$ defined by $g^{AC}(x)g_{CB}(x)=\delta^A_B$ becomes then

$$\begin{aligned}&g^{\mu\nu}(x)=\left(\frac{x^5}{R}\right)^{-2}\eta^{\mu\nu}\,,\\ &g^{66}(x)=4\frac{x^6}{R},\quad g^{56}(x)=2\frac{x^5}{R}\,,\\ &g^{\mu5}(x)=g^{\mu6}(x)=g^{55}(x)=0\,.\end{aligned} \tag{4.7}$$

The conjugate momenta P_A to x^A are related to Π_A through the formula[5]

$$\Pi_A=\frac{\partial}{\partial x^B}y_A(x)g^{BC}(x)P_C\,. \tag{4.8}$$

Explicitly we find

$$\begin{aligned}&\Pi_\mu=\left(\frac{x^5}{R}\right)^{-1}P_\mu+2\frac{x^5}{R^2}\eta_{\mu\nu}x^\nu P_6\,,\\ &\Pi_5=-P_5+\frac{x^\mu P_\mu}{x^5}+\left[\frac{x^5}{R}\left(1+\eta_{\mu\nu}\frac{x^\mu x^\nu}{R^2}\right)-\frac{x^6}{x^5}\right]P_6\,,\end{aligned} \tag{4.9}$$

$$\Pi_6 = P_5 - \frac{x^\mu P_\mu}{x^5} + \left[\frac{x^5}{R}\left(1-\eta_{\mu\nu}\frac{x^\mu x^\nu}{R^2}\right) + \frac{x^6}{x^5}\right] P_6 .$$

By means of these relations it is easily shown that (4.1) and (4.2) become

$$H = \frac{1}{2m}\left(\frac{x^5}{R}\right)^{-2} \eta^{\mu\nu} P_\mu P_\nu = 0 , \tag{4.10}$$

$$P_5 = 0, \quad x^6 = 0 , \tag{4.11}$$

which is the promised linearized form. Since the system is independent of P_6, x^6 is trivially eliminated. P_5 is eliminated by means of a gauge choice on x^5. The class of meaningful choices are parametrized as

$$x^5 = R\phi(x^\mu) , \tag{4.12}$$

where ϕ is any nonsingular function of x^μ. The system (4.10) and (4.11) becomes then

$$H = \frac{1}{2m}\phi^{-2}(x)\eta^{\mu\nu} P_\mu P_\nu = 0 , \tag{4.13}$$

which describes a "massless" particle in a space which is conformal to the compactified Minkowski space. In particular for $\phi = 1$ we have a massless particle in Minkowski space itself and for

$$\phi(x) = \frac{1}{1 - x_\mu x^\mu/4L^2}$$

we have a massless particle in a de Sitter space with radius L.

We end this section by stating the transformation properties of x^A, P_A under conformal transformations generated by (2.9) [cf. (2.8)]. For x^A we have

$$\delta x^\mu = \epsilon^\mu{}_\nu x^\nu + \rho x^\mu + \epsilon^\mu - C_\nu x^\nu x^\mu - 2C^\mu \eta_{\rho\nu} x^\rho x^\nu , \tag{4.14}$$

$$\delta x^5 = -\rho x^5 + 4x^5 C_\mu x^\mu, \quad \delta x^6 = 0 , \tag{4.15}$$

where $\rho \equiv \epsilon_{65}$, $\epsilon^\mu \equiv \frac{1}{2}R(\epsilon^{5\mu} + \epsilon^{6\mu})$, and $C^\mu = (1/4R) \times (\epsilon^{6\mu} - \epsilon^{5\mu})$. x^6 is set equal to zero. We conclude that $J_{\mu\nu}$, J_{65}, $J_{5\mu} + J_{6\mu}$, and $J_{6\mu} - J_{5\mu}$ generate Lorentz, scale, translation, and special conformal transformations, respectively. Notice that x^5 is Poincaré invariant. P_A transforms as follows ($P_5 = x^6 = 0$):

$$\begin{aligned} \delta P_\mu &= \epsilon_\mu{}^\nu P_\nu - \rho P_\mu + 2C_\nu x^\nu P_\mu + 4C_\mu x^\nu P_\nu \\ &\quad + (2C_\nu x^\nu \eta_{\mu\rho} x^\rho + 4C_\mu \eta_{\nu\rho} x^\nu x^\rho)\left(\frac{x^5}{R}\right)^2 P_6 , \\ \delta P_5 &= 0, \quad \delta P_6 = 2R\left(\frac{R}{x^5}\right)^2 C^\mu P_\mu . \end{aligned} \tag{4.16}$$

V. MANIFESTLY CONFORMALLY COVARIANT DESCRIPTION OF A RELATIVISTIC CHARGE PARTICLE IN INTERACTION WITH AN EXTERNAL FIELD

In this section we shall show that a relativistic charged particle in interaction with an external field may be described by the Lagrangian (we set the charge $e = 1$)

$$L(\tau) = L_0(\tau) + \dot{y}^B A_B(y) , \tag{5.1}$$

where $L_0(\tau)$ is given by (2.1) $A_B(y)$ is a six-component external vector field which satisfies the subsidiary conditions[6]

$$y^B A_B(y) = 0 , \tag{5.2}$$

$$y^B \partial_B A_C(y) = -A_C(y) . \tag{5.3}$$

After the gauge choice

$$V = \frac{1}{m}, \quad W = D = 0 \tag{5.4}$$

the equations of motion reduce to

$$m\ddot{y}^A = \dot{y}_B F^{AB}(y) , \tag{5.5}$$

$$\dot{y}^2 = \dot{y}\cdot y = y^2 = 0 , \tag{5.6}$$

where

$$F^{AB}(y) = \partial^A A^B - \partial^B A^A . \tag{5.7}$$

The corresponding Hamiltonian formulation of (5.1) after the gauge choice (5.4) is given by

$$H = \frac{1}{2m}[\Pi - A(y)]^2 = 0 , \tag{5.8}$$

$$\Pi_A y^A = y^2 = 0 , \tag{5.9}$$

where Π_A is the conjugate momentum to y^A. Notice that $\Pi_A y^A = 0$ is equivalent to $(P-A)_A y^A = 0$ owing to Eq. (5.2). The constraints (5.8) and (5.9) form a closed Poisson algebra since, e.g.,

$$\{[\Pi - A(y)]^2, \Pi\cdot y\} = -2\Pi^2 + 2\Pi\cdot A - 2(\Pi - A)_B y^A \partial_A A^B = -2(\Pi - A)^2 \tag{5.10}$$

owing to (5.3). Hence, the conditions (5.2) and (5.3) are necessary in order to make the minimal coupling possible.

The reduction to Minkowski space is here rather nontrivial, however. One obvious difficulty is that $A_B(y)$ depend on too many coordinates which may not be reduced by means of gauge choices. On the other hand, owing to the condition (5.3), we may define new fields which effectively depend on only four coordinates after y^2

$=0$ is imposed. They are given by[6]

$$Ra_\mu(x) = (y^6 - y^5)A_\mu - y_\mu(A_5 + A_6)\,,$$

$$Ra_5(x) = (y^6 - y^5)A_5 - y^\mu A_\mu + \frac{y^\mu y_\mu}{2(y^6 - y^5)}(A_5 + A_6)\,,$$

$$Ra_6(x) = (y^6 - y^5)A_6 + y^\mu A_\mu - \frac{y^\mu y_\mu}{2(y^6 - y^5)}(A_5 + A_6)\,, \qquad (5.11)$$

where $x^\mu = (y^6 - y^5)^{-1} y^\mu R$, $y^2 = 0$, and R is a constant with dimensions of length. The inverse relations are in terms of the coordinates x^A in (4.3),

$$A_\mu(y) = \left(\frac{x^5}{R}\right)^{-1}\left[a_\mu + \eta_{\mu\nu}\frac{x^\nu}{R}(a_5 + a_6)\right]\,,$$

$$A_5(y) = \left(\frac{x^5}{R}\right)^{-1}\left[a_5 + \eta_{\mu\nu}\frac{x^\mu x^\nu}{2R^2}(a_5 + a_6) + \frac{x^\mu}{R}a_\mu\right]\,, \qquad (5.12)$$

$$A_6(y) = \left(\frac{x^5}{R}\right)^{-1}\left[a_6 - \eta_{\mu\nu}\frac{x^\mu x^\nu}{2R^2}(a_5 + a_6) - \frac{x^\mu}{R}a_\mu\right]\,.$$

The condition (5.2) becomes

$$a_5(x) = a_6(x)\,. \qquad (5.13)$$

These relations obviously suggest the point transformation of Sec. IV. The system (5.8) and (5.9) becomes in terms of x^A, P_A

$$H = \frac{1}{2m}\left(\frac{x^5}{R}\right)^{-2}\eta^{\mu\nu}(P_\mu - a_\mu)(P_\nu - a_\nu)\,, \qquad (5.14)$$

$$P_5 = 0, \quad x^6 = 0\,, \qquad (5.15)$$

which describes a massless charged particle in interaction with an external electromagnetic field in a space-time which is conformal to the Minkowski space (the gauge condition $x^5 = R$ yields the Minkowski space itself).

VI. CONCLUDING REMARKS

We have shown that the description of relativistic particles is possible to embed in a manifestly conformal formalism. In the case of massless particles interaction with an electromagnetic field is introduced through the trivial generalization of the conventional minimal coupling, which is also known to be possible in the pure field theory case.[6]

In paper I we showed that one may add a fifth component to the particle position when local scale invariance is included and in the present paper we have shown that also a sixth component can be included together with a further local invariance. One may ask if it is possible to add even further components to the position. However, one easily realizes that there are no further local invariances which can make these components unobservable. [The constraints (2.6b) are the only manifestly invariant ones one may construct out of P_m and y^n.] Hence conformal invariance is the maximal invariance we can have [O(2,4) is the maximal global O(m,n) symmetry of an action describing particles].

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