Regge-pole behavior from perturbative scalar-field theories

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In the planar approximation, we consider the large-energy-s, fixed-transfer-t limit of the four-point vertex function. In ϕ^3 theory (6 space-time dimensions) and in ϕ^4 theory (4 space-time dimensions), for any essentially and crossed planar graph, we analytically calculate the coefficients of all powers of logarithms of s for the leading power of s. After summing the series in logarithms obtained when the four-point function is considered, we discuss the existence of Regge trajectories from a Riccati-type differential equation. In $g(\phi^3)_6$ theory we find a discrete family of Regge-pole trajectories with g-dependent intercepts accumulating at $\alpha = -1$. In $g(\phi^4)_4$ theory the solution of the Riccati equation may be easily found if there exists a fixed point $g = g^*$; the result then is a fixed g-independent cut and an infinite number of pole trajectories (the square-root branch point is above the intercepts at $g = g^*$).

I. INTRODUCTION

In looking at the Regge behavior of field theories, one goal would be to elucidate the situation in quantum chromodynamics and to find out whether hadrons can be considered as bound states of quarks. A first step in this direction was presented in a preceding paper,[1] where the existence of Regge trajectories is proved in the case of a superrenormalizable field theory.

Such a perturbative approach raises many unsolved questions; the most important of them is the following: Does the perturbative series of Feynman graphs define an acceptable field theory, eventually with a nontrivial S matrix? Until this question could be answered, we tend towards a more physical theory by presenting here the second step, the case of renormalizable, but non-superrenormalizable field theories. This case has already been studied in many papers. Before we recall the current situation in the literature, let us summarize the set of new results we obtain:

For any planar (and crossed-planar) graph contributing to the four-point function, we give the complete asymptotic behavior at large s, fixed t, with the coefficients of all powers of ln$s$ [Eq. (3.17)].

The geometric organization of these coefficients and the use of renormalization-group techniques allow us to perform the summation over the whole set of these contributions and to get a general formula generating the possible Regge trajectories [Eqs. (5.1) and (5.2)].

The actual existence of Regge poles and cuts is discussed from a nonlinear Riccati-type equation [Eq. (5.3)], which for this asymptotic limit plays the same role that the Callan-Symanzik equation plays in the scaling limit.

The technique used in this work is based on a desingularization operator, called the R operator, and which has already been used in the literature to take care of ultraviolet divergences,[2] to define the analytic continuation in dimension of Feynman amplitudes,[3] to describe scaling behavior of Feynman amplitudes,[4] to determine the Bjorken limit of Feynman amplitudes,[5] etc. Such an operator acts directly on the integrand of the Feynman amplitude or on the integrand of its Mellin transform adapted to a given asymptotic limit, written in the α-parametric language. This operator, which is an extension of the renormalization technique of Appelquist,[6] and is in α-space what Zimmermann's[7] R operation is in momentum space, achieves in compact form the recurrent procedure of subtraction initiated by Bogoliubov and Parasiuk,[8] reexamined later by Hepp.[9] It turns out that the same operator which is able to take care of ultraviolet divergences also allows in its original form the investigation of many asymptotic behaviors and presumably all of them in a generalized form. One generalization to be found should take care of nonplanar amplitudes and lead to the study of Regge cuts.

Let us now try to summarize and classify the very large literature which deals with Regge behavior in quantum field theory. We observe mainly two different kinds of approaches to this problem.

First, the structure of the Bethe-Salpeter integral equation is used and applied in the early papers to evaluate the asymptotic behavior of the ladder graphs in $g(\phi^3)_4$ theory,[10] and later in $g(\phi^4)_4$ theory.[10] Although the existence of Regge trajectories was proved in the case of $(\phi^3)_4$ theory, the presence of square-root branch points was discovered[11] in $(\phi^4)_4$ theory. Then many papers using Fredholm-type techniques dealt with the description of the trajectory obtained from ladder graphs in $(\phi^3)_4$ theory.[12] More recently, people realized that ultraviolet divergences have to play an important role in strictly renormalizable field theories. Lovelace[13] and Cardy[14] introduced an asymptotic kernel in the Bethe-Salpeter equation, and after partial-wave expansion found that in $g(\phi^3)_6$ theory the integral equation was of the Fredholm type (existence of Regge trajectories), while it was not the case in $g(\phi^4)_4$ theory because of the presence of a square-root branch point. Lovelace was more explicit and proved that (-1) was an accumulation point of g-independent intercepts in $g(\phi^3)_6$ theory, and the square-root branch point was g-independent in $g(\phi^4)_4$ theory. Although this approximation is better than the "leading-logarithm approximation," it is incorrect to believe that the asymptotic behavior of the complete four-point function is the solution of the Bethe-Salpeter equation with the asymptotic kernel. For $g(\phi^3)_6$ theory Lovelace found[13] g-independent intercepts (owing to an asymptotic kernel which reduces to the Born term of the perturbative expansion with an effective coupling constant and a zero effective mass), while our solution gives g-dependent intercepts.

In fact, the Callan-Symanzik equation[15] may be used to find the large-s behavior of the four-point vertex function at $t=p_i{}^2=0$, since s and m^2 are the only dimensioned variables; of course, we are in the case of exceptional momenta[15] and the right-hand side of the equation is not negligible. One purpose of this paper is to evaluate this right-hand side. It is found that in the Mellin transform space (where products correspond to convolutions in logs), the Callan-Symanzik equation has to be replaced by a Riccati differential equation. The solutions to this equation may have g-dependent poles which define the intercepts of the trajectories, or in the case of a nontrivial ultraviolet attractive fixed point $g=g^*$, g-independent square-root branch points. Now at t and $p_i{}^2$ different from zero, trajectories are generated essentially from the nonleading powers of logarithms. In the presence of a fixed point $g=g^*$, an anomalous behavior persists and we obtain

$$s^{\text{const}}[\beta_0(t)\log^{-3/2}s+\beta_1(t)\log^{-5/2}s+\cdots],$$

where the constant is g and t independent. This fixed cut is accompanied by Regge trajectories, and although the branch point dominates over the intercepts of the trajectories at $g=g^*$, it is not known, for t away from zero and g different from g^*, whether these relative locations persist.

The second technique which is largely seen in the current literature is to look for the asymptotic behavior of independent Feynman graphs. Some authors tried to obtain precise rules to determine the leading power of s. First, in ϕ^3 theory it was found[16] that this power has to do with the length of the shortest path from one side to the opposite side of the diagram (if planar), but very soon these rules had to be modified because of divergent subgraphs. To our knowledge, it was Zavyalov[17] and Zavyalov-Stepanov[17] who first gave the exact rules to obtain the leading power of s for any (convergent or divergent) graph, and who generalized this rule to give the largest power of logarithm of s. They introduced the notion of what we call essential subgraphs $\mathcal{S}$ (such that the reduced graph $[G/\mathcal{S}]$ looses its s dependence) and of what we call leading subgraphs (those essential subgraphs with largest superficial degree Ω of divergence). The leading power is then Ω, and the largest power of logarithm of s is related to the largest number of nonoverlapping leading subgraphs.

These rules known, the next problem is to sum the logarithms. The first attempt has been to sum the leading powers of logarithms for special classes of graphs. Again, the ladder graphs are privileged; for the ladders of $(\phi^3)_4$ theory leading logs and subleading logs are summed,[18] and finally all powers of logs are summed[19] to obtain the Regge trajectory. In $(\phi^4)_4$ theory and in $(\phi^3)_6$ theory leading logs of ladder graphs are summed[20] to obtain a g-dependent square-root branch point which is even confirmed if the same technique is applied to the so-called truss-bridge graphs.[21] More general results are also obtained in Ref. 22. Another class of graphs are also privileged in this analysis of leading logarithms which is the class of graphs generated at low orders of perturbation. In particular, in the study of non-Abelian gauge fields, the perturbation has been explored up to the 12th order,[23] and except for Ref. 24 the results do not seem to indicate the presence of Regge trajectories in quantum chromodynamics.

Our result differs from the current results presented in the literature because we solve completely and analytically the problem of finding the coefficients of all powers of logarithms. The structure of these coefficients is quasigeometric; the coefficients are built from the knowledge of the so-called leading subgraphs organized into forests (set of nonoverlapping subgraphs) and they factorize into absolutely convergent Feynman-type integrals attached to reduced subgraphs defined from the forests. This structure explains why the summation of all the logarithms is possible. It is easy to understand why the "leading-logarithm approximation" cannot describe Regge trajectories; the leading power of logarithms is given by the forests which have the largest number of elements. Since the complete graph G is itself a leading subgraph (among others), the forests with the largest number of elements necessarily contain the complete graph G. The corresponding coefficient which is attached to the reduced graph G into itself $[G/G]$ is really attached to the trivial Born-term graph and thus has no t and p_i^2 dependence. The leading-logarithm approximation can only provide an asymptotic behavior of the type $s^{\alpha(g)}$ up to a possible negative power of log$s$ and has very little to do with the exact behavior.

The end of this section is devoted to the introduction of technical definitions such as "renormalized Feynman amplitude" in parametric representation, or to the classification of graphs with regard to their topology. We also state a general theorem on asymptotic behavior of Feynman amplitudes and discuss its relevance to our problem. In Sec. II we introduce the Mellin transform of a Feynman amplitude with respect to the large variable s and determine the leading power of s, the essential and leading subgraphs, and the source of logarithms. In Sec. III we introduce the desingularization operator R and compute the coefficients of all power of logarithms for any essentially and crossed-planar graph. In Sec. IV we sum all the logarithms over all essentially and crossed-planar graphs. We first obtain the equation which determines the possible trajectories, and second, the Riccati-type differential equation which gives the possible intercepts (or the square-root branch points in the case of the existence of a fixed point $g=g^*$). In Sec. V we discuss the existence of trajectories in $(\phi^3)_6$ theory and in $(\phi^4)_4$ theory; all results are gathered in the Conclusion. We now quickly remind the reader of a certain number of important properties related to Regge behavior in field theory, and which have been already exposed in detail in Ref. 1.

(*a*) *Classifications of the graphs with respect to their topological properties*. The connected four-point function $G^c_{(4)}(p_i, m, g)$ is a function of the invariants p_i^2 and of the Mandelstam variables

$$\begin{aligned} s &= (p_1+p_2)^2\,, \\ t &= (p_1+p_3)^2\,, \\ u &= (p_1+p_4)^2\,, \end{aligned} \tag{1.1}$$

where the external momenta are ingoing and such that

$$s+t+u=\sum_{i=1}^{4} p_i^2\,. \tag{1.2}$$

Any two-tree of a graph G splits the external momenta into two parts and therefore is specific for one of the seven invariants:

the graphs contributing to $G^c_{(4)}$ and which contain only p_i^2 and t-two-trees contribute to $G^1_{(4)}(t, p_i^2, m, g)$,

the graphs contributing to $G^c_{(4)}$ and which contain at least one s-two-tree but no u-two-trees contribute to $G^2_{(4)}(s, t, p_i^2, m, g)$,

the graphs contributing to $G^c_{(4)}$ and which contain at least one u-two-tree but no s-two-trees contribute to $G^3_{(4)}(t, u, p_i^2, m, g)$,

the graphs contributing to $G^c_{(4)}$ and which contain at least one s and at least one u-two-trees contribute to $G^4_{(4)}(s, t, u, p_i^2, m, g)$.

The graphs contributing to $G_4^{(1)}$ and $G_4^{(2)}$ are called essentially planar graphs. The graphs contributing to $G^3_{(4)}$ are called crossed-planar graphs and they are obtained from the graphs of $G^2_{(4)}$ by exchanging the external legs p_2 and p_4. The graphs contributing to $G^4_{(4)}$ have a third double spectral function and are susceptible to generating moving Regge cuts.[19]

(*b*) *The renormalized Feynman amplitude*. We find it convenient to define a Feynman amplitude by its Schwinger-integral representation

$$I_G^\epsilon = -(-g)^n (ie^{-i\epsilon})^{-\omega(G)/2} \int_0^\infty \prod_{a=1}^{l(G)} d\alpha_a \exp\left[-\left(ie^{-i\epsilon}\sum_{a=1}^{l(G)}\alpha_a m^2\right)\right] R\left(\frac{\exp\{ie^{-i\epsilon}[k_i(\epsilon)d_{ij}(\alpha)k_j(\epsilon)]\}}{P_G{}^{D/2}(\alpha)}\right), \tag{1.3}$$

where $k(\epsilon)=(k_0 e^{i\epsilon}, \vec{k})$.

The Euclidean-space amplitude is obtained at $\epsilon = \pi/2$ (with the negative metric) and the Minkowski-space amplitude is the limit $\epsilon \to 0$ of I_G^ϵ and is known to be a distribution. In this paper we purposely failed to write the ϵ dependence of I_G^ϵ and we assume, at least for essentially planar graphs, that the limits $s\to\infty$ and $\epsilon\to 0$ commute. The superficial degree of divergence of G is

$$\omega(G) = 4L(G) - 2l(G)\,, \tag{1.4}$$

where $L(G)$ and $l(G)$ are, respectively, the number of independent loops and lines of G. $\omega(G)$ is found to be zero for all four external-leg graphs of $(\phi^4)_4$ theory and minus two for the corresponding graphs of $(\phi^3)_6$ theory. The functions $d_{ij}(\alpha)$ and $P_G(\alpha)$ are characteristic of the topology of the graph; the dimension of space-time is called D [6 in $(\phi^3)_6$ theory and 4 in $(\phi^4)_4$ theory].

The operator R (Ref. 2) is a subtraction operator which acts directly upon the variables α and ensures the ultraviolet convergence. We define

$$R = \prod_{S\subseteq G} (1 - \tau_S{}^{-2l(S)})\,, \tag{1.5}$$

where the operators τ are generalized Taylor operators, $l(\mathcal{S})$ is the number of lines of the subgraph $\mathcal{S}$, and the product runs over the $(2^{l(G)}-1)$ subgraphs of G. Another expression for R is (Ref. 2)

$$R=1+\sum_{\mathfrak{F}}\prod_{\mathcal{S}\in\mathfrak{F}}(-\tau_{\mathcal{S}}{}^{-2l(\mathcal{S})}), \tag{1.6}$$

where $\mathfrak{F}$ is any Zimmermann's forest[7] of proper (connected, one-particle irreducible) divergent subgraphs. The generalized Taylor operators τ are defined as follows: Given a function $f(x)$ such that $x^{-\nu}f(x)$ is infinitely differentiable for ν complex, then

$$\tau_x{}^n f(x)=x^{-\lambda-\epsilon}T^{n+\lambda}[x^{\lambda+\epsilon}f(x)]. \tag{1.7}$$

This definition is λ independent provided that $\lambda \geq -E'(\nu)$ where $E'(\nu)$ is the integer part of Reν and $E'(\nu)\geq$ Reν, $\epsilon=E'(\nu)-\nu$. This definition is generalizable to the case of several variables α:

$$\tau_{\mathcal{S}}{}^n f(\alpha)=[\tau_\rho{}^n f(\alpha)|_{\alpha_a\to\rho^2\alpha_a,\ \forall a\in\mathcal{S}}]_{\rho=1}. \tag{1.8}$$

(*c*) *Description of the method used to find the large-s behavior of* $G^2_{(4)}(s,t,p_i{}^2,m,g)$. First, we examine the large-s behavior for a given Feynman amplitude. The following theorem has been proved[25]:

$I_G(s,t,p_i{}^2,m,g)$ has an asymptotic behavior at large s of the type

$$I_G(s,t,p_i{}^2,m,g)=\sum_{p=p_{\max}}^{-\infty} s^p \sum_{q=0}^{q_{\max}(p)} (\ln^q s) g_{pq}(t,p_i{}^2,m,g), \tag{1.9}$$

where p runs over the rational values of a decreasing arithmetic progression and q, for a given p, over a finite number of non-negative integers.

This theorem, which is a generalization of Weinberg's theorem[26] on scaling, has now been extended to a very wide class of asymptotic behaviors (all asymptotic behaviors of renormalized Feynman amplitudes in Euclidean space and those asymptotic behaviors in Minkowski space, where Landau singularities are only spectators of the behaviors without contributing specifically). The above theorem can be applied to all graphs contributing to $G^i_{(4)}$ with $i=1$, 2, and 3 by crossing, but fails for the graphs contributing to $G^{(4)}_4$ because Landau singularities are responsible for moving Regge cuts. We define

$$I^{as}_G(s,t,p_i{}^2,m,g) = s^{p_{\max}} \sum_{q=0}^{q_{\max}(p_{\max})} (\ln^q s) g_{p_{\max}q}(t,p_i{}^2,m,g). \tag{1.10}$$

The first problem to solve is to determine $p_{\max}$, $q_{\max}(p_{\max})$, and the coefficients $g_{p_{\max}q}$ for all q. The technique is to introduce the Mellin transform $M_G(x)$ of $I_G(s)$ with respect to s. The function $M_G(x)$ is initially defined in a certain analyticity domain in x and the right-most singularity on the left (leading singularity) is at $x=p_{\max}$. The expansion (1.10) corresponds to a meromorphic function $M_G(x)$ and the multiplicity of the leading singularity is related to $q_{\max}(p_{\max})$. The residues of the poles determine the coefficients g_{pq}. We now explain how we may obtain the residues for the leading singularity (that is, the coefficients $g_{p_{\max}q}$ for all q).

Two cases may appear. Either the integrand of the Mellin transform expressed in the variables α has a "simultaneous Taylor expansion" in every Hepp's sector defined as an ordering of the variables α; then the operator R (Ref. 2) defines an analytic continuation of the Mellin transform beyond the leading singularity and extracts the residue at the pole. Let us mention that the property of Taylor expansion in every Hepp's sector is equivalent to the validity of the naive power counting (as it is used for ultraviolet divergences[2] and for scaling asymptotic behavior[4]). Or, the integrand does not have a simultaneous Taylor expansion in every Hepp's sector [which is the case for Regge behavior, zero-mass limit of gluons (photons) in quantum chromodynamics (QCD) (QED) when fermions are on their mass-shell, etc.]. We then use a multiple Melling transform which is the sum of analytic functions in tubes, the real part of which are convex polyhedrons.[25] The asymptotic behavior is then obtained from an extremal point of the polyhedrons. A simple example of this method is exposed in the appendices of Ref. 1.

Once we have obtained the coefficients $g_{p_{\max}q}$, that is, I^{as}_G, we sum all the logarithms over all graphs G contributing to $G^{1,2,3}_{(4)}$ in order to obtain the asymptotic behavior of $G^c_{(4)}$ in the planar approximation. Let us point out that we assume in this result that the infinite summation of logarithms of subleading powers ($p<p_{\max}$) does not destroy the leadership of the expression ($\sum_G I^{as}_G$); this is an assumption similar to neglecting the right-hand side of the Callan-Symanzik equation in the case of scaling at nonexceptional momenta.

II. ESTIMATION OF THE LEADING POWER IN s FOR A GRAPH CONTRIBUTING TO $G^2_{(4)}$

To any graph contributing to $G^2_{(4)}$ there corresponds a graph contributing to $G^3_{(4)}$, and the estimation in s of the first amplitude is the same as the estimation in u of the second one. We thus restrict our discussion to a graph of $G^2_{(4)}$.

The quadratic form $[k_i d_{ij}(\alpha)k_j]$ in (1.3) can be written

$$sA_s(\alpha)+tA_t(\alpha)+\sum_{i=1}^{4}p_i{}^2A_i(\alpha). \tag{2.1}$$

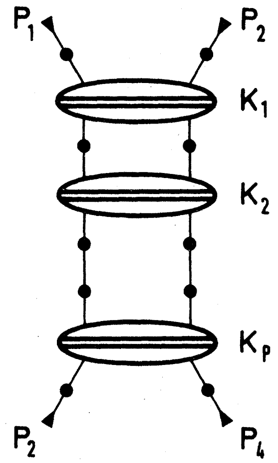

FIG. 1. A graph which contributes to $G^2_{(4)}$.

The function $A_s(\alpha)$ is the ratio $N_s(\alpha)/P_G(\alpha)$ of two polynomials and is homogeneous in all α's of degree 1. We now give an expression for $N_s(\alpha)$.

We define an s-cut as a set of lines such that if these lines are cut, the graph G becomes two-connected with one connected part containing the external legs p_1 and p_2, the other connected part containing the external legs p_3 and p_4, and such that no subset of lines has the same property. An s-cut defines two connected subgraphs G_L and G_R and we have

$$N_S(\alpha) = \sum_{\{s\text{-cuts}\}} \prod_{\{a \in s\text{-cuts}\}} \alpha_a P_{G_L}(\alpha) P_{G_R}(\alpha) . \qquad (2.2)$$

If the graph G is drawn as in Fig. 1, where each black dot represents a graph which contributes to the complete propagator and where each kernel K_i represents a graph two-line irreducible in the t channel, then

$$N_s(\alpha) = \prod_{i=1}^{P} N_{K_i}(\alpha) \prod_{\text{dots}} P_{\text{dot}}(\alpha) . \qquad (2.3)$$

Given a subgraph $\mathcal{S}$ with $\chi_{\mathcal{S}}$ connected parts and an s-cut c, this s-cut splits $\mathcal{S}$ into two subgraphs $\mathcal{S}_L$ and $\mathcal{S}_R$ with, respectively, $\chi_{\mathcal{S}_L}$ and $\chi_{\mathcal{S}_R}$ connected parts (some of them being eventually reduced to single vertices). From topological considerations it is easy to show that when all α variables corresponding to lines of $\mathcal{S}$ vanish like ρ, the expression

$$A_c(\alpha) = \prod_{\{a \in c\}} \alpha_a P_{G_L}(\alpha) P_{G_R}(\alpha) / P_G(\alpha) \qquad (2.4)$$

vanishes like $\rho^{\mathcal{Y}_c(\mathcal{S})}$ where

$$\mathcal{Y}_c(\mathcal{S}) = \chi_{\mathcal{S}_L} + \chi_{\mathcal{S}_R} - \chi_{\mathcal{S}} . \qquad (2.5)$$

Consequently, $A_s(\alpha)$ vanishes like $\rho^{\mathcal{Y}(\mathcal{S})}$ where

$$\mathcal{Y}(\mathcal{S}) = \operatorname*{Inf}_{\{c\}} \mathcal{Y}_c(\mathcal{S}) . \qquad (2.6)$$

A subgraph $\mathcal{S}$ is said to be essential if $\mathcal{Y}(\mathcal{S}) \geq 1$. An essential subgraph is such that the reduced subgraph $[G/\mathcal{S}]$ obtained from G when $\mathcal{S}$ is shrunk into $\chi_{\mathcal{S}}$ points has an s-independent Feynman amplitude. Let us note that all essential subgraphs in $(\phi^3)_6$ theory are convergent while there exist logarithmically divergent essential subgraphs in $(\phi^4)_4$ theory.

The Mellin transform of $I_G(s)$ with respect to s is defined as

$$M_G(x) = \int_0^\infty ds\, s^{-x-1} I_G(s) \qquad (2.7)$$

in $(\phi^3)_6$ field theory and

$$M_G(x) = \int_0^\infty ds\, s^{-x-1} [I_G(s) - I_G(0)] \qquad (2.8)$$

in $(\phi^4)_4$ field theory. The reason for this subtraction is that the Mellin transform (2.7) does not exist in $(\phi^4)_4$ field theory because of ultraviolet renormalization. The subtractions over logarithmically divergent essential subgraphs make the integrand of $I_G(s)$ the sum of a function exponentially decreasing at large s and of an s-independent constant which is not allowed in (2.7) and which is conveniently subtracted in (2.8).

In the primitive region of x where $M_G(x)$ is defined (see below) we may interchange the s integration with the α integrations. We may also interchange the s integration with the subtraction R operator. The reason for this is that in $(\phi^3)_6$ theory and in $(\phi^4)_4$ theory once the subtraction (2.8) is performed, the Taylor operators τ included in R subtract only nonessential divergent subgraphs. We are left with the integral

$$\int_0^\infty ds\, s^{-x-1} (e^{isA_s(\alpha)} - \delta) = \Gamma(-x) e^{-i\pi x} [iA_s(\alpha)]^x , \qquad (2.9)$$

where $\delta = 0$ in $(\phi^3)_6$ theory and $\delta = +1$ in $(\phi^4)_4$ theory. The above integral exists for $\operatorname{Re} x < 0$ in $(\phi^3)_6$ theory and for $0 < \operatorname{Re} x < 1$ in $(\phi^4)_4$ theory, provided that we use the $+i\epsilon s$ rule defined in (1.3) for I_G^ϵ. In (2.9), i^x is $\exp(i\pi x/2)$. The combination $[+iA_s(\alpha)]$ is chosen in such a way that the α integrals are real in the Euclidean region and the factor $e^{-i\pi x}$ is reminiscent of the fact that the amplitude has a cut in the complex s plane for s larger than the first threshold. The Mellin transform $M_G(x)$ is thus found to be

$$M_G(x) = -(-g)^n (i)^{-\omega(G)/2} \Gamma(-x) e^{-i\pi x} \int_0^\infty \prod d\alpha \exp\left(-i \sum \alpha m^2\right) R \left\{ \frac{[iA_s(\alpha)]^x \exp[i(tA_t(\alpha) + \sum_{i=1}^4 p_i^2 A_i(\alpha))]}{P_G^{D/2}(\alpha)} \right\} \qquad (2.10)$$

in the region

$$p_{\max} < \mathrm{Re}\,x < 0 \quad \text{for } (\phi^3)_6 \text{ theory},$$
$$\sup\{0, p_{\max}\} < \mathrm{Re}\,x < 1 \quad \text{for } (\phi^4)_4 \text{ theory}. \tag{2.11}$$

In this band of definition the subtraction operator in (2.10) subtracts only nonessential divergent subgraphs.

In the remaining part of this section we wish to find $p_{\max}$, $q_{\max}(p_{\max})$, and the leading subgraphs defined as those essential subgraphs which are responsible for a pole at $x = p_{\max}$. As already mentioned in part (c) of the Introduction, the derivation of these results needs the introduction of multiple Mellin transform, Hepp's sectors, equivalent classes of nested subgraphs, convex polyhedrons, etc., as can be read, for instance, in the appendices of Ref. 1 for the Regge-pole behavior of $(\phi^3)_4$ theory. We prefer to postpone the writing of such a lengthy proof to some later time and simply give the results which are used in Sec. III:

(a) $p_{\max} = -1$ in $(\phi^3)_6$ theory,

$$p_{\max} = 0 \quad \text{in } (\phi^4)_4 \text{ theory}. \tag{2.12}$$

(b) In contrast to what happens in $(\phi^3)_4$ theory, all graphs in $(\phi^3)_6$ theory and in $(\phi^4)_4$ theory contributing to $G^2_{(4)}$ generate a singularity at $x = p_{\max}$. In fact, the connected leading subgraphs are the four-external-leg essential subgraphs shown in Fig. 2 up to G itself.

(c) In addition to the singularities at $x = p_{\max}$ which are generated by leading subgraphs, the subtraction R operator is responsible for other pole singularities at $x = p_{\max}$ associated with proper divergent nonessential subgraphs.

(d) If ν is the number of subgraphs in the largest forest (set of nonoverlapping subgraphs) of connected leading and of proper divergent nonessential subgraphs such that the maximal elements of the forest are leading, then $q_{\max}(p_{\max})$ is $\nu - 1 + \delta$, where δ is zero in $(\phi^3)_6$ theory and +1 in $(\phi^4)_4$ theory.

(e) Although the integrand of $M_G(x)$ does not have a simultaneous Taylor series in every Hepp's sector, it may be shown from the structure of the convex polyhedrons[1,25] that the R operator defines an analytic continuation of $M_G(x)$ for $\mathrm{Re}\,x < p_{\max}$ and extracts the residue of the pole at $x = p_{\max}$.

The results (a), (b), (c), (d) were known from the work of Zavyalov and Stepanov,[17] but their derivation relies on naive power counting. Let us finally mention that the above results describe how $A_s(\alpha)$ vanishes when a subset of variables α vanish. $A_s(\alpha)$ is strictly positive when the α's are all strictly positive (this should not be the case for a graph contributing to $G^4_{(4)}$, where $A_s(\alpha)$ have to be replaced by $[A_s(\alpha) - A_u(\alpha)]$ which vanishes inside the α domain of integration).

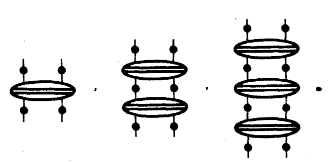

FIG. 2. The leading subgraphs.

III. ASYMPTOTIC EXPANSION OF FEYNMAN AMPLITUDES ASSOCIATED WITH ESSENTIALLY AND CROSSED-PLANAR GRAPHS

In this section we calculate the coefficients $g_{p_{\max}q}(t, p_i^{\,2}, m, g)$ for any q and for all the graphs contributing to $G^2_{(4)}$ and $G^3_{(4)}$. We apply the property (e) of Sec. II to define a function $\overline{M}_G(x)$ which in the band

$$a < \mathrm{Re}\,x < p_{\max} \tag{3.1}$$

takes on the form (2.10). This function is different from the analytic continuation of $M_G(x)$ in this band because the subtraction operator R in $\overline{M}_G(x)$ subtracts not only the nonessential divergent subgraphs as in $M_G(x)$ but also the leading subgraphs. What property (e) of Sec. II means is that $\overline{M}_G(x)$ is analytic in x for

$$a < \mathrm{Re}\,x < b\,, \quad p_{\max} < b\,. \tag{3.2}$$

We do not intend to be precise for a and b; it is important to note that $\overline{M}_G(x)$ has no singularity at $x = p_{\max}$ and that the pole structure at $x = p_{\max}$ appears in the difference $M_G(x) - \overline{M}_G(x)$, which may be computed, for instance, in the common band of analyticity $p_{\max} < \mathrm{Re}\,x < \mathrm{Inf}[b, \delta]$ with $\delta = 0$ in $(\phi^3)_6$ theory and +1 in $(\phi^4)_4$ theory. The R operator in $\overline{M}_G(x)$ may be written as a sum over all forests of connected leading and of proper divergent, nonessential subgraphs. We must compute

$$\sum_{\mathfrak{F}}\left[\prod_{S\in\mathfrak{F}}\left(-\tau_S^{-2l(S)}\right) - \prod_{S\in\mathfrak{F}}\left(-\tau_S^{-2l(S)+\delta(S)}\right)\right]\{\ \}\,, \tag{3.3}$$

where $\delta(\mathcal{S})$ is +2 for connected leading subgraphs and zero otherwise.

The curly brackets $\{\ \}$ in (3.3) is the same as in (2.10). The calculation (3.3) is now familiar to us and is a generalization of the technique we use to prove Zimmermann's identity.[4] We give the main steps of this calculation in Appendix A. We obtain

$$M_G(x) = \overline{M}_G(x) - \Gamma(-x)e^{-i\pi x} \times \sum_{\{S_i\}}\prod_i F_{S_i}(x)\overline{F}_{[G/\cup S_i]}(x, t, p_i^{\,2})\,. \tag{3.4}$$

In (3.4) we sum over all sets of disjoint connected leading subgraphs $\mathcal{S}_i$. The functions $F_{S_i}(x)$ attached to the subgraphs $\mathcal{S}_i$ are defined as

$$F_{S_i}(x)=(-g)^{n(S_i)}(i)^{-\omega(S_i)/2}\int_0^\infty \prod d\alpha \exp\left(-i\sum \alpha m^2\right) R\left\{\frac{[iA_{S_i}(\alpha)]^x}{P_{S_i}{}^{D/2}(\alpha)}\right\}, \tag{3.5}$$

where $A_{S_i}(\alpha)$ is the function $A_s(\alpha)$ for the graph $\mathcal{S}_i$, and where the R operator subtracts only the divergent nonessential subgraphs. The function $\bar{F}_{[G/US_i]}$ attached to the reduced graph $[G/U\mathcal{S}_i]$ is defined as

$$\bar{F}_{[G/US_i]}(x,t,p_i{}^2)=(-g)^{n(G)-\Sigma_i n(S_i)}(i)^{-[\omega(G)-\Sigma_i\omega(S_i)]/2}$$
$$\times\int_0^\infty \prod d\alpha \exp\left(-i\sum \alpha m^2\right) R\left(\frac{[(i)^{1-\nu_{S_i}}A_{[G/US_i]}(\alpha)]^x}{P_{[G/US_i]}{}^{D/2}(\alpha)}\exp\left[i\left(tA_t(\alpha)+\sum_{i=1}^4 p_i{}^2A_i(\alpha)\right)_{[G/US_i]}\right]\right), \tag{3.6}$$

where the bar above F means that the subtraction operator R subtracts not only the divergent nonessential subgraphs but also the leading subgraphs which remain in $[G/U\mathcal{S}_i]$. The factors i are written in such a way that the amplitudes are real in the Euclidean region. In (3.5), because of the absence of external momentum, we may omit all factors i by Wick rotation. There are no singularities in x in the band (3.2) for the function $\bar{F}$. On the other hand, the functions $F_{S_i}(x)$ contain all the singularities at $x=p_{\max}=\frac{1}{2}\omega(\mathcal{S}_i)$. It is easy to show by homogeneity that

$$F_{S_i}(x)=\frac{1}{x-p_{\max}}\hat{F}_{S_i}(x), \tag{3.7}$$

where

$$\hat{F}_{S_i}(x)=(-g)^{n(S_i)}\int_0^\infty \prod d\alpha \exp\left(-\sum \alpha m^2\right)\left(\sum\alpha m^2\right) R\left(\frac{[A_{S_i}(\alpha)]^x}{P_{S_i}{}^{D/2}(\alpha)}\right). \tag{3.8}$$

The function $\hat{F}_{S_i}(x)$ is now attached to the subgraph $\mathcal{S}_i$ with all possible "hard-mass" insertions and still contains singularities at $x=p_{\max}$ because of leading subgraphs of $\mathcal{S}_i$ and because of nonessential divergent subgraphs of $\mathcal{S}_i$. To desingularize $F_{S_i}(x)$ completely around $x=p_{\max}$ we first intend to find a recursive formula between $F_{S_i}(x)$ and similar functions attached to its leading subgraphs. First, we use Zimmermann's identity, as given in Appendix A, to transform "hard-mass" insertions into "soft-mass" insertions:

$$\hat{F}_{S_i}(x)=\tilde{F}_{S_i}(x)+\sum_T \tilde{\beta}_T^{\chi_{\omega(T)}}F_{[S_i/T]_{\chi_{\omega(T)}}}(x), \tag{3.9}$$

where we sum over all proper, divergent, nonessential subgraphs T of $\mathcal{S}_i$. The function $\tilde{F}_{\ i}(x)$ is now given as

$$\tilde{F}_{S_i}(x)=(-g)^{n(S_i)}\int_0^\infty \prod d\alpha \exp\left(-\sum \alpha m^2\right) R\left(\frac{(\sum\alpha m^2)[A_{S_i}(\alpha)]^x}{P_{S_i}{}^{D/2}(\alpha)}\right). \tag{3.10}$$

The coefficients $\tilde{\beta}_T^{\chi_{\omega(T)}}$ are the coefficients obtained when we calculate the scaling (all momentum scaled to infinity) asymptotic behavior,[4] and they are momentum, mass, and x independent. They are defined as

$$\tilde{\beta}_T^{\chi_{\omega(T)}}=\frac{1}{\omega(T)!}\left.\frac{\partial\tilde{I}_T(p,m)}{\partial p_{i_1}\cdots\partial p_{i_{\omega(T)}}}\right|_{p_i=0}, \tag{3.11}$$

where $\omega(T)$ is the superficial degree of divergence of T and $\chi_{\omega(T)}$ is a sequence of $\omega(T)$ external legs to T. $\tilde{I}_T$ is the Feynman amplitude attached to T with all possible soft-mass insertions. Finally $[\mathcal{S}_i/T]_{\chi_{\omega(T)}}$ is the reduced graph obtained from $\mathcal{S}_i$ by shrinking the subgraphs T into a point and by inserting $\omega(T)$ derivative couplings on the legs defined in the contracted point by $\chi_{\omega(T)}$. In (3.9) we sum implicitly over all possible sequences $\chi_{\omega(T)}$. By definition the function $F_{[S_i/T]}$ contains a factor $(-g)^{n(S_i)-n(T)}=(-g)^{n(S_i/T)-1}$.

Second, we calculate $\tilde{F}_{S_i}(x)-F^+_{S_i}(x)$ in exactly the same way as in the calculation of $M_G(x)-\bar{M}_G(x)$, the superscript plus meaning simultaneously that the corresponding integral has a soft mass insertion and has been subtracted to be regular at $x=p_{\max}$. We obtain the recursive formula

$$F_{S_i}(x)=\frac{1}{x-p_{\max}}\left[F^+_{S_i}(x)+\sum_{\{S_j\}}\prod_j F_{S_j}(x)F^+_{[S_i/US_j]}(x)+\sum_T\tilde{\beta}_T^{\chi_{\omega(T)}}F_{[S_i/T]_{\chi_{\omega(T)}}}(x)\right]. \tag{3.12}$$

The sum $\{\mathcal{S}_j\}$ is over all sets of disjoint, connected, leading subgraphs inside $\mathcal{S}_i$. This recursive relation which associates the same type of functions $(F_{S_i}, F_{S_j}, F_{[S_i/T]_{\chi_{\omega(T)}}})$ can be solved step by step, in terms of functions $\tilde{\beta}$ and F^+ regular at $x=p_{\max}$. We obtain

$$F_{S_i}(x)=\sum_{\mathfrak{F}} \chi_{\mathfrak{F}} \prod_{T\in\mathfrak{F}} \frac{\tilde{\beta}_{[T]_{\mathfrak{F}}}^{\chi_{\omega(T)}}}{x-p_{\max}} \prod_{S\in\mathfrak{F}} \frac{F^+_{[S]_{\mathfrak{F}}}(x)}{x-p_{\max}} \frac{F^+_{[S_i]_{\mathfrak{F}}}(x)}{x-p_{\max}}. \tag{3.13}$$

In (3.13) we sum over all forests $\mathfrak{F}$ of connected leading subgraphs and of proper divergent nonessential subgraphs inside $\mathcal{S}_i$. Each of these subgraphs contributes for one power of $(x-p_{\max})^{-1}$. The numerical coefficient $\chi_{\mathfrak{F}}$ is found to be

$$\chi_{\mathfrak{F}}=\frac{\mathcal{R}_{\mathfrak{F}(S)}[p_S;p_{S_1},\ldots,p_{S_n}]}{\prod_T[\nu(T)+1]}, \tag{3.14a}$$

where the subgraphs T are the proper divergent nonessential elements of $\mathfrak{F}$, $\nu(T)$ is the number of proper divergent nonessential elements inside T, $\mathfrak{F}(\mathcal{S})$ is the forest of connected leading subgraphs $\mathcal{S},\mathcal{S}_1,\ldots,\mathcal{S}_n$ induced by $\mathfrak{F}$ and such that $\mathcal{S}$ is the unique maximal element [$\mathcal{S}_i$ in Eq. (3.13)], and p_S, $p_{S_1},\ldots,p_{S_n}$ are respectively the numbers of proper, divergent nonessential subgraphs in $[\mathcal{S}]_{\mathfrak{F}(S)}$, $[\mathcal{S}_1]_{\mathfrak{F}(S)},\ldots,[\mathcal{S}_n]_{\mathfrak{F}(S)}$. Here, as in (3.13), the subgraphs $[T]_{\mathfrak{F}}$, $[\mathcal{S}]_{\mathfrak{F}}$, or $[\mathcal{S}_i]_{\mathfrak{F}}$ are the reduced subgraphs obtained from T, $\mathcal{S}$, or $\mathcal{S}_i$ by shrinking into points all the graphs of $\mathfrak{F}$ which are, respectively, inside T, $\mathcal{S}$, or $\mathcal{S}_i$. Finally, the numerator $\mathcal{R}$ is given recurrently in a forest-wise expression

$$\mathcal{R}_{\mathfrak{F}(S)}[p_S;p_{S_1},\ldots,p_{S_n}]=\sum_{k_{S_1}=0}^{p_{S_1}}\cdots\sum_{k_{S_n}=0}^{p_{S_n}}\left(p_S+\sum_{i=1}^{n}k_{S_i}\right)! \times\prod_{i=1}^{n}\binom{p_{S_i}}{k_{S_i}}\prod_{S_{\max}}\mathcal{R}_{\mathfrak{F}(S_{\max})}[p_{S_{\max}}-k_{S_{\max}};p_{S_{j_1}}-k_{S_{j_1}},\ldots,p_{S_{j_r}}-k_{S_{j_r}}], \tag{3.14b}$$

where the set $\{\mathcal{S}_{\max}\}$ is the set of subgraphs of $\mathfrak{F}(\mathcal{S})$ maximal in $\mathcal{S}$, and where for each $\mathcal{S}_{\max}$, the subgraphs $\mathcal{S}_{j_1},\ldots,\mathcal{S}_{j_r}$ are the subgraphs of $\mathfrak{F}(\mathcal{S}_{\max})$ inside $\mathcal{S}_{\max}$. We may now substitute (3.13) into (3.4) and we realize that all the singularities of $M_G(x)$ at $x=p_{\max}$ are now extracted. To obtain the quantity I_G^{as} defined in (1.10) we use the inverse Mellin transform

$$I_G(s)-\delta I_G(0)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}dx\, s^x M_G(x), \tag{3.15}$$

with $-1+\delta<\sigma<\delta$ and $\delta=0$ for $(\phi^3)_6$ theory and $+1$ for $(\phi^4)_4$ theory. Now the presence of the function $\Gamma(-x)e^{-i\pi x}$ in $M_G(x)$ makes $M_G(x)$ exponentially small at $\epsilon>0$ when $\mathrm{Im}x\to\pm\infty$. This allows us to push the contour of integration in (3.15) to the left beyond the point $(-1+\delta)$:

$$\int_{\sigma>-1+\delta}=\int_{\sigma<-1+\delta}+\oint_c, \tag{3.16}$$

where the contour c goes around the point $x=-1+\delta$. The integral for $\sigma<-1+\delta$ gives a contribution in s smaller than or equal to s^σ which is nonleading by a power of s. The Cauchy integral around $x=-1+\delta$ gives

$$I_G^{as}(s,t,p_i^2,m,g)=(-1)^{1-\delta}\sum_{\mathfrak{F}}\frac{\chi'_{\mathfrak{F}}}{(q_{\mathfrak{F}}+\delta-1)!}\left(\frac{d}{dx}\right)^{q_{\mathfrak{F}}+\delta-1}\left\{\Gamma(-x+\delta)e^{-i\pi x}s^x\prod_{T\in\mathfrak{F}}\tilde{\beta}_{[T]_{\mathfrak{F}}}^{\chi_{\omega(T)}}\prod_{S\in\mathfrak{F}}F^+_{[S]_{\mathfrak{F}}}(x)\bar{F}_{[G/\mathfrak{F}]}(x,t,p_i^2)\right\}_{x=-1+\delta}, \tag{3.17}$$

where again $\delta=0$ in $(\phi^3)_6$ theory and $+1$ in $(\phi^4)_4$ theory. In (3.17) we sum over all nonempty forests $\mathfrak{F}$ of connected leading subgraphs and of proper divergent nonessential subgraphs with the condition that the maximal elements of the forest $\mathfrak{F}$ are leading. The symbol $[G/\mathfrak{F}]$ means $[G]_{\mathfrak{F}}$ if $\mathfrak{F}$ does not contain G itself and means a single vertex if $\mathfrak{F}$ contains G itself. $\bar{F}_{[G/G]}$ equals 1 by convention. The total number of elements in $\mathfrak{F}$ is $q_{\mathfrak{F}}$ and the numerical coefficient $\chi'_{\mathfrak{F}}$ is given by

$$\chi'_{\mathfrak{F}}=\prod_{i=1}^{p}\chi_{\mathfrak{F}_i}, \tag{3.18}$$

where $\mathfrak{F}$ has p maximal leading subgraphs $\mathcal{S}_i$ and $\mathfrak{F}_i$ is the forest induced by $\mathfrak{F}$ in $\mathcal{S}_i$. Let us give some properties of (3.17).

Differentiating s^x with respect to x n times generates $\ln^n s$, so, by performing the derivatives $d/$

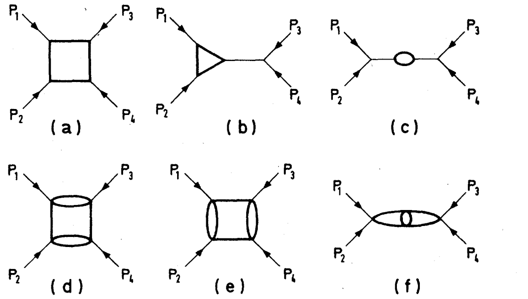

FIG. 3. Leading-logarithm contribution for the graphs (a), (b), (c) of $(\phi^3)_6$ theory and (d), (e), (f) of $(\phi^4)_4$ theory.

dx in (3.17) and by summing over all forests, we obtain the coefficients of all powers of logarithms for the given graph G.

The largest power of logarithm is obtained from the forests with the largest number $q_{\mathfrak{F}}$ of elements and when all derivatives d/dx are applied on s^x. Such forests necessarily contain the graph G itself and the reduced subgraph $[G/\mathfrak{F}]$ is a single vertex. By convention $\overline{F}_{[G/\mathfrak{F}]}$ equal 1 for such forests. Consequently, the leading-logarithm approximation gives a t- and p_i^2-independent coefficient.

From the definition (3.6) it may be seen that $\overline{F}_{[G/\mathfrak{F}]}(x, t=0, p_i^2=0)$ is zero by subtraction for any forest $\mathfrak{F}$ which does not contain G. At $t=p_i^2=0$ all the forests contributing to (3.17) contain the graph G itself.

The function $\overline{F}_{[G/\mathfrak{F}]}$ factorizes into several functions because the reduced graph $[G/\mathfrak{F}]$ is an n-vertex reducible graph if n is the number of maximal leading elements of $\mathfrak{F}$ (see Sec. IV).

By definition, the function $F^+_{[S]_{\mathfrak{F}}}(x)$ is zero if the subgraph $[\mathcal{S}]_{\mathfrak{F}}$ is one-vertex reducible.

For crossed-planar graphs the exchange of the external momentum $p_2 \to p_4$ results in the replacement of $e^{-i\pi x}$ by 1 in (3.17).

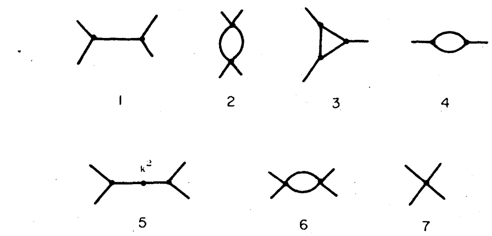

FIG. 4. Subgraphs and reduced graphs for the example of Fig. 3.

We now end this section by giving the coefficient for the leading power of logarithm of the graphs of Fig. 3:

(a) $\frac{1}{2}[F_1^+(-1)]^2 F_2^+(-1) s^{-1} \ln^2 s + O(s^{-1}\ln s)$,

(b) $\tilde{\beta}_3 F_1^+(-1) s^{-1} \ln s + O(s^{-1})$,

(c) $\tilde{\beta}_4 F_5^+(-1) s^{-1} \ln s + O(s^{-1})$,

(d) $\frac{1}{6}[F_6^+(0)]^2 F_2^+(0) s^{-1} \ln^3 s + O(s^{-1}\ln^2 s)$,

(e) $\frac{1}{3}[\tilde{\beta}_2]^2 F_6^+(0) s^{-1} \ln^3 s + O(s^{-1}\ln^2 s)$,

(f) $\frac{1}{3}\tilde{\beta}_2 \tilde{\beta}_6 F_6^+(0) s^{-1} \ln^3 s + O(s^{-1}\ln^2 s)$,

where the indices 1 to 6 corresponds to the graphs of Fig. 4 and where $\overline{F}_7 = 1$. In the literature, most calculations are performed at the leading-logarithm approximation which means that for a given order of perturbation, only the contributions coming from the largest power of logarithm are kept. From (a), (b), (c), and this is true at all orders, this approximation selects ladder graphs in $(\phi^3)_6$ theory. From (d), (e), and (f) it is clear that this is not at all the case in $(\phi^4)_4$ theory. Let us anticipate on Sec. V and already state here that the asymptotic behavior of ladder graphs[20] in $(\phi^3)_6$ theory has nothing to do with the asymptotic behavior of the complete vertex function.

IV. SUMMATION OF THE INFINITE SERIES IN LOGARITHMS OF s

Once we have obtained the coefficient of all logarithms of s for all graphs contributing to $G_4^{(2)}$, we intend to sum the series in logarithms of s defined by the sum of (3.17) over all graphs of $G_4^{(2)}$. The calculation is more easily performed in Mellin space. The first sum we are interested in comes from (3.4). For a given graph G we obtain contributions from one connected leading $\mathcal{S}_i$ and two disjoint connected leading $\mathcal{S}_i$ and $\mathcal{S}_j$, etc. We wish to sum these successive contributions separately over all graphs G.

The first partial sum is obtained by combining all possible graphs $\mathcal{S}_i$ of $G_4^{(2)}$ with all possible graphs $[G/\mathcal{S}_i]$ which are one-vertex reducible (each irreducible part may happen to be a single vertex). We define

$$\overline{F}_V(x, t, p_1^2, p_3^2, m, g)$$

$$= (-g)^{n(K)} i^{-\omega(V)/2} \int_0^\infty \prod d\alpha \exp\left(-i\sum \alpha m^2\right) R\left(\frac{[N_V(\alpha)]^x}{[P_V(\alpha)]^{x+D/2}} \exp\{i[tA_t(\alpha) + p_1^2 A_1(\alpha) + p_3^2 A_3(\alpha)]\}\right), \quad (4.1)$$

where the integral is attached to the vertex graph V given in Fig. 5.

The functions $A_t(\alpha)$, $A_1(\alpha)$, and $A_3(\alpha)$ are characteristic of the graph V. The subgraph K is one-line irreducible in the t channel and the function $N_V(\alpha)$ is obtained from (2.2), where each s cut passes through

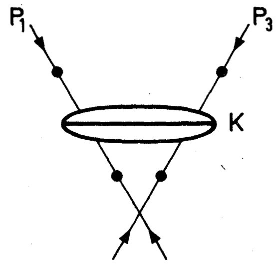

FIG. 5. The graph V.

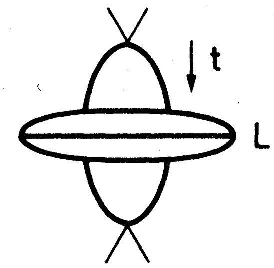

FIG. 6. The graph T.

the reduced vertex. Similarly, $\overline{F}_V(x,t,p_2^2,p_4^2,m,g)$ is obtained as a function of the momenta p_2 and p_4. By convention $\overline{F}_7=1$, where the index 7 corresponds to the trivial Born graph of Fig. 4. Setting

$$\overline{V}_{\rm up}(x,t,p_1^2,p_3^2,m,g)=\sum_V \theta_V \overline{F}_V(x,t,p_1^2,p_3^2,m,g)\,, \tag{4.2}$$

where θ_V is a numerical factor [and a similar definition for $V_{\rm down}(x,t,p_2^2,p_4^2,m,g)$], and

$$S(x,m,g)=\sum_{G\in\mathcal{S}_4^{(2)}}\theta_G F_G(x)\,, \tag{4.3}$$

then the first partial sum we calculate is nothing but $\overline{V}_{\rm up}\overline{V}_{\rm down}S$.

In a similar fashion, we may introduce the second partial sum where we have the contribution of two disjoint connected leading subgraphs. Now, the graph $[G/\mathcal{S}_1 U\mathcal{S}_2]$ is a two-vertex reducible graph. We define

$$\overline{F}_T(x,t,m,g)=(-g)^{n(L)}i^{-\omega(T)/2}\int_0^\infty\prod d\alpha\,\exp\left(-i\sum\alpha m^2\right)R\left(\frac{[-iN_T(\alpha)]^x}{[P_T(\alpha)]^{x+D/2}}e^{itA_t(\alpha)}\right), \tag{4.4}$$

where the functions $A_t(\alpha)$, $N_T(\alpha)$, and $P_T(\alpha)$ have definitions similar to those functions present in (4.1) but are now attached to the graph T given in Fig. 6. The subgraph L is a one-line irreducible subgraph in the t channel. Setting

$$\overline{T}(x,t,m,g)=\sum_T \theta_T \overline{F}_T(x,t,m,g)\,, \tag{4.5}$$

the second partial sum is then given by $\overline{V}_{\rm up}\overline{T}\overline{V}_{\rm down}S^2$. More generally, the partial sum corresponding to the contribution of n disjoint connected leading subgraphs is given by $\overline{V}_{\rm up}[\overline{T}]^{n-1}\overline{V}_{\rm down}S^n$. Consequently, the sum over all graphs of the relation (3.4) gives a geometrical series and we obtain

$$\sum_{G\in G_4^{(2)}}\theta_G M_G(x)=\sum_{G\in G_4^{(2)}}\theta_G\overline{M}_G(x)-\Gamma(-x)e^{-i\pi x}\frac{\overline{V}_{\rm up}(x,t,p_{\rm up}^2,m,g)\overline{V}_{\rm down}(x,t,p_{\rm down}^2,m,g)S(x,m,g)}{1-\overline{T}(x,t,m,g)S(x,m,g)}\,. \tag{4.6}$$

The functions with an overbar are regular around $x=p_{\max}$ and only the function S is singular in perturbation theory at $x=p_{\max}$ [of course, the infinite sum (4.3) is going to transform this singularity]. Let us give some properties of the form (4.6).

The form (4.6) is characteristic of Regge-pole behavior in field theory and was already obtained in $(\phi^3)_4$ theory (see Ref. 1). In this reference the function $S(x,m,g)$ was attached to the single rung and was essentially equal to $m^{-2(x+1)}\Gamma(x+1)$. Thus, the singularity obtained there in perturbation theory at $x=-1$ was shifted to $x=x_0(t,m,g)$, where x_0 is the solution of the equation $1-\overline{T}S=0$. In strictly renormalizable field theory, the function $S(x,m,g)$ is more complicated since the leading subgraphs themselves contain subleading subgraphs and divergent nonessential subgraphs. This function will be studied in the following:

If the theory contains Regge-Pole trajectories, they are given by $x=x_0(t,m,g)$, where x_0 is the solution of the equation $1-\overline{T}S=0$.

The "leading-logarithm approximation" is t and p_i^2 independent and gives $\overline{V}=1$, $\overline{T}=0$, and $S=S$ (leading log). This approximation cannot generate Regge trajectories but only t-independent singularities.

By construction, $\overline{V}(x,t=0,p_i^2=0,m,g)=1$ and $\overline{T}(x,t=0,m,g)=0$. Consequently, at $t=0$, the intercepts are given by the singularities of $S(x,m,g)$.

In $(\phi^4)_4$ theory the pole at $x=0$, present in $\overline{M}_G(x)$ and in $\Gamma(-x)$ [see (4.6)], is not a fixed (t-independent) pole which might give trouble with unitarity. In Sec. V it is shown that $S(0,m,g)=-g$, so that the residue of the pole at $x=0$ simply cancels the subtraction $G^c_{(4)}(s=0,t,p_i^2,m,g)$ made at $s=0$ over the complete vertex function in order to define its Mellin transform with respect to s (a complete explanation requires some anticipation over the treatment of nonplanar graphs).

The remaining part of this section is devoted to the function $S(x,m,g)$. All functions with an overbar are trivial at $t=p_i^2=0$ by renormalization [$\overline{M}_G(x)$ is zero in that case]. $[-\Gamma(-x)e^{-i\pi x}S(x,m,g)]$ is the Mellin transform of $G^2_{(4)}(s,t=p_j^2=0,m,g)$ and $[-\Gamma(-x)S(x,m,g)]$ is the Mellin transform of $G^3_{(4)}(u=-s,t=p_i^2=0,m,g)$. Since we have only two-dimensioned variables (s and m^2) in $G^c_{(4)}(s,t=0,u=-s,p_i^2=0,m,g)$ we have by homogeneity

$$\left(m^2\frac{\partial}{\partial m^2}+s\frac{\partial}{\partial s}-p_{\max}\right)G^c_{(4)}(s,t=0,u=-s,p_i^2=0,m,g)=0\,. \tag{4.7}$$

Then $m^2\partial G^c_{(4)}/\partial m^2$ may be replaced in the Callan-Symanzik equation[15] by the expression obtained from (4.7), which leads to

$$\left(-s\frac{\partial}{\partial s}+\beta(g)\frac{\partial}{\partial g}-2\gamma(g)+p_{\max}\right)G^c_{(4)}(s,m,g)=-\alpha(g)\tilde{G}_{(4)}(s,m,g)\,. \tag{4.8}$$

Unfortunately, even in the planar approximation, the right-hand side of (4.8) is not negligible at large s and nothing may be concluded as long as we do not know the large-s behavior of $\tilde{G}_{(4)}$. The answer to this question is obtained in the planar approximation from Eq. (3.12), which we write in the form

$$[x-p_{\max}]F_{\mathcal{S}_i}(x)-\sum_T\tilde{\beta}_T^{\chi_{\omega(T)}}F_{[\mathcal{S}_i/T]_{\chi_{\omega(T)}}}(x)=F^+_{\mathcal{S}_i}(x)+\sum_{\{\mathcal{S}_j\}}\prod_{\mathcal{S}_j}F_{\mathcal{S}_j}(x)F^+_{[\mathcal{S}_i/\cup\mathcal{S}_j]}(x)\,. \tag{4.9}$$

The sum of (4.9) over all graphs $\mathcal{S}_i$ contributing to $G_4^{(2)}$ is performed in Appendix B (in this appendix we have anticipated on the complete treatment of nonplanar graphs and in the sum over T and $\mathcal{S}_i$, we have included the contributions of nonplanar graphs $\mathcal{S}_i$ and T such that $[\mathcal{S}_i/T]$ is planar). The technique used in this sum may also be found in Ref. 27.

The sum over the coefficients $\tilde{\beta}_T$ on the left-hand side generates the functions $\beta(g)$ and $\gamma(g)$ defined as follows:

$$\beta_{\log}(g)=-\sum_{T\in G_4}\tilde{\beta}_T^{\chi_0},\quad \beta_{\text{quad}}(g)=\sum_{T\in G_2}\tilde{\beta}_T^{\chi_2}, \tag{4.10}$$

where G_4 and G_2 are, respectively the four-point and the two-point vertex functions:

$$\beta(g)=\frac{\beta_{\log}(g)+\frac{1}{2}(3+\delta)g\beta_{\text{quad}}(g)}{1+\beta_{\text{quad}}(g)},\quad \delta=\begin{cases}0 \text{ in } (\phi^3)_6 \text{ theory}\\ 1 \text{ in } (\phi^4)_4 \text{ theory}\end{cases} \tag{4.11a}$$

$$\gamma(g)=\frac{\beta_{\text{quad}}(g)}{1+\beta_{\text{quad}}(g)}, \tag{4.11b}$$

$$\alpha(g)=[1+\beta_{\text{quad}}(g)]^{-1}=1-\gamma(g)\,. \tag{4.11c}$$

The right-hand side of (4.9) is summed in a way similar to the sum of the right-hand side of (3.4). We define

$$S^+(x,m,g)=\sum_{G\in G_4^{(2)}}\theta_G F^+_G(x)\,, \tag{4.12a}$$

$$V^+_{\text{up}}(x,m,g)=\sum_V\theta_V F^+_V(x)\,, \tag{4.12b}$$

and similarly V^+_{down}. We note that $F^+_7=0$.

$$T^+(x,m,g)=\sum_T\theta_T F^+_T(x)\,, \tag{4.12c}$$

where V and T are, respectively, given in Fig. 5 and Fig. 6.

Then Eq. (4.9) can be summed over all graphs $\mathcal{S}_i$ under the form

$$\left(-x+\beta(g)\frac{\partial}{\partial g}-2\gamma(g)+p_{\max}\right)S(x,m,g)$$
$$=-\alpha(g)\{S^+(x,m,g)+[\overline{V}^+_{\rm up}(x,m,g)+V^+_{\rm down}(x,m,g)]S(x,m,g)+T^+(x,m,g)S^2(x,m,g)\}. \quad (4.13)$$

Equation (4.13) is the Mellin transform of Eq. (4.8) in the planar approximation ($sd/ds \Rightarrow x$), where the right-hand side is expanded in terms of $S(x,m,g)$ itself with coefficients regular in perturbation theory at $x=p_{\max}$ (notation +). The reason for the term $S^2(x,m,g)$ and for the absence of S^3, etc. on the right-hand side of (4.13) is due to the fact that in the right-hand side of (4.9) we may have two disjoint leading subgraphs $\mathcal{S}_{j_1}$ and $\mathcal{S}_{j_2}$, but we cannot have more than two of them since by construction $F_{[\mathcal{S}_i/U\mathcal{S}_j]}(x)$ is zero if $[\mathcal{S}_i/U\mathcal{S}_j]$ is one-vertex reducible. We note finally that $V^+_{\rm up}(x,m,g)$ and $V^+_{\rm down}(x,m,g)$ are equal.

Equations (4.6) and (4.13) resume the situation for strictly renormalizable scalar field theory with respect to Regge-pole behavior. We believe that similar types of equations will be valid in gauge field theories; the complication there is mainly due to kinematics, spin, gauge invariance, and internal symmetries.

V. EXISTENCE OF REGGE TRAJECTORIES

The existence of Regge-pole trajectories is now related to the solutions of the equation

$$1-\overline{T}\left(x,\frac{t}{m^2},\ m=1,g\right)S(x,m=1,g)=0\ . \quad (5.1)$$

We may obtain for each solution $x=\alpha(t/m^2,\ g)$ a trajectory. For each solution we may use the inverse Mellin transform of Eq. (4.6) (as well as for crossed planar graphs $e^{-i\pi x}\to 1$), and we obtain by Cauchy's theorem around $x=\alpha(t/m^2,g)$ the asymptotic behavior

$$\frac{-\Gamma(-\alpha)\overline{V}_{\rm up}\left(\alpha,\frac{t}{m^2},\frac{p_{\rm up}^2}{m^2},\ m=1,g\right)\overline{V}_{\rm down}\left(\alpha,\frac{t}{m^2},\frac{p_{\rm down}^2}{m^2},\ m=1,\ g\right)S(\alpha,\ m=1,g)(1+e^{-i\pi\alpha})\left(\frac{s}{m^2}\right)^{\alpha(t/m^2,g)}}{\frac{d}{dx}\left[1-\overline{T}\left(x,\frac{t}{m^2},\ m=1,g\right)S(\alpha,\ m=1,g)\right]_{x=\alpha(t/m^2,g)}} \quad (5.2)$$

By continuity in the transfer t and because $\overline{T}(x,0,m,\ g)=0$, the intercepts $x=\alpha(0,g)$ are such that $S(x,\ m,\ g)$ is infinite. The function $S(x,\ m,g)$ satisfies the Riccati equation

$$\beta(g)\frac{\partial S}{\partial g}=-\alpha(g)T^+(x,\ m,g)S^2+[x-p_{\max}+2\gamma(g)-2\alpha(g)V^+(x,\ m,g)]S-\alpha(g)S^+(x,\ m,g)\ . \quad (5.3)$$

Let us discuss the possible singularities of $S(x,\ m,g)$. We expand it around an algebraic singularity at $x=\phi(g)$:

$$S(x,\ m,g)=[x-\phi(g)]^{\nu}S_\nu(g)+[x-\phi(g)]^{\nu'>\nu}S_{\nu'}(g)+\cdots\ , \quad (5.4)$$

and insert this expansion in (5.3). Then the only negative value for ν' is $\nu=-1$, showing the possibility of having simple poles in S. The residue $S_\nu(g)$ of these poles is related to $\phi(g)$ by

$$S_\nu(g)=\frac{-\beta(g)\dot{\phi}(g)}{\alpha(g)T^+(\phi,m,g)}\quad . \quad (5.5)$$

Equation (5.5) indicates that the intercepts at $x=\phi(g)$ are necessarily g dependent and that if the theory possesses a fixed point g^* where $\beta(g^*)=0$, the residues of the poles $S_\nu(g^*)$ vanish *a priori*. Now if the expansion (5.4) contains a noninteger positive power, then $\phi(g)$ is necessarily g independent. At the branch point, $S(\phi,m,g)$ is finite and, consequently, such branch points are not the intercept of any trajectory defined in (5.1). Such singularities, if any, are t and g independent.

To study the Riccati equation, it is convenient to transform Eq. (5.3) into a second-order linear differential equation. We write

$$S=\Sigma+\sigma(x,g)\ , \quad (5.6)$$

$$\sigma(x,g)=\frac{x-p_{\max}+2\gamma(g)-2\alpha(g)V^+(x,\ m,g)}{2\alpha(g)T^+(x,\ m,\ g)}\ , \quad (5.7)$$

$$z(x)=\int^g\frac{\alpha(g')T^+(x,m,g')}{\beta(g')}dg'\ , \quad (5.8)$$

$$\Sigma=U^{-1}\frac{dU}{dz}\ , \quad (5.9)$$

and we obtain for $U(z)$ the "Schrödinger-type"

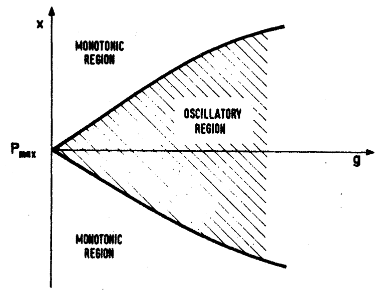

FIG. 7. The lines of turning points at small g.

equation

$$\frac{d^2U}{dz^2} - V(x,z)U = 0 , \tag{5.10}$$

where the "potential" $V(x,z)$ is found to be

$$V(x,z) = \sigma^2(x, g(z)) - \frac{d\sigma}{dz}(x, g(z)) - \frac{S^+(x,m,g(z))}{T^+(x,m,g(z))} . \tag{5.11}$$

We now look for the zeros or singularities of U. We shall consider this the potential at small coupling constant. From the definition of the various functions (4.11) and (4.12), $\alpha(g) = 1 + O(g^2)$, $\gamma(g) \simeq O(g^2)$, $V^+ = O(g^2)$, $T^+ = t_0(x) + O(g^2)$ with $t_0(x) > 0$ and $S^+(x,m,g) = \rho(x)g^2 + O(g^3)$ with $\rho(x) > 0$ we get

$$\sigma(x,g) \simeq \frac{x - p_{\max} + O(g^2)}{2t_0(x)[1 + O(g^2)]} . \tag{5.12}$$

Using (5.8), we have

$$z(x) \simeq \frac{c(x)}{g^{2-\delta}} , \tag{5.13}$$

where $\delta = 0$ in $(\phi^3)_6$ theory and $+1$ in $(\phi^4)_4$ theory, and where $c(x)$ is positive in $(\phi^3)_6$ theory and negative in $(\phi^4)_4$ theory. The potential $V(x,z)$ when $|z|$ is large may be approximated by

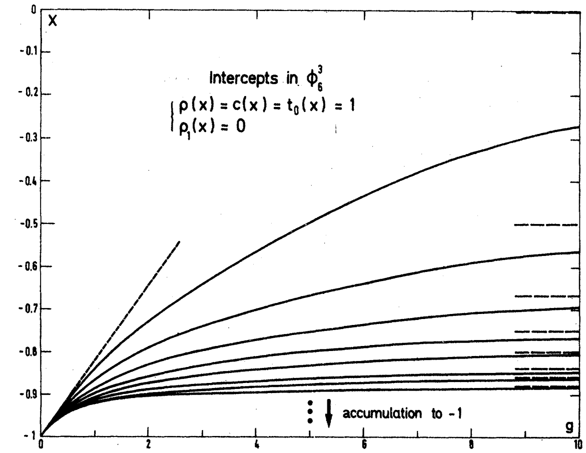

FIG. 8. The intercepts of the $(\phi^3)_6$ Regge trajectories at small g.

$$V(x,z) \simeq \left(\frac{x - p_{\max}}{2t_0(x)}\right)^2 + \rho_1(x)\frac{(x - p_{\max})}{2t_0(x)}\left(\frac{c(x)}{z}\right)^{2/(2-\delta)} - \frac{\rho(x)}{t_0(x)}\left(\frac{c(x)}{z}\right)^{2/(2-\delta)} , \tag{5.14}$$

where $\rho_1(x)$ has not been calculated.

The so-called turning points which separate the monotonic regime from the possibly oscillatory regime are given by $V(x,z) = 0$ or

$$\begin{aligned} x - p_{\max} &= \pm 2[\rho(x)t_0(x)]^{1/2}\left(\frac{c(x)}{z(x)}\right)^{1/(2-\delta)} \\ &= \pm 2[\rho(x)t_0(x)]^{1/2} g \end{aligned} \tag{5.15}$$

and are plotted in Fig. 7. The functions $\rho(x)$, $\rho_1(x)$, $t_0(x)$, $c(x)$, and $z(x)$ are smooth in x around $x = p_{\max}$ and *a priori* different from zero; we may consider them as a constant in x in first approximation. The condition that $S(x,m,g)$ vanishes like g^2 when $g \to 0$ makes $U(x, z \to (-1)^\delta \infty)$ of the form $\exp\{-z[(x - p_{\max})/2t_0(x)]\}$ up to powers of z.

(1) The case of the $(\phi^3)_6$ coupling. The potential $V(x,z)$ is of the Coulomb type, and the solution for U is given in terms of confluent hypergeometric functions ψ.[28] In Appendix C we show that

$$U(x,z) = \exp\{-[(x+1)/2t_0(x)]z\}\,\frac{(x+1)}{t_0(x)}\,z\psi\left(\frac{1 + c(x)\rho_1(x)}{2} - \frac{\rho(x)c(x)}{x+1},\ 2,\ \frac{(x+1)z}{t_0(x)}\right). \tag{5.16}$$

The number of zeros of $U(x,z)$ depends on the integer part of

$$\frac{\rho(x)c(x)}{x+1} - \frac{c(x)\rho_1(x)}{2} - 1.$$

Taking $t_0(x)$, $c(x)$, $\rho(x)$, and $\rho_1(x)$ constant in x in first approximation, we may apply Appendix C and find the lines of zeros which describe the intercepts in $(\phi^3)_6$ theory. This is given in Fig. 8. We see that for any value of the coupling constant g we obtain an accumulation of intercepts around $x = -1$ and there exists a dominant intercept. When g becomes large, the small-g approximations of the intercepts tend to constants larger than -1 [conversely to the intercepts of the ladder graphs in $(\phi^3)_4$ theory]. Of course, a better knowledge of the potential V at large coupling constant g is needed to conclude this study, but this is not the purpose of this paper. Let us remark finally that for ladder graphs in $(\phi^3)_6$ theory, Eq. (5.3) holds true with $\alpha(g)=1$, $\beta(g)=\gamma(g)=0$ and we naturally obtain a square-root branch point in S.[20]

(2) The case of $(\phi^4)_4$ *coupling.* At small coupling constant ($z\to-\infty$) we obtain for U the solution

$$U=\left(\frac{xz}{\pi t_0(x)}\right)^{1/2} K_\nu\left(\frac{xz}{2t_0(x)}\right) , \tag{5.17}$$

where K_ν is a modified Bessel function and

$$\nu=\left(\frac{1}{4}-\frac{\rho(x)c^2(x)}{t_0(x)}+\frac{x\rho_1(x)c^2(x)}{2t_0(x)}\right)^{1/2} . \tag{5.18}$$

We wish to argue that this solution is meaningless because $g=0$ is an infrared-stable fixed point. If, in the same approximation, we wish to solve the Callan-Symanzik equation (in the Mellin version)

$$\left(-x+cg^2\frac{\partial}{\partial g}-2\gamma_0 g^2\right)S(x,g)=-\tilde{c}g^2 \quad (c>0) \tag{5.19}$$

we obtain (up to terms with an identically zero Taylor expansion at $g=0$)

$$S(x,g)=-\frac{\tilde{c}}{c}\exp\left(-\frac{x}{cg}+\frac{2\gamma_0 g}{c}\right)\int_0^g du\,\exp\left(\frac{x}{cu}-\frac{2\gamma_0}{c}u\right) . \tag{5.20}$$

This function has a cut for $\mathrm{Re}\,x>0$ and describes asymptotic freedom in the infrared limit. We remind the reader that the singularities of the Mellin transform for $x\to-\infty$ $(+\infty)$ describe the asymptotic behavior of the amplitude for large (small) momentum. The situation is similar in (5.17), z is negative and U has a cut for $\mathrm{Re}\,x>0$ and has no zero for $x<0$. This simply means that the small-g approximation of the potential V cannot describe the large-s behavior of the amplitude. What happens here tells us that we were not allowed to consider the infinite sum of logarithms of s, which is an infinite sum of inverse Mellin transform, as the inverse Mellin transform of an infinite sum. This is only true when the contour of integration can be distorted in such a way that the infinite sum is convergent everywhere on the contour.

(3) The case of $(\phi^4)_4$ *around a fixed point* $g=g^*$. It is known from the solutions of the Callan-Symanzik equation that large-energy-momentum behavior is given (even at small g) by the first nontrivial positive zero of the function $\beta(g)$ (fixed point $g=g^*$) if any, or by $g\to\infty$. In the same way Regge limit should be described by the behavior of the potential V at $g\simeq g^*$ or at $g\to\infty$. Unfortunately, since V is known only in formal power series in g, nothing can be said at $g\to\infty$ and very little at $g=g^*$. In this latter case, the Riccati equation becomes an algebraic equation (if $\partial S/\partial g\,|_{g=g^*}<\infty$) and we have

$$S(x,m,g^*)=\frac{x+2\gamma^*-2\alpha^*V^{*+})}{2\alpha^*T^{*+}}\left[1-\left(1-\frac{4\alpha^{*2}T^{*+}S^{*+}}{(x+2\gamma^*-2\alpha^*V^{*+})^2}\right)^{1/2}\right], \tag{5.21}$$

which exhibits two square-root branch points at

$$x_\pm=2[\alpha^*V^{*+}-\gamma^*\pm\alpha^*(T^{*+}S^{*+})^{1/2}] \tag{5.22}$$

and a cut in between.

It is easy to see that two formal power series in g satisfy Eq. (5.3), one which is nonzero when $S^+(x,\ m,\ g)\to 0$ and another one which is zero in this limit. The first one is such that $S(x,\ m,\ g=0)=\alpha_0 t_0/x$, while the second one is of order g^2 when $g\to 0$. It is clear from the definition of $S(x,\ m,\ g)$ in terms of graphs in perturbation theory that we must consider only the second case. This explains the reason for the way (5.21) is written.

In order to discuss the position of the branch points, it is again necessary to anticipate on the treatment of nonplanar graphs and to consider the complete functions S^+, T^+, and V^+ (including nonplanar contributions). Then we may note that since

$$\beta(g)=\{\alpha(g)T^+(x,g)g^2+2[\gamma(g)-\alpha(g)V^+(x,g)]g+\alpha(g)S^+(x,g)\}_{x=0}, \tag{5.23}$$

$S(0,\ m,\ g)=-g$ is a solution of (5.3).

If there exists a fixed point $g=g^*>0$, then

$$\frac{(\alpha^*V^{*+}-\gamma^*)+[(\gamma^*-\alpha^*V^{*+})^2-\alpha^{*2}S^{*+}T^{*+}]^{1/2}}{\alpha^*T^{*+}} \tag{5.24}$$

is positive, which implies either S^{*+} is negative, or if it is positive, $(\alpha^*V^{*+}-\gamma^*)>\alpha^*(S^{*+}T^{*+})^{1/2}$ [$\alpha(g)$ and $T^+(x=0,g)$ are positive by spectral decomposition of the two-point function]. If we assume that the functions S^+, T^+, and V^+ are smooth enough in x around $x=0$, we see that the cut (5.21) of $S(x,\ m,\ g^*)$ is either in the com-

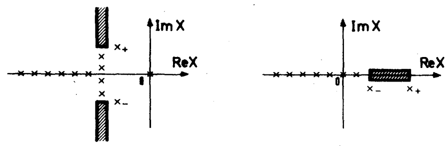

FIG. 9. Singularities in x of $S(x,g)$ in $(\phi^4)_4$ theory for g and around g^*. (a) $S^{*+}<0$. (b) $S^{*+}>0$.

plex and self-conjugate or we have $x_\pm > 0$ (see Fig. 9).

We finally discuss what happens when g is close to g^* but different from it. We suppose that the functions which enter the Riccati equation have a Taylor expansion around $g=g^*$. We have from (5.8)

$$g^* - g \simeq \eta \exp(\beta'^* z/\alpha^* T^{*+}) , \tag{5.25}$$

where η is a positive constant. Since $\beta'^* < 0$, when $g \to g^*$, $z \to +\infty$. The potential $V(x,z)$ becomes

$$V(x,z) \simeq V^* + \eta(x) \exp(\beta'^* z/\alpha^* T^{*+}) , \tag{5.26a}$$

where

$$(V^*)^{1/2} = \left(\frac{x+2\gamma^* - 2\alpha^* V^{*+}}{2\alpha^* T^{*+}}\right)\left(1 - \frac{4\alpha^{*2} T^{*+} S^{*+}}{(x+2\gamma^* - 2\alpha^* V^{*+})^2}\right)^{1/2} \tag{5.26b}$$

and $\eta(x)$ is an unknown quadratic form in x. V^* is positive for real x outside the cut. We may now solve Eq. (5.10) and we obtain

$$U(x,z) = J_\lambda\left[\frac{-\alpha^* T^{*+}[-\eta(x)]^{1/2}}{\beta'^*} \exp\left(\frac{\beta'^* z}{2\alpha^* T^{*+}}\right)\right] , \quad \lambda = [-2\alpha^* T^{*+} (V^*)^{1/2}/\beta'^*] , \tag{5.27}$$

where J is a Bessel function. The above function $U(x,z)$ has plenty of simple zeros and especially for $z \to +\infty$ we may use the convergent expansion[28]

$$J_\nu(x) = (\tfrac{1}{2}x)^\nu \sum_{m=0}^{\infty} (-1)^m (\tfrac{1}{2}x)^{2m} \frac{1}{m!\,\Gamma(m+\nu+1)} , \tag{5.28}$$

which gives for ν close to a negative integer a zero at small x (approximately $\nu = -n + [(-1)^{n-1}/(n-1)!](\frac{1}{2}x)^{2n}$ for $n = 1, 2, 3, \ldots$).

We get, consequently, zeros of U for

$$\frac{-2\alpha^* T^{*+} (V^*)^{1/2}}{\beta'^*} = -n - \frac{(-1)^{n-1}}{(n-1)!}\left(\frac{\alpha^* T^{*+}}{\beta'^*}\right)^{2n}\left(\frac{\eta(x)}{4}\right)^n \exp(n\beta'^* z/\alpha^* T^{*+}) . \tag{5.29}$$

At $z \to +\infty$ these poles of S have a limit; we find for $n = 1, 2, 3, \ldots$

$$x_n(g=g^*) = 2\alpha^* V^{*+} - 2\gamma^* - (n^2 \beta'^{*2} + 4\alpha^{*2} T^{*+} S^{*+})^{1/2} . \tag{5.30}$$

These poles are at the left of the cut if $S^{*+} > 0$, and $\mathrm{Re}\,x_n \leq 2\alpha^* V^{*+} - 2\gamma^*$ if $S^{*+} < 0$ (see Fig. 9). As we know from (5.5), their residues vanish at $g = g^*$. The behavior for $g \neq g^*$ is difficult to evaluate because of the unknown function $\eta(x)$. We may write

$$x_n(g) = x_n(g^*) + C_n(x_n)(g-g^*)^n \tag{5.31}$$

with $C_n(x_n)$ positive for n even. By (5.5) the residue of the pole at $x = x_n$ is of order $O((g-g^*)^n)$.

We do not try to interpret the result for $S^{*+} < 0$. To resume the situation in the case $S^{*+} > 0$ [at small g, $S^+(g) > 0$], for $g \simeq g^*$, the leading singularity is a square-root branch point at $x = x_+$ with x_+ independent of the coupling constant. This leads to a leading behavior in $s^{x_+}[\ln^{-3/2} s + O(\log^{-5/2} s)]$, which is also the leading behavior of a function of s obtained by taking a Cauchy contour around the complete cut. Then comes an infinite number of g-dependent poles which generate by continuity in the transfer t an infinite number of trajectories. Although these poles are *a priori*, not leading, it is not known (especially for n even) whether, at g far enough from g^* and at t far enough from 0, they could or not dominate over the g- and t-independent branch point.

VI. CONCLUSION

We wanted to know the large-s fixed-t asymptotic behavior of the four-point vertex function in ϕ^3 field theory (6 dimensions) and ϕ^4 field theory (4 dimensions) in the planar approximation. In this paper we solved completely and analytically the problem of finding, for the renormalized Feynman amplitude of any essentially and crossed planar graph, the leading power of s and the coefficients of all the powers of logarithms of s. The structure of these coefficients were found to be such that the infinite sum of logarithms, obtained when we sum the amplitudes over all graphs, can be performed.

In $(\phi^3)_6$ theory we did find an accumulation of Regge trajectories at $\alpha=-1$, with a well-defined leading trajectory. The intercepts were described at small coupling constant g and found to be g dependent (going to -1 when $g\to 0$). Although we could not discuss the solution at large g, the extrapolation of our results at small g may indicate that the intercepts remain bounded when $g\to\infty$. We explained why, in $(\phi^3)_6$ theory, the leading-logarithm approximation could not give a pole trajectory; in fact, this approximation leads to the same approximation for the ladder graphs which generates a g-dependent fixed cut.[20] A better approximation was given by Lovelace.[13] He took the asymptotically free approximation of the Bethe-Salpeter kernel and consequently generated ladder graphs with effective coupling constant $\bar{g}(s)$ and effective mass $\bar{m}(s)=0$, and found an accumulation point of intercepts above -1, but the consequence of having a zero mass was to obtain g-independent intercepts.

In $(\phi^4)_4$ theory, little can be said because the small-g approximation of the solution is inconsistent (like the small-g approximation of the solution of the Callan-Symanzik equation), and the large-g approximation is not available. In case of the existence of a fixed point g^*, we found for g around g^* a fixed g-independent cut, and below, an infinite number of g-dependent intercepts (with zero residue at $g=g^*$) of Regge-pole trajectories. It is not known for $g\neq g^*$ and t far from zero whether the square-root branch point is leading over the Regge poles, or not.

Let us conclude with three remarks:

Although moving Regge cuts may be numerically as or more important than moving Regge poles, qualitatively, we understand that in this asymptotic behavior there exist two kinds of objects (poles and cuts?). A complete study of nonplanar amplitudes should be performed in the future. Let us mention that in the small-g approximation of $(\phi^3)_6$ theory, nonplanar amplitudes do not contribute, being of higher order in g.

Regge trajectories are not a peculiarity of the ladders of ϕ^3 theory in four dimensions but really seem to be an intrinsic feature of quantum field theory, and the presence of fixed cuts might simply be an anomalous behavior due to the eventual presence of fixed points $[\beta(g^*)=0]$.

Finally, non-Abelian gauge fields, which are our next objective, should also, in this asymptotic limit, obey a (matrix) Riccati differential equation; because of the property of asymptotic freedom, the solutions should also generate Regge trajectories.

ACKNOWLEDGMENTS

We wish to express our gratitude to Dr. J. Bartels and to Dr. J. des Cloizeaux for useful discussions about the Riccati equation. We also thank N. Tichit for her computation of zeros of confluent hypergeometric function.

APPENDIX A

(a) Example of calculations performed with the operator R. Suppose that we have to calculate the difference

$$\sum_{\mathfrak{F}}\left[\prod_{\mathcal{S}\in\mathfrak{F}}(\tau_{\mathcal{S}}^{-2l(\mathcal{S})})-\prod_{\mathcal{S}\in\mathfrak{F}}(-\tau_{\mathcal{S}}^{-2l(\mathcal{S})+\delta(\mathcal{S})})\right]\{\;\},\tag{A1}$$

where $\{\;\}$ is a given expression of the variables α which has a simultaneous Taylor expansion in regard to any forest of subgraphs [at least up to the order needed in (A1)]; we sum over all forests $\mathfrak{F}$. The quantities $\delta(\mathcal{S})$ are non-negative integers.

We write the Taylor operators (which commute when the subgraphs form a forest) in a given order chosen in such a way that if $\mathcal{S}'\subset\mathcal{S}$, $\tau_{\mathcal{S}'}$ is written at the left of $\tau_{\mathcal{S}}$. For this order we have

$$\left[\prod_{\mathcal{S}\in\mathfrak{F}}(-\tau_{\mathcal{S}}^{-2l(\mathcal{S})})-\prod_{\mathcal{S}\in\mathfrak{F}}(-\tau_{\mathcal{S}}^{-2l(\mathcal{S})+\delta(\mathcal{S})})\right]=\sum_{\mathcal{S}\in\mathfrak{F}}\prod_{\mathcal{S}'<\mathcal{S}}(-\tau_{\mathcal{S}'}^{-2l(\mathcal{S}')})\left[-\tau_{\mathcal{S}}^{-2l(\mathcal{S})}+\tau_{\mathcal{S}}^{-2l(\mathcal{S})+\delta(\mathcal{S})}\right]\prod_{\mathcal{S}''>\mathcal{S}}(-\tau_{\mathcal{S}''}^{-2l(\mathcal{S}'')+\delta(\mathcal{S}'')}),\tag{A2}$$

where $\mathcal{S}'<(>)\mathcal{S}$ means $\tau_{\mathcal{S}'}$ was at the left (right) of $\tau_{\mathcal{S}}$ in the chosen order. On the right-hand side of (A2), we have on the left of the square brackets not only subgraphs $\{\mathcal{S}_1'\}$ which are inside $\mathcal{S}$, but eventually subgraphs $\{\mathcal{S}_2'\}$ which are disjoint from $\mathcal{S}$. For the set of subgraphs $\{\mathcal{S}_2'\}$ which are disjoint from $\mathcal{S}$ we may write again

$$\prod_{\{\mathcal{S}_2'\}}(-\tau_{\mathcal{S}_2'}^{-2l(\mathcal{S}_2')})=\prod_{\{\mathcal{S}_2'\}}(-\tau_{\mathcal{S}_2'}^{-2l(\mathcal{S}_2')+\delta(\mathcal{S}_2')})+\sum_{\{\mathcal{S}_2'\}}\prod_{\mathcal{S}'<\mathcal{S}_2'}(-\tau_{\mathcal{S}'}^{-2l(\mathcal{S}')})\left[-\tau_{\mathcal{S}_2'}^{-2l(\mathcal{S}_2')}+\tau_{\mathcal{S}_2'}^{-2l(\mathcal{S}_2')+\delta(\mathcal{S}_2')}\right]\prod_{\mathcal{S}''>\mathcal{S}_2'}(-\tau_{\mathcal{S}''}^{-2l(\mathcal{S}'')+\delta(\mathcal{S}'')}),\tag{A3}$$

where in (A3), S' and $S''\in\{S_2'\}$. We have obtained for (A2) the expression

$$\sum_{S\in\mathfrak{F}}\prod_{S'\subset S}(-\tau_{S'}^{-2I(S')})[-\tau_S^{-2I(S)}+\tau_S^{-2I(S)+\delta(S)}]\prod_{\substack{S''\supset S\\ \text{or } S''\cap S=\phi}}(-\tau_{S''}^{-2I(S'')+\delta(S'')})$$
$$+\sum_{\substack{S_1<S_2\in\mathfrak{F}\\ S_1\cap S_2=\phi}}\prod_{\left\{\substack{S'<S_1\\ \text{or } S'\subset S_2}\right\}}(-\tau_{S'}^{-2I(S')})[-\tau_{S_1}^{-2I(S_1)}+\tau_{S_1}^{-2I(S_1)+\delta(S_1)}][-\tau_{S_2}^{-2I(S_2)}+\tau_{S_2}^{-2I(S_2)+\delta(S_2)}]\prod_{\substack{S''>S_1\\ S''\not\subset S_2}}[-\tau_{S''}^{-2I(S'')+\delta(S'')}]. \tag{A4}$$

Again, we may find in (A4) subgraphs S' which are disjoint from S_1 and S_2. We may apply (A3) again and again, up to the case where the sum of terms obtained will be such that on the left of the square brackets $[\ ]_{S_1}\cdots[\ ]_{S_n}$ we will have only some subgraphs $\{S_r'\}$, each of them being inside one of the subgraphs $S_1,\ldots,S_n$.

Then we have obtained the relation

$$\left[\prod_{S\in\mathfrak{F}}(-\tau_S^{-2I(S)})-\prod_{S\in\mathfrak{F}}(-\tau_S^{-2I(S)+\delta(S)})\right]=\sum_{\{S_1,\ldots,S_n\}}\left\{\prod_{i=1}^{n}\left[\prod_{S'\subset S_i}(-\tau_{S'}^{-2I(S')})(-\tau_{S_i}^{-2I(S_i)}+\tau_{S_i}^{-2I(S_i)+\delta(S_i)})\right]\right.$$
$$\left.\times\prod_{S''\not\subset U_i S_i}[-\tau_{S''}^{-2I(S'')+\delta(S'')}]\right\}, \tag{A5}$$

where we sum over all sets of disjoint subgraphs $\{S_1,\ldots,S_n\}$.

Now we may finally sum (A5) over all forests $\mathfrak{F}$ to calculate (A1). For a given set of disjoint subgraphs $\{S_1,\ldots,S_n\}$, we sum over all forests inside each S_i and over all forests of $[G/U_iS_i]$. We obtain

$$\sum_{\mathfrak{F}}\left[\prod_{S\in\mathfrak{F}}(-\tau_S^{-2I(S)})-\prod_{S\in\mathfrak{F}}(-\tau_S^{-2I(S)+\delta(S)})\right]$$
$$=\sum_{\{S_1,\ldots,S_n\}}\left\{\prod_{i=1}^{n}\left[\left(1+\sum_{\mathfrak{F}(S_i)}\prod_{S'\in\mathfrak{F}(S_i)}(-\tau_{S'}^{-2I(S')})\right)(-\tau_{S_i}^{-2I(S_i)}+\tau_{S_i}^{-2I(S_i)+\delta(S_i)})\right]\right.$$
$$\left.\times\left(1+\sum_{\mathfrak{F}(G/US_i)}\prod_{(S''/US_i)\in\mathfrak{F}(G/US_i)}(-\tau_{S''}^{-2I(S'')+\delta(S'')})\right)\right\}. \tag{A6}$$

This proof remains valid if we consider only the sum over all forests of connected subgraphs.

(b) Application 1: proof of (3.4). We apply (A6) to the integrand $\{\ \}$ written in (2.10). In this calculation, $\delta(S_i)=+2$ for connected leading subgraphs [we use (A6) with the sum over forests of connected subgraphs] and 0 otherwise. Then we calculate

$$\prod_i[-\tau_{S_i}^{-2I(S_i)}+\tau_{S_i}^{-2I(S_i)+\delta(S_i)}]\{\ \}. \tag{A7}$$

It is important to know that when all α variables of S_i are dilated by ρ_i^2, we have

$$P_G(\alpha,\rho_i^2\alpha)=\prod_i\rho_i^{2L(S_i)}\left[\prod_i P_{S_i}(\alpha)P_{[G/US_i]}(\alpha)+O(\rho^2)\right], \tag{A8}$$

where $L(S_i)$ is the number of independent loops of S_i. Also, if S_i is a connected leading subgraph, then

$$A_s(\alpha,\rho_i^2\alpha)=\prod_i\rho_i^2\left[\prod_i A_{S_i}(\alpha)A_{G/US_i}(\alpha)+O(\rho^2)\right], \tag{A9a}$$

$$A_t(\alpha,\rho_i^2\alpha)=[A_t(\alpha)]_{[G/U(S_i)]}+O(\rho^2), \tag{A9b}$$

$$A_j(\alpha,\rho_i^2\alpha)=[A_j(\alpha)]_{[G/US_i]}+O(\rho^2)\quad\text{for } j=1,\ldots,4. \tag{A9c}$$

If we apply the operators τ_{S_i} over the curly bracket $\{\ \}$ in (2.10), according to the definition (1.8), for $p_{\max}<\text{Re}\,x<p_{\max}+\eta$ (where η is small enough and positive), we obtain

$$\prod_i[-\tau_{S_i}^{-2I(S_i)}+\tau_{S_i}^{-2I(S_i)+\delta(S_i)}]\{\ \}=\prod_i\left\{\frac{[iA_{S_i}(\alpha)]^x}{P_{S_i}{}^{D/2}(\alpha)}\right\}\frac{[i^{1-\nu_{S_i}}A_{[G/US_i]}(\alpha)]^x}{P_{[G/US_i]}{}^{D/2}(\alpha)}\exp\left\{i\left[tA_t(\alpha)+\sum_1^4 p_j^2A_j(\alpha)\right]_{[G/US_i]}\right\}. \tag{A10}$$

The factorization which occurs in (A10) is of great importance for the structure of the coefficients of the logarithms and explains the possible exponentiation of these logarithms.

Now each expression

$$\left[1+\sum_{\mathfrak{F}(S_i)}\prod_{S'\in\mathfrak{F}(S_i)}(-\tau_{S'}{}^{-2l(S')})\right]$$

operates in the right-hand side of (A10) on the corresponding curly brackets $\{\ \}$ related to S_i and reconstructs the subtracted integrand for $\mathfrak{F}_{S_i}(x)$. Then the expression

$$\left[1+\sum_{\mathfrak{F}(G/US_i)}\prod_{[S''/US_i]\in\mathfrak{F}(G/US_i)}(-\tau_{S''}{}^{-2l(S'')+\delta(S'')})\right]$$

operates on the complete right-hand side of (A10). The product

$$\prod_i\left\{\frac{[iA_{S_i}(\alpha)]^x}{P_{S_i}{}^{D/2}(\alpha)}\right\}$$

is completely homogeneous for the operators $\tau_{S''}{}^{-2l(S'')+\delta(S'')}$ of degree $\sum_{i\{S_i\subset S''\}}[x-L(S_i)D/2]$, and may be taken from right to left of the operator $\tau_{S''}$ which becomes $\tau_{[S''/US_i]}{}^{-2l(S''/US_i)+\delta(S'')}$. Consequently, we reconstruct the subtracted integrand for $\overline{F}_{[G/US_i]}(x,t,p_i{}^2)$. This proves Eq. (3.4).

(c) Application 2: proof of (3.9) (Zimmermann's identity for graphs). The relation between $\hat{F}_S(x)$ defined in (3.8) and $\tilde{F}_S(x)$ defined in (3.10) comes from the calculation of

$$\left(\sum_{a=1}^{l}\alpha_a m_a{}^2\right)R\{\ \}-R\left[\left(\sum_{a=1}^{l}\alpha_a m_a{}^2\right)\{\ \}\right].$$

For each forest $\mathfrak{F}$ we have to calculate

$$\left[\prod_{T\in\mathfrak{F}}(-\tau_T{}^{-2l(T)+\delta(T)})-\prod_{T\in\mathfrak{F}}(-\tau_T{}^{-2l(T)})\right]\{\alpha_a\{\ \}\},\tag{A11}$$

where $\delta(T)=+2$ if T contains the line a and 0 otherwise. The relation (A6) may be applied in this simpler case; here the graphs T such that $\delta(T)=+2$ are necessarily nested. When all α_a's for a belonging to a nonessential subgraph T are dilated by ρ^2, we have (A8) and (A9b)–(A9c), but (A9a) is replaced by

$$A_S(\alpha,\rho^2\alpha)=[A_s(\alpha)]_{[G/T]}+O(\rho^2)\ .\tag{A12}$$

It has been shown in Ref. 4 that

$$[-\tau_T{}^{-2l(T)+2}+\tau_T{}^{-2l(T)}]\left(\frac{\alpha_a[A_S(\alpha)]_S^x}{P_S^{D/2}}\right)=-(\alpha_a\mathcal{Y}_T^{\chi_{\omega(T)}}(\alpha))\left(\frac{[A_S(\alpha)]^x}{P^{D/2}(\alpha)}\right)_{[S/T]\chi_{\omega(T)}},\tag{A13}$$

where

$$\mathcal{Y}_T^{\chi_{\omega(T)}}(\alpha)=\left.\frac{\partial(P_T{}^{-D/2}(\alpha)\exp\{-[k_i d_{ij}(\alpha)k_j]_T\})}{\partial k_{i_1}\cdots\partial k_{i_{\omega(T)}}}\right|_{k_i=0}\tag{A14}$$

In (A14) the four-vectors k are the external momentum of T and $\chi_{\omega(T)}$ is a choice of $\omega(T)$ (the superficial degree of divergence of T) external momentum $\{k_{i_1},\ldots,k_{i_{\omega(T)}}\}$. In (A13) we sum over all possible choice $\chi_{\omega(T)}$.

The application of (A6) and (A13) explains (3.9). Let us note that the factorization (A13) is important for the structure of the renormalization group.

APPENDIX B

In this appendix we wish to sum Eq. (4.9) over all graphs contributing to the essentially planar vertex function $G_4^{(2)}$. The expression on the right-hand side is summed in exactly the same way as (3.4), except that here, $F^{+}_{[S_i/US_j]}(x)$ is equal to zero if $[S_i/US_j]$ is one-vertex reducible and, consequently, the sum $\sum_{\{S_j\}}$ runs only over single elements S_j and over couples of elements $\{S_{j_1},S_{j_2}\}$. The expression in (4.9) which is left to sum over S is

$$\sum_T\tilde{\beta}_T^{\chi_{\omega(T)}}F_{[S/T]\chi_{\omega(T)}}(x)$$

where T is a nonessential, proper divergent subgraph.

First, we suppose that T is logarithmically divergent. For a given graph $[S/T]$, we may sum over all T the coefficients $\tilde{\beta}_T$ to obtain the function $[-\beta_{\log}(g)]$ defined in (4.10). Now the amplitude for

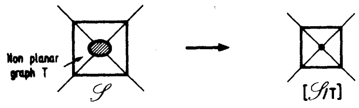

FIG. 10. Examples of nonplanar graphs $\mathcal{S}$ and T such that $[\mathcal{S}/T]$ is planar.

$[\mathcal{S}/T]$ has the coupling constant $(-g)$ at each vertex except at the reduced vertex obtained from T. If $n(\mathcal{S}/T)$ is the number of vertices of $(\mathcal{S}/T)$, we have a factor $(-g)^{n(\mathcal{S}/T)-1}$. Finally, the graph $[\mathcal{S}/T]$ may be obtained from a larger graph by reducing T at any of the vertices of $[\mathcal{S}/T]$; when we sum over all graphs $[\mathcal{S}/T]$, that is, all graphs of $G_4^{(2)}$, we have a factor $n(\mathcal{S}/T)(-g)^{n(\mathcal{S}/T)-1}$ and this explains that

$$\sum_{\mathcal{S}} \sum_{T \text{ log div}} \beta_T^{\chi_0} F_{[\mathcal{S}/T]\chi_0}(x) = \beta_{\log}(g) \frac{\partial}{\partial g} S(x,m,g). \qquad \text{(B1)}$$

This expression is effectively obtained in $(\phi^3)_6$ theory because all possible divergent subgraphs are effectively present in the set of essentially planar graphs. This is not strictly the case in $(\phi^4)_4$ theory where nonplanar logarithmically divergent subgraphs may be excluded. At this step, we anticipate the treatment of nonplanar graphs: The amplitudes for nonplanar graphs have two kinds of singularities, first the singularities inside the α-integration domain which are supposed to generate moving Regge cuts, then the singularities when a subset of variables $\alpha \to 0$. In this last case, what is done in this paper for essentially planar graphs may be extended, and, for instance, a nonplanar logarithmically divergent subgraph T inside a nonplanar graph $\mathcal{S}$ contributes a power of logarithm of s with a residue generated from $\tilde{\beta}_T F_{[\mathcal{S}/T]}$. When $[\mathcal{S}/T]$ is planar, as shown in Fig. 10, the coefficients $\tilde{\beta}_T$ complete the function $[-\tilde{\beta}_{\log}(g)]$ by nonplanar contributions.

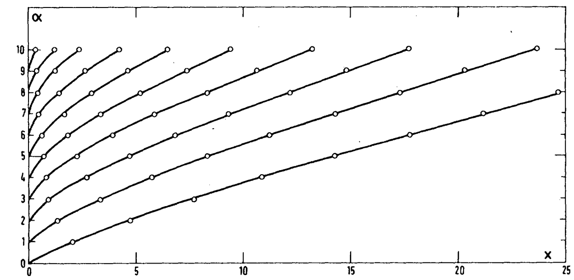

FIG. 11. Zeros of the hypergeometric confluent function $\psi(-\alpha, 2, x)$.

Next we sum over the quadratic divergent subgraphs T. The same technique is applied, but here at each reduced T vertex we insert a quadratic coupling derivative k^2, where k is the momentum which flows through the self-energy. This k^2 factor is written as $(k^2-m^2)+m^2$. For a given $[\mathcal{S}/T]_{\chi_2}$ we sum over all T the coefficients $\tilde{\beta}_T^{\chi_2}$ to obtain the function $\beta_{\text{quad}}(g)$ defined in (4.10). Then we group together all the graphs $[\mathcal{S}/T]_{\chi_2}$ which differ only by the line upon which the insertion χ_2 is performed. We also use the topological relation for any graph contributing to G_4,

$$4+2l=(3+\delta)n, \qquad \text{(B2)}$$

where l and n are, respectively, the number of lines and vertices of the graph and where $\delta=0$ in $(\phi^3)_6$ theory and $+1$ in $(\phi^4)_4$ theory. We thus obtain by summing over all $[\mathcal{S}/T]$ in $G_4^{(2)}$

$$\sum_{\mathcal{S}} \sum_{T \text{ quad}} \tilde{\beta}_T^{\chi_2} F_{[\mathcal{S}/T]\chi_2}(x) = \beta_{\text{quad}}(g)\left(-2+\frac{3+\delta}{2} g\frac{\partial}{\partial g}+m^2\frac{\partial}{\partial m^2}\right)S(x,m,g). \qquad \text{(B3)}$$

It is also useful to realize from (3.5) and (4.3) that by homogeneity

$$S(x,m,g)=(m^2)^{p_{\max}-x} S(x,1,g). \qquad \text{(B4)}$$

Then from (4.9), (B1), (B3), and (B4) we obtain Eq. (4.13) via the definitions (4.11). We want to point out that a generalization of the technique used in this appendix has been used in Ref. 27 to define a representation of the renormalization group.

APPENDIX C

Most of what is said in this appendix may be found in Ref. 28.

The solution of the equation.

$$\ddot{U}-\left(A+\frac{B}{z}\right)U=0 \quad \text{for } A \geq 0,\quad B<0. \qquad \text{(C1)}$$

Taking into account the exponential form $e^{-\sqrt{A}z}$ for U at large positive z, we find for solution

$$U=e^{-\sqrt{A}z}\, 2\sqrt{A}z\,\psi(1+B/2\sqrt{A},\,2,\,2\sqrt{A}z), \qquad \text{(C2)}$$

where ψ is the confluent hypergeometric function

$$\psi(a,c,x)=\frac{1}{\Gamma(a)}\int_0^\infty e^{-xt}t^{a-1}(1+t)^{c-a-1}dt\,. \tag{C3}$$

For $1+B/2\sqrt{A}=-n$,

$$U_n=e^{-\sqrt{A}z}\,2\sqrt{A}z(-)^n n!\,L_n^1(2\sqrt{A}z)\,, \tag{C4}$$

where L_n^1 is a generalized Laguerre polynomial.

If we wish to obtain the zeros of U (different from $z=0$), we must first realize that $\psi(a,\ 2,\ x)$ has $-E(a)$ single positive zeros in x, for $a\leqslant 0$ and 0 otherwise [$E(a)$ means integer part smaller or equal to a]. We may first find the zeros of the Laguerre polynomials; especially for large n [and using $2\sqrt{A}=-B/(n+1)$] we have

$$\lim_{n\to\infty} L_n^1(2\sqrt{A}z)=\frac{e^{\sqrt{A}z}}{(2\sqrt{A}z)^{1/2}}(n+1)^{1/2}J_1(2(-Bz)^{1/2})\,, \tag{C5}$$

so that the zeros of L_n^1 tend to the zeros of the Bessel function J_1. Now the way the zeros evolve, when n is noninteger negative, may be obtained on the computer from subtracted integral representations of the type (C3). The result is shown in Fig. 11. It is important to note that the lines of zeros go from $z=0$ to $z=\infty$, since the zeros of J_1 are periodic in $\sqrt{z}$ at large z. At large $\sqrt{z}$ and large n all the lines of zeros have the same slope.

The last point we wish to determine is the limit of the lines of zeros when $x\to 0$. This may be done from

$$\psi(a,c,x)=\frac{\Gamma(1-c)}{\Gamma(a-c+1)}\phi(a,c,x)+\frac{\Gamma(c-1)}{\Gamma(a)}x^{1-c}\phi(a-c+1,2-c,x) \tag{C6}$$

with

$$\phi(a,c,x)=1+(a/c)x+O(x^2)\,. \tag{C7}$$

We obtain

$$\psi(a,2,x)=\frac{1}{\Gamma(a)x}+\frac{\ln x}{\Gamma(a-1)}-\left(\frac{\Gamma'(1)}{\Gamma(a-1)}+\frac{(a-1)\Gamma'(1)}{\Gamma(a)}+\frac{1}{\Gamma(a-1)}-\frac{\Gamma'(a-1)}{\Gamma^2(a-1)}-\frac{1}{\Gamma(a)}\right)+O(x\ln x)\,. \tag{C8}$$

At small x the zeros are given by

$$\frac{1}{x}+(a-1)\ln x+\frac{\Gamma(a)\Gamma'(a-1)}{\Gamma^2(a-1)}-2\Gamma'(1)(a-1)+2-a=0\,. \tag{C9}$$

It is clear that $x\to 0$ makes $a\to -n$ for $n=0,1,2,\ldots$. Using $\Gamma'(a-1)/\Gamma^2(a-1)\to(-1)^n(n+1)!$ when $a\to -n$, we set at small x

$$\alpha=n+(n+1)x\,, \tag{C10}$$

where $\alpha=-a$ in Fig. 11.

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