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On-shell counterterms and nonlinear invariances

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For theories with nonlinear invariances it is not sufficient to restrict one's attention to on-shell invariants in order to discuss the infinities of the S matrix. We give examples of models in which S-matrix divergences cannot be absorbed by invariant on-shell counterterms, and describe the general situation. We point out the possible relevance of the nonclosure of the algebra of symmetry transformations.

In the analysis of quantum divergences the background-field method, and the invariance of the background-field functional under the symmetries of the classical action, plays an important role.[1] In particular, in the presence of local gauge symmetries this method is more convenient than the conventional Feynman-diagram calculations of individual graphs, which do not manifestly maintain the symmetries.

The background-field functional for a field theory described by a Lagrangian $\mathcal{L}(\phi)$ with corresponding action $S[\phi]=\int dx\, \mathcal{L}(\phi(x))$ is defined by a functional integral:

$$\Omega[\phi]=\int [d\chi]\exp\{i\hbar^{-1}(S[\phi+\chi]-S[\phi]-S_{,k}[\phi]\chi_k)\}. \tag{1}$$

In this definition ϕ stands for the set of background fields ϕ_k, and χ for the corresponding set of quantum fields. The index k labels both space-time points and type of fields, while $S_{,k}[\phi]$ denotes the variational derivative of the action with respect to ϕ_k. In the one-loop approximation, $\Omega[\phi]$ generates precisely all the one-particle irreducible vertices; in higher orders this correspondence is not so direct.

The original action $S[\phi]$ is assumed to be invariant under transformations

$$\phi\to\phi'=g(\phi)\,, \tag{2}$$

and this implies certain properties of the background field functional $\Omega[\phi]$. To describe the effect of the transformations (2) on $\Omega[\phi]$ it is convenient to introduce separate background-field and quantum-field transformations:

$$\phi\to\phi'=g(\phi)\,, \tag{3a}$$

$$\chi\to\chi'=g(\phi+\chi)-g(\phi)\,. \tag{3b}$$

Except for the last term in Eq. (1) proportional to the classical field equations $S_{,k}[\phi]$, the functional $\Omega[\phi]$ is manifestly invariant under (3). The term $S_{,k}[\phi]\chi_k$ is in general not invariant, in particular because the quantum-field transformation (3b) is usually no longer covariant. Of course, in the case that the transformation (2) is at most linear in the fields, so that (3b) is linear and homogeneous in the quantum fields, the term proportional to the field equation preserves the invariance (3). The functional $\Omega[\phi]$ is then invariant off-shell, i.e., without need for imposing the classical field equations $S_{,k}[\phi]=0$.[2]

If $S_{,k}[\phi]\chi_k$ is not invariant under (3) we recover the invariance of $\Omega[\phi]$ only after the field equations are imposed. It is then tempting to assume that $\Omega[\phi]$ can be written as the sum of an invariant functional and terms proportional to the field equations

$$\Omega[\phi]=I[\phi]+X^k[\phi]S_{,k}[\phi]\,. \tag{4}$$

When studying the (one-loop) quantum divergences, the divergent part of $I[\phi]$ will define the required on-shell counterterm, whereas the divergences in the remaining term can be absorbed into field redefinitions. Therefore, the existence of invariants $I[\phi]$ which are not already contained in the original action is an indication that the S matrix of the theory is infinite. Conversely, the absence of such invariants has been given as an argument for the finiteness of scattering amplitudes.

However, Eq. (4) does not represent the general situation: The decomposition of $\Omega[\phi]$ can also contain additional noninvariant terms, denoted by $H[\phi]$, which do not vanish on-shell, although their variations do. In this note we will exhibit local quantities of this type in the framework of two specific models. It is important to realize that some of these quantities are required as actual counterterms to absorb one-loop divergences. We will then discuss the general circumstances under which such terms can occur. In order to establish the finiteness of the S matrix under these circumstances, it is not sufficient to prove that no local on-shell invariants can be constructed. We argue that this situation is particularly relevant for theories where the algebra of infinitesimal symmetry transformations only closes upon use of the field equations.

We now discuss two explicit examples. The models we consider are invariant under supersymmetry transformations and contain a scalar, a pseudoscalar, and a Majorana spinor field, denoted by A, B, and χ, respectively. The first model, due to Wess and Zumino, is of the renormalizable type, and its Lagrangian is given by[3,4]

$$\begin{aligned}\mathcal{L}_1 = &-\tfrac{1}{2}(\partial_\mu A)^2 - \tfrac{1}{2}(\partial_\mu B)^2 - \tfrac{1}{2}\bar{\psi}\not{\partial}\psi - \tfrac{1}{2}m^2(A^2+B^2)\\ &-\tfrac{1}{2}m\bar{\psi}\psi - \tfrac{1}{2}g^2(A^2+B^2)^2 - gmA(A^2+B^2)\\ &-g\bar{\psi}(A-i\gamma_5 B)\psi\,. \end{aligned}\tag{5}$$

The corresponding action is invariant under transformations characterized by a constant anticommuting Majorana spinorial parameter ϵ:

$$\begin{aligned}&\delta A = \bar{\epsilon}\psi\,,\\ &\delta B = i\bar{\epsilon}\gamma_5\psi\,,\\ &\delta\psi = \not{\partial}(A+i\gamma_5 B)\epsilon - m(A+i\gamma_5 B)\epsilon - g(A+i\gamma_5 B)^2\epsilon\,.\end{aligned}\tag{6}$$

The second Lagrangian, which has been discussed by Lang and Wess,[5,6] contains a dimensional coupling constant, and is not renormalizable by power counting:

$$\begin{aligned}\mathcal{L}_2 = &-\tfrac{1}{2}(\partial_\mu A)^2 - \tfrac{1}{2}(\partial_\mu B)^2 - \tfrac{1}{2}\bar{\psi}\not{\partial}\psi - \tfrac{1}{2}m^2(A^2+B^2)\\ &-\tfrac{1}{2}m\bar{\psi}\psi - \lambda m(A^4-B^4) - \tfrac{1}{2}\lambda^2(A^2+B^2)^3\\ &-\tfrac{3}{2}\lambda\bar{\psi}(A-i\gamma_5 B)^2\psi\,.\end{aligned}\tag{7}$$

This theory is invariant under a set of similar transformations:

$$\begin{aligned}&\delta A = \bar{\epsilon}\psi\,,\\ &\delta B = i\bar{\epsilon}\gamma_5\psi\,,\\ &\delta\psi = \not{\partial}(A+i\gamma_5 B)\epsilon - m(A+i\gamma_5 B)\epsilon - \lambda(A+i\gamma_5 B)^3\epsilon\,.\end{aligned}\tag{8}$$

In both models the transformations are nonlinear.

In the first model the following quantity, which depends on an arbitrary parameter τ, does not vanish on-shell:

$$\begin{aligned}H^1[\phi] = \int dx\,[&-\tfrac{1}{2}(\partial_\mu A)^2 - \tfrac{1}{2}(\partial_\mu B)^2 - \tfrac{1}{2}\bar{\psi}\not{\partial}\psi\\ &+\tfrac{1}{2}m^2(1-2\tau)(A^2+B^2) - \tfrac{1}{2}m\tau\bar{\psi}\psi\\ &+gm(1-\tau)A(A^2+B^2) + \tfrac{1}{2}g^2(A^2+B^2)^2]\,.\end{aligned}\tag{9}$$

It is straightforward to verify that this quantity is not invariant under the transformations (6). Its variation is given by

$$\begin{aligned}\delta H^1[\phi] = \int dx\,\bar{\epsilon}[&m(1-\tau)(A+i\gamma_5 B)\\ &+g(A+i\gamma_5 B)^2]\delta S/\delta\bar{\psi}\end{aligned}\tag{10}$$

which does vanish on-shell by virtue of the ψ-field equation:

$$\delta S/\delta\bar{\psi} = \not{\partial}\psi + m\psi + 2g(A-i\gamma_5 B)\psi\,.\tag{11}$$

The situation is similar in the second model. The quantity $H^2[\phi]$ given by

$$\begin{aligned}H^2[\phi] = \int dx\{&(A^2+B^2)[m^2(A^2+B^2) - (\partial_\mu A)^2 - (\partial_\mu B)^2 - \bar{\psi}\not{\partial}\psi]\\ &+2\lambda m(A^2+B^2)(A^4-B^4) + \lambda^2(A^2+B^2)^4 + m(A^2+B^2)\bar{\psi}\psi - 2mAB\,i\bar{\psi}\gamma_5\psi\\ &+\lambda(A^4-B^4)\bar{\psi}\psi - 2\lambda AB(A^2+B^2)i\bar{\psi}\gamma_5\psi + iA\overleftrightarrow{\partial}_\mu B\bar{\psi}\gamma_\mu\gamma_5\psi + \tfrac{1}{2}(\bar{\psi}\psi)^2\}\end{aligned}\tag{12}$$

does not vanish on-shell, but its variation does:

$$\delta H^2[\phi] = \int dx\,\bar{\epsilon}\{2(A^2+B^2)[m(A+i\gamma_5 B) + \lambda A(A^2-3B^2) + i\lambda\gamma_5 B(3A^2-B^2)] + \bar{\psi}\psi(A+i\gamma_5 B) + \bar{\psi}\gamma_5\psi(A+i\gamma_5 B)\gamma_5\}\delta S/\delta\bar{\psi}\,.\tag{13}$$

In this case the ψ-field equation is given by

$$\delta S/\delta\bar{\psi} = \not{\partial}\psi + m\psi + 3\lambda(A+i\gamma_5 B)^2\psi\,.\tag{14}$$

Clearly, the on-shell invariance of the background-field functionals does not exclude contributions proportional to H^1 or H^2. In fact H^1 (with $\tau=0$) and H^2 turn out to represent precisely the quantum divergences of these models in the one-loop approximation. Hence, the S matrix is infinite in both models, although in the first one the divergences can be absorbed by standard renormalization procedures. This is not the case in the second model, which is truly unrenormalizable. Thus, these examples show that to analyze the possible absence of infinities of the S matrix, it is not sufficient to argue that invariant (on-shell) counterterms cannot be constructed. For instance, using this type of argument would lead to the erroneous conclusion that the second model is on-shell finite (after a possible coupling-constant renormalization).

The occurrence of terms such as H^1 and H^2 is somewhat troublesome. Unlike the case of true invariants, which can usually be constructed in a systematic way be using some form of tensor calculus, there is no general construction available for noninvariant quantities with vanishing on-shell variations. The following argument shows that these quantities exist in many field theories: Assume that the action $S[\phi]$ is invariant under some group of transformations, with corresponding field variations $\delta\phi$. Hence, we may write

$$S_{,k}[\phi]\delta\phi_k = 0\,. \tag{15}$$

The first-order derivatives of the action with respect to the various parameters contained in this action (masses and coupling constants) are then in general no longer invariant (at least not for nonlinear transformations). However, their variations under the symmetry transformations are proportional to the field equations, as follows directly from differentiating Eq. (15) with respect to such parameters.

Although this argument shows that quantities such as (9) and (12) do exist in general, it only allows the construction of those quantities that are already generically present in the classical action. As mentioned above, a systematic procedure for generally constructing these noninvariant terms is lacking. At any rate, it is of more importance to understand under which circumstances such terms will actually contribute to the background-field functional (1), and in particular to its divergent part. Clearly, the nonlinearity of the transformations is important. As mentioned above, the standard invariance arguments are valid in the case of linear transformations, since the background-field functional is manifestly invariant, even off-shell.

In addition to nonlinearity, the transformations (6) and (8) exhibit a second property which might have some relevance to this matter, namely the nonclosure of the algebra of (infinitesimal) symmetry transformations: The commutator of two supersymmetry transformations, δ_1 and δ_2, yields a translation δ_T plus further field-dependent transformations δ', such that $\delta'\phi$ is proportional to the field equations. These transformations still leave the action invariant, without need to impose the field equations. Moreover, by considering repeated commutators, we are in general able to generate a large set of new field-dependent transformations, which all vanish on-shell, and still leave the action invariant off-shell. Because this whole set is constructed from the original symmetry transformations, every true invariant of the theory must be invariant under these new transformations as well. This severely restricts the possibility of having invariants other than the action in nontrivial theories.[7] Thus, in cases that the symmetry algebra fails to close, most of the contributions to the background-field functional (1) cannot be decomposed according to Eq. (4). Instead, they are generated by terms such as (9) and (12), which are not invariant for general background-field configurations, but only for those that satisfy the field equations. Therefore, in formulations with an open symmetry algebra, it is in principle not meaningful to discuss the finiteness of the S matrices within the context of invariants.

For the models discussed above, both the nonlinearity and the nonclosure of the transformations can be eliminated by introducing so-called auxiliary fields. The background-field functional, which then depends on the auxiliary fields as well, is manifestly invariant in that case, so that the quantum divergences can be fully discussed in the context of invariant counterterms.[4,6] However, there are theories with nonlinear symmetry transformations that have a closed algebra. In such cases we know of no arguments to determine in advance whether terms such as (9) and (12) will actually occur to represent the quantum divergences of the theory.

In the nonlinear σ model, where the nonlinear symmetry transformations close, the on-shell part of the one-loop divergences turns out to be a true invariant.[8] In supergravity theories where auxiliary-field formulations with a close algebra exist,[9] but where the transformations are still nonlinear, the actual situation is not known, and merits further investigation. The original discussion of the one-, two-, and three-loop counterterms[10] was not in the context of an auxiliary-field formulation and was mainly devoted to finding true invariants, although some attention was paid to the possibility that noninvariant on-shell counterterms of the type discussed above could exist. However, the linearized counterterms are not expected to lead to true invariants when completed to all orders in the gravitational coupling constant because of the nonclosure of the gauge algebra. This seems to suggest that the analysis presented without auxiliary fields is not complete.

In conclusion, as our examples have demonstrated, a discussion of the background-field functional based on the decomposition (4) may be misleading, and in particular arguments for the finiteness of the S matrix purely based on the absence of true (on-shell) invariants are incomplete. This is especially relevant in supergravity, where conventional methods to investigate the finiteness of the S matrix are difficult to apply. No actual indications that these complications will occur are known in low orders of perturbation theory, and up to the two-loop approximation independent arguments based on S-matrix invariance,[11,12] as well as explicit calculations,[11,13] have indeed demonstrated the finiteness of certain scattering amplitudes. Of course, the existence of auxiliary-field formulations and a corresponding tensor calculus[14] makes the analysis of quantum divergences much more transparent. Nonetheless, the nonlinearity of the supersymmetry transformations remains, so that the situation at the one- and two-loop level needs further clarification.

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