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Asymptotic behavior of the Sudakov form factor

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The asymptotic behavior of the form factor in quantum electrodynamics is calculated in case the photon has a mass. The technique used takes into account all logarithms in q^2 but neglects inverse powers of q^2. The result is essentially that found by Sudakov. In order to obtain this result the large-n behavior of γ_n is obtained.

I. INTRODUCTION

In this paper the asymptotic behavior of the Sudakov[1-4] form factor is derived in an Abelian gauge theory having photon mass μ and fermion mass m. The result holds for the on-mass-shell form factor or for a particular component of the off-shell vertex function. In case there is no coupling-constant renormalization, that is vacuum polarization graphs are omitted, we find

$$\ln\Gamma(q^2) = -\frac{g^2}{16\pi^2}\ln^2\left(-\frac{q^2}{M^2}\right) + O(1)\,,$$

where M^2 is a scale depending on μ^2/m^2 and the coupling g. In the case where $g^2 < 0$, an asymptotically free theory but not a unitary theory,

$$\ln\Gamma = -\tfrac{3}{2}\ln\left(-\frac{q^2}{\mu^2}\right)\ \ln\ln\left(-\frac{q^2}{\mu^2}\right) + O\left(\ln\left(-\frac{q^2}{\mu^2}\right)\right).$$

In order to obtain these results two technical results are needed. The first of these is the behavior of γ_n for large n where γ_n is the anomalous dimension of certain composite operators. The second technical result is a factorized form for a certain off-shell vertex function. The large-n behavior of γ_n is derived in Sec. II, the result being

$$\gamma_n(g) \underset{n\to\infty}{\simeq} \frac{g^2h(g^2)}{4\pi^2}\ln n + 2c(g)\,.$$

The factorization of the vertex function is given in Sec. III, although a more detailed discussion is given in Ref. 5. This factorization is not a rigorous result of quantum field theory although its validity is on the same basis as the factorization results in inclusive processes.

It is apparently important that $\mu \neq 0$. Although the derivations given in Sec. II explicity depend on $\mu \neq 0$, the large-n behavior of γ_n presumably is a property of the zero-mass theory and so will be unaltered if $\mu = 0$. However, the factorization discussed in Sec. III and in the Appendix depends crucially on $\mu \neq 0$. The reader may recall that the leading-logarithm result for the off-shell form factor when $\mu = 0$ is

$$\ln\Gamma(q^2) = -\frac{g^2}{8\pi^2}\ln^2(-q^2)\,.$$

There is a factor of 2 difference between $\ln\Gamma$ for $\mu = 0$ and $\mu \neq 0$.

In this paper all terms of the form $(g^2)^m(\ln q^2)^n$ with $2m \geq n$ are kept; however, inverse powers of q^2 are not kept. Thus one should be wary of taking the rapid decrease of the form factor too seriously since we have no reason to believe that the form factor is not simply of order $(1/q^2)$.

II. A CALCULATION OF γ_σ FOR LARGE σ

In this section γ_σ will be calculated for large values of the parameter σ.[6] We begin with a calculation of the lowest-order graphs contributing to γ_σ in order to establish notation and to set the general idea for the procedure of computation. The theory beging discussed is an Abelian gauge theory with photon mass μ and fermion mass m. γ_σ is not dependent on μ/m. We shall use a form for γ_σ that is motivated by a previous discussion of cut vertices.[7] However, when $\sigma = n$ the expression which will be used is equivalent to the standard expression.

A. An order-g^2 computation of γ_σ

In order g^2 the graphs which contribute to γ_σ are shown in Figs. 1 and 2. The $\times$ on the photon and fermion lines in these figures indicates a soft mass insertion which will always be normalized so that the mass derivative in the Callan-Symanzik equation is of the form $\mu^2\partial/\partial\mu^2 + m^2\partial/\partial m^2$. In addition there are

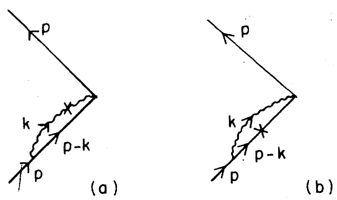

FIG. 1. Some order-g^2 contributions to γ_σ.

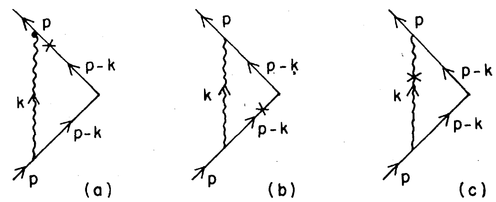

FIG. 2. Further-order g^2 contributions to γ_σ.

two graphs identical to those shown in Fig. 1 except that the photon connects to the outgoing fermion line rather than to the incoming one. Let us begin with the graphs shown in Fig. 1(a). Here

$$\gamma_\sigma = -\frac{i\mu^2 p_-^{-\sigma+1}}{4}\frac{g^2}{(2\pi)^4}\int \frac{d^4k\, \mathrm{tr}\gamma_+\gamma_-[\gamma\cdot(p-k)+m]\gamma_-}{(k^2-\mu^2+i\epsilon)^2[(p-k)^2-m^2+i\epsilon]}\frac{(p-k)_-^{\sigma-1}-p_-^{\sigma-1}}{k_-}. \tag{1}$$

In the above $p_+ = \underline{p} = (p_1, p_2) = 0$. This expression is most easily evaluated by performing the k_+ contour integration in such a way as to pick up the pole at $k_+ = -[(\underline{k}^2+m^2)/2(p-k)_-]+i\epsilon$. Then

$$\gamma_\sigma = -\frac{g^2}{8\pi^3}p_-^{-\sigma+1}\mu^2\int_0^{p_-} dk_- \int \frac{d^2\underline{k}\,(p-k)_-^{\,2}}{[p_-(\underline{k}^2+\mu^2)+k_-(m^2-\mu^2)]^2}\frac{(p-k)_-^{\sigma-1}-p_-^{\sigma-1}}{k_-}.$$

The $d^2\underline{k}$ integration can be done exactly. After a rescaling of the dk_- integral one finds

$$\gamma_\sigma = \frac{g^2\mu^2}{8\pi^2}\int_0^1 \frac{dx}{x}\frac{(1-x)^2[1-(1-x)^{\sigma-1}]}{\mu^2+(m^2-\mu^2)x}. \tag{2}$$

For large σ, $\gamma_\sigma = (g^2/8\pi^2)\ln\sigma$.

Now examine the contribution of the graph shown in Fig. 1(b). In this case

$$\gamma_\sigma = -\frac{imp_-^{-\sigma+1}}{8}\frac{g^2}{(2\pi)^4}\int \frac{d^4k\,\mathrm{tr}\gamma_+\gamma_-[\gamma\cdot(p-k)+m][\gamma\cdot(p-k)+m]\gamma_-}{(k^2-\mu^2+i\epsilon)[(p-k)^2-m^2+i\epsilon]^2}\frac{(p-k)_-^{\sigma-1}-p_-^{\sigma-1}}{k_-}. \tag{3}$$

If one closes the k_+ contour around the pole at $k_+ = [(\underline{k}^2+\mu^2)/2k_-)]-i\epsilon$, then

$$\gamma_\sigma = -\frac{g^2}{8\pi^3}m^2 p_-^{-\sigma+1}\int_0^{p_-} dk_- \int \frac{d^2\underline{k}\,(p-k)_-}{[p_-(\underline{k}^2+\mu^2)+k_-(m^2-\mu^2)]^2}[(p-k)_-^{\sigma-1}-p_-^{\sigma-1}].$$

Evaluating the $d^2\underline{k}$ integral one finds

$$\gamma_\sigma = \frac{g^2}{8\pi^2}m^2\int_0^1 dx\frac{(1-x)[1-(1-x)^{\sigma-1}]}{\mu^2+(m^2-\mu^2)x}. \tag{4}$$

For large σ this integral contributes a constant term. Combining (2) and (4) one finds

$$\gamma_\sigma = \frac{g^2}{8\pi^2}\int_0^1\frac{dx}{x}[(1-x)-(1-x)^\sigma]$$
$$= \frac{g^2}{8\pi^2}\left[\frac{\Gamma'(\sigma+1)}{\Gamma(\sigma+1)}-\Gamma'(1)-1\right]. \tag{5}$$

When σ is an integer n

$$\gamma_n = \frac{g^2}{8\pi^2}\sum_{j=2}^{n}\frac{1}{j}$$

which is the usual result.

In view of the arguments to come later it is important to realize that for the one-loop graphs the dominant term for large σ is determined completely from the graph shown in Fig. 1(a). So long as $\mu \neq 0$ the graph shown in Fig. 1(b) only contributes a constant term for large σ while the graphs shown in Fig. 2 go as $1/\sigma^2$ for large σ. There is an easy way to see how these large σ results emerge. A possible $\ln\sigma$ can arise only from small values of k_- of size $k_-/p_- \approx 1/\sigma$. When k_- goes to zero in the graph shown in Fig. 1(b) the singularities in the k_+ plane are located at $k_+ = [(\underline{k}^2+\mu^2)/2k_-)]-i\epsilon$ from the photon propagator, and at $k_+ = -[(\underline{k}^2+m^2)/2p_-)]+i\epsilon$ from the fermion propagator. It is thus possible to distort the k_+ contour down into the complex plane so as to remain a distance on the order of $\mu^2/2k_-$ away from the origin. During this distortion the photon propagator remains effectively $-1/(\underline{k}^2+\mu^2)$ while the fermion propagator $1/[(p-k)^2-m^2+i\epsilon]$ becomes approximately $1/(-2p_-k_+ + i\epsilon)$. Since the fermion propagator occurs quadratically the small-k_- region is suppressed by the fact that k_+ is becoming large. In fact the k_- integral is linearly convergent for large σ and so the region $k_-/p_- = O(1/\sigma)$ only contributes a term of order $1/\sigma$ to γ_σ.

B. γ_σ for QED with no vacuum polarization

Now γ_σ will be computed for large values of σ in case no vacuum polarization graphs are allowed. We begin by considering the contribution of the graphs illustrated in Fig. 3. This contribution is

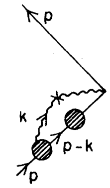

FIG. 3. A photon-mass-inserted contribution to γ_σ.

$$\gamma_\sigma = -\frac{i\mu^2 p_-^{-\sigma+1}}{4}\frac{g^2}{(2\pi)^4}\int \frac{d^4k\, \mathrm{tr}\gamma_+\gamma_-[-iS(p-k)]\Gamma_-(p,k)}{(k^2-\mu^2+i\epsilon)^2}\,\frac{(p-k)_-^{\sigma-1}-p_-^{\sigma-1}}{k_-}\,. \qquad (6)$$

Now use the Ward identity to write

$$\Gamma_-(p,k)=\frac{1}{k_+-i\epsilon}[-k_-\Gamma_+(p,k)+k_i\Gamma_i(p,k)-iS^{-1}(p-k)+iS^{-1}(p)]\,. \qquad (7)$$

Let us designate the four terms in γ_σ which appear when (7) is substituted into (6) as γ_σ^a, γ_σ^b, γ_σ^c, γ_σ^d. Then

$$\gamma_\sigma^a=\frac{i\mu^2 p_-^{-\sigma+1}}{4}\frac{g^2}{(2\pi)^4}\int\frac{d^4k}{k_+-i\epsilon}\,\frac{\mathrm{tr}\gamma_+\gamma_-[-iS(p-k)]\Gamma_+(p,k)}{(k^2-\mu^2+i\epsilon)^2}[(p-k)_-^{\sigma-1}-p_-^{\sigma-1}]\,. \qquad (8)$$

Now (8) has no possible singularity in k_- when σ becomes large, so γ_σ^a has no $\ln\sigma$ term for large σ. Also,

$$\gamma_\sigma^b=-\frac{i\mu^2 p_-^{-\sigma+1}}{4}\frac{g^2}{(2\pi)^4}\int\frac{d^4k}{k_+-i\epsilon}\,\frac{\mathrm{tr}\gamma_+\gamma_-[-iS(p-k)]k_i\Gamma_i(p,k)}{(k^2-\mu^2+i\epsilon)^2}\,\frac{(p-k)_-^{\sigma-1}-p_-^{\sigma-1}}{k_-}\,. \qquad (9)$$

The possible singularity in (9) when σ becomes large is in the region $k_-/p_-=O(1/\sigma)$. Now when k_- becomes small we may distort the k_+ contour into the lower half plane as discussed previously. This is apparent since the singularity structure of S and Γ is determined from the invariants $k^2+i\epsilon$ and $(p-k)^2+i\epsilon$. All that is needed is to estimate the large-k_+ behavior of S and Γ_i. This is given by

$$\gamma_-S(p-k)\underset{k_+\to\infty}{=}O(1/k_+)\,,$$
$$k_i\Gamma_i(p,k)\underset{k_+\to\infty}{=}O(1)\,.$$

Thus the integrand in (9) behaves as $(1/k_+)^2$ for large k_+ and the resulting k_- integral is linearly convergent for large σ.

Now

$$\gamma_\sigma^c=-\frac{i\mu^2 p_-^{-\sigma+1}}{4}\frac{g^2}{(2\pi)^4}\int\frac{d^4k}{k_+-i\epsilon}\,\frac{\mathrm{tr}\gamma_+\gamma_-[-iS(p-k)][-iS^{-1}(p-k)]}{(k^2-\mu^2+i\epsilon)^2}\,\frac{(p-k)_-^{\sigma-1}-p_-^{\sigma-1}}{k_-}\,. \qquad (10)$$

The calculation here is almost the same as for the graphs shown in Fig. 1(a) and one finds

$$\gamma_\sigma^c=\frac{g^2}{8\pi^2}\int_0^1\frac{dx}{x}[1-(1-x)^{\sigma-1}]=\frac{g^2}{8\pi^2}\left(\frac{\Gamma'(\sigma)}{\Gamma(\sigma)}+C_E\right) \qquad (11)$$

where C_E is the Euler-Mascheroni constant, $C_E=0.577\cdots$. For large $\sigma,\gamma_\sigma=(g^2/8\pi^2)\ln\sigma$. Finally,

$$\gamma_\sigma^d=-\frac{i\mu^2 p_-^{-\sigma+1}}{4}\frac{g^2}{(2\pi)^4}\int\frac{d^4k}{k_+-i\epsilon}\,\frac{\mathrm{tr}\gamma_+\gamma_-[-iS(p-k)][iS^{-1}(p)]}{(k^2-\mu^2+i\epsilon)^2}\,\frac{(p-k)_-^{\sigma-1}-p_-^{\sigma-1}}{k_-}\,. \qquad (12)$$

Again, in (12) one may distort the k_+ contour for small k_- to obtain a linearly convergent k_- integral for large σ. Thus the total contribution to γ_σ of the graphs shown in Fig. 3 is $(g^2/8\pi^2)\ln\sigma$ as far as the $\ln\sigma$ term is concerned. The constant term appears difficult to calculate.

The graphs shown in Figs. 4(a) and 4(b) have no $\ln\sigma$ term for the same reason that the graph in Fig. 1(b) had no such term. It is important to realize, however, that one must include along with the graphs of Fig. 4(b) all the counterterms (subtractions), including many photon composite vertices, necessary to make these graphs converge. If we now add the graph which is identical to that of Fig. 3 except that the photon interaction occurs with the outgoing fermion line we obtain the result

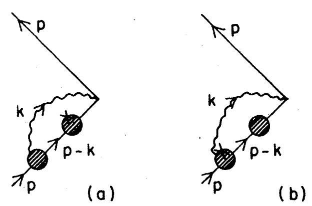

FIG. 4. Subdominant contributions to γ_σ.

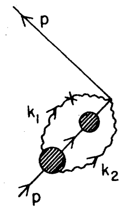

FIG. 5. A higher-vertex contribution to γ_σ.

$$\gamma_\sigma \underset{\sigma\to\infty}{\sim} \frac{g^2}{4\pi}\ln\sigma + 2C(g)$$

for all graphs having one photon at the bare composite vertex.

Consider now graphs having two photons at the bare composite vertex. Begin with the graph shown in Fig. 5. The expression for γ_σ is

$$\gamma_\sigma = \frac{i\mu^2 p_-^{-\sigma+1}}{4}\left(\frac{g^2}{(2\pi)^4}\right)^2 \int \frac{d^4k_1 d^4k_2 \,\mathrm{tr}\gamma_+\gamma_-[-iS(p-k_1-k_2)]\Gamma_{--}(p,k_1,k_2)}{(k_1^2-\mu^2+i\epsilon)^2(k_2^2-\mu^2+i\epsilon)} \times[(p-k_1-k_2)_-^{\sigma-1}-(p-k_1)_-^{\sigma-1}-(p-k_2)_-^{\sigma-1}+p_-^{\sigma-1}]\frac{1}{k_{1-}k_{2-}}\,. \tag{13}$$

The resulting integral is ultraviolet divergent and so subtraction terms, shown in Fig. 6 and Fig. 7, must be included. The term shown in Fig. 6(b) is assumed to be fully renormalized. The precise rule for the counterterm in Fig. 6 is

$$\gamma_\sigma^{\mathrm{ct}} = \frac{\mu^2 p_-^{-\sigma+1}}{4}\left(\frac{g^2}{(2\pi)^4}\right)\int \frac{d^4k_1 d^4k_2\,\mathrm{tr}\gamma_+\gamma_-[-iS(p-\hat{k}_1-k_2)]\Gamma_-(p-\hat{k}_1,k_2)}{(k_1^2-\mu^2+i\epsilon)^2(k_2^2-\mu^2+i\epsilon)}\,\frac{[-iS(p-k_1)]\Gamma_-(p,k_1)}{k_{1-}k_{2-}} \times[(p-k_1-k_2)_-^{\sigma-1}-(p-k_1)_-^{\sigma-1}-(p-k_2)_-^{\sigma-1}+p_-^{\sigma-1}] \tag{14}$$

where the many-photon subtractions in (14) are suppressed. In (14) $\hat{k}_+ = \underline{\hat{k}} = 0$ while $\hat{k}_- = k_-$. The rules for subtractions are the Bogolubov-Parasiuk-Hepp-Zimmermann (BPHZ) rules given in Ref. 7. There are now three regions which must be considered as possible sources of $\ln\sigma$ and $\ln^2\sigma$ terms. These regions are (i) $k_{1-}\to 0$, k_{2-} finite as $\sigma\to\infty$, (ii) $k_{2-}\to 0$, k_{1-} finite as $\sigma\to\infty$, and (iii) $k_{1-}\to 0$, $k_{2-}\to 0$ as $\sigma\to\infty$. (It is useful to note that k_{1-} and k_{2-} may be restricted to the region $k_{1-}>0$, $k_{2-}>0$, $k_{1-}+k_{2-}<p_-$ in (13) and (14).)

Begin with region (i). Write

$$\Gamma_{--}(p,k_1,k_2) = \frac{1}{k_{1+}-i\epsilon}[-k_{1-}\Gamma_{+-}(p,k_1,k_2)+k_{1i}\Gamma_{i-}(p,k_1,k_2)-iS^{-1}(p-k_1-k_2)S(p-k_2)\Gamma_-(p,k_2) +i\Gamma_-(p-k_1,k_2)S(p-k_1)S^{-1}(p)]. \tag{15}$$

Substituting (15) into (13) we again get four contributions to γ_σ. Arguing exactly as before, the only possible $\ln\sigma$ term comes from the third term on the right-hand side of (15). From this term we obtain a contribution $\tilde{\gamma}_\sigma$ to γ_σ where

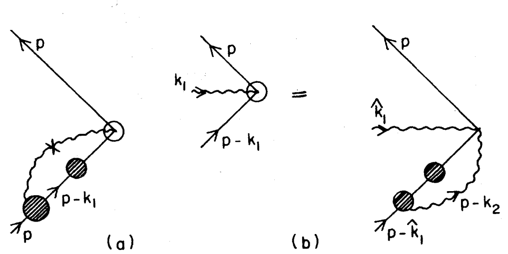

FIG. 6. A renormalization contribution to γ_σ.

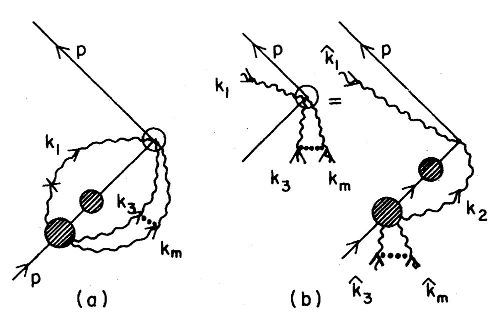

FIG. 7. A renormalization contribution to γ_σ.

$$\tilde{\gamma}_\sigma = -\frac{i\mu^2 p_-^{-\sigma+1}}{4}\left(\frac{g^2}{(2\pi)^4}\right)^2 \int \frac{d^4k_1 d^4k_2 \,\mathrm{tr}\gamma_+\gamma_- S(p-k_2)\Gamma_-(p,k_2)}{(k_{1+}-i\epsilon)(k_1^2-\mu^2+i\epsilon)^2(k_2^2-\mu^2+i\epsilon)}$$

$$\times[(p-k_1-k_2)^{\sigma-1}-(p-k_1)^{\sigma-1}-(p-k_2)^{\sigma-1}+p_-^{\sigma-1}]\frac{1}{k_{1-}k_{2-}}\,. \tag{16}$$

The d^4k_2 integral in (16) is divergent. In order to obtain a finite result we must keep the contribution in γ_σ which could cancel this divergence. (Other divergences in the k_2 variable are canceled by the counterterms shown in Fig. 7.) In γ_σ write $\Gamma_-(p,k_1)$ as

$$\Gamma_-(p,k_1)=\frac{1}{k_{1+}-i\epsilon}[-k_{1-}\Gamma_+(p,k_1)+k_{1i}\Gamma_i(p,k_1)-iS^{-1}(p-k_1)+iS^{-1}(p)]\,. \tag{17}$$

The only term on the right-hand side of (17) which can give a $\ln\sigma$ for $k_{1-}\to 0$ is the $S^{-1}(p-k_1)$ term. This contribution is γ_σ is

$$\tilde{\gamma}_\sigma = \frac{i\mu^2 p_-^{-\sigma+1}}{4}\left(\frac{g^2}{(2\pi)^4}\right)^2 \int \frac{d^4k_1 d^4k_2 \,\mathrm{tr}\gamma_+\gamma_- S(p-\hat{k}_1-k_2)\Gamma_-(p-\hat{k}_1,k_2)}{(k_{1+}-i\epsilon)(k_1^2-\mu^2+i\epsilon)^2(k_2^2-\mu^2+i\epsilon)}$$

$$\times[(p-k_1-k_2)^{\sigma-1}-(p-k_1)_-^{\sigma-1}-(p-k_2)_-^{\sigma-1}+p_-^{\sigma-1}]\frac{1}{k_{1-}k_{2-}}\,. \tag{18}$$

However, the small-k_{1-} regions of (16) and (18) cancel. The net contribution of the small-k_{1-} region is linearly convergent in k_- for large σ and so no $\ln\sigma$ can emerge.

Now consider region (ii). In this case $k_{2-}\to 0$ which allows one to distort the k_{2+} contour far into the lower half plane. When $k_{2+}\to\infty$ the possible large momentum flows in $\Gamma_{--}(p,k_1,k_2)$ are exactly those which are subtracted out by the graphs shown in Figs. 6 and 7. Thus there is a $1/k_{2+}^2$ decrease, so long as $k_{2+}k_{2-}$ is not large, when the k_{2+} contour is distorted into the lower half plane. After multiplying by the range of k_{2+} integration one finds a linearly convergent k_{2-} integral for large σ. Thus again there is no $\ln\sigma$.

Now that regions (i) and (ii) have been dealt with, it is clear that region (iii), where k_{1-} and k_{2-} both go to zero for large σ, also cannot give an $\ln\sigma$ behavior. If, for example, $k_{1-}>k_{2-}$, one easily sees that the small values of k_{1-} do not enhance the large k_{2+} regions. If $k_{2-}>k_{1-}$ one observes that the small values of k_{2-} do not stop the cancellation of Eqs. (16) and (18) as discussed above.

Graphs having two photons at the bare composite vertex and having mass insertions other than as shown in Fig. 5 do not give $\ln\sigma$ terms for exactly the same reasons that we have just given. When one of the momenta, say k_{1-}, becomes small as σ becomes large the contour distortion in k_{1+} gives a rapidly convergent resultant k_{1-} integral. As in the case of region (ii) above, the role of the renormalization terms is crucial.

Graphs with higher numbers of photons at the composite vertex work in a similar way. Consider, for example, the graphs shown in Fig. 8. Suppose $k_{1-}\to 0$ as $\sigma\to\infty$. From our previous discussion it is clear that the k_1^2 branch points of $\Gamma_{--\cdots-}(p,k_1,k_2,\ldots,k_n)$ give singularities in k_{1+} in the lower half plane $-i\epsilon$, at real values such that $k_{1+}\gtrsim \mu^2/2k_-$ while all the other k_{1+} singularities of Γ occur in the upper half plane $+i\epsilon$. [We may view $\Gamma_{--\cdots-}(p,k_1,k_2,\ldots,k_n)$ as depending on the $4n-3$ invariants $k_1^2+i\epsilon$, $k_2^2+i\epsilon,\ldots,k_n^2+i\epsilon$, $(p-k_1)^2+i\epsilon$, $(p-k_2)^2+i\epsilon,\ldots,(p-k_n)^2+i\epsilon$, $(p-k_1-k_2)^2+i\epsilon$, $(p-k_1-k_3)^3+i\epsilon,\ldots,$ $(p-k_1-k_n)^2+i\epsilon$, $(p-k_1-k_2-k_3)^2+i\epsilon$, $(p-k_1-k_2-k_4)^2+i\epsilon,\ldots,$ $(p-k_1-k_2-k_n)^2+i\epsilon$. $k_{i-}>0$ and $\sum_{i=1}^n k_{i-}<p_-$.] Thus the k_{1+} contour can be distorted into the lower half plane a distance greater than or equal to

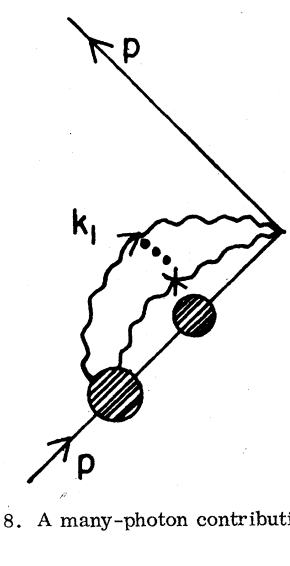

FIG. 8. A many-photon contribution to γ_σ.

$\mu^2/2k_-$ from the origin. Now the large k_{1+} momentum flows through the graph in exactly those ways which determine the subtractions necessary to give a convergent d^4k_1 integral. Again the renormalizations cancel the dominant k_{1+} behavior leaving a linearly convergent k_{1-} integration for large σ. In case the mass insertion is on the k_1 line one uses the Ward identity as was used for the small k_{1-} region of the graphs shown in Fig. 5.

Thus our final result, when vacuum polarization is neglected is

$$\gamma_\sigma \underset{\sigma\to\infty}{\sim} \frac{g^2}{4\pi^2}\ln\sigma + 2c(g)\,. \tag{19}$$

C. γ_σ in massive QED including vacuum polarization

Vritually all of the arguments which have been given in Sec. II B extend trivially to the case where vacuum polarization graphs are included. Consider the graph shown in Fig. 9. The contribution to γ_σ of this graph is

$$\gamma_\sigma = \frac{p_-^{-\sigma+1}}{4}\frac{g^2}{(2\pi)^4}\int \frac{d^4k\,\mathrm{tr}\gamma_+\gamma_-\gamma\cdot(p-k)\gamma_-}{(p-k)^2-m^2+i\epsilon} \times \hat{D}_{+-}(k)\frac{(p-k)_-^{\,\sigma-1}-p_-^{\,\sigma-1}}{k_-}\,, \tag{20}$$

where $\hat{D}_{\mu\nu}(k)$ is the mass-inserted photon propagator. It is easy to see that for large σ

$$\gamma_\sigma \underset{\sigma\to\infty}{\sim} \frac{g^2}{8\pi^2}\ln\sigma \int d^2\underline{k}\, i\hat{D}_{+-}(k)\Big|_{k_+=k_-=0} + \text{constant terms}\,. \tag{21}$$

If we write

$$D_{\mu\nu} = g_{\mu\nu}\frac{-i}{(k^2-\mu^2)[1+\pi(k^2)]}$$

where $\pi(\mu^2)=0$, then

$$i\hat{D}_{+-}(k)\Big|_{k_+=k_-=0} = -\left(\mu^2\frac{\partial}{\partial\mu^2}+m^2\frac{\partial}{\partial m^2}+\beta\frac{\partial}{\partial g}\right) \times\frac{1}{(\underline{k}^2+\mu^2)[1+\pi(-\underline{k}^2)]}\,. \tag{22}$$

In general it does not seem possible to completely evaluate Eq. (21): however we may say that

$$\gamma_\sigma \underset{\sigma\to\infty}{\sim} \frac{g^2}{8\pi^2}h(g)\ln\sigma$$

for the contribution of the graphs shown in Fig. 9. Here $h(g)=1+\sum_{n=1}^{\infty}(g^2)^n h_n$. The discussion of radiative corrections and higher-order vertices proceeds just as before. We thus find in general that

$$\gamma_\sigma \underset{\sigma\to\infty}{\sim} \frac{g^2}{4\pi^2}h(g)\ln\sigma + 2c(g) \tag{23}$$

where

$$h(g) = -\int d\underline{k}^2\left(\mu^2\frac{\partial}{\partial\mu^2}+m^2\frac{\partial}{\partial m^2}+\beta\frac{\partial}{\partial g}\right) \times\frac{1}{(\underline{k}^2+\mu^2)[1+\pi(-\underline{k}^2)]}\,. \tag{24}$$

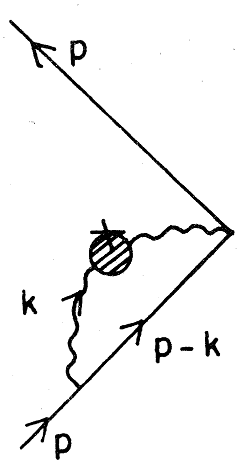

FIG. 9. The dominant contribution to γ_σ for large σ.

III. A FACTORIZATION OF THE VERTEX FUNCTION

In this section a factorized form for the vertex function for two off-mass-shell fermions in a massive Abelian gauge theory will be given. Consider the vertex function shown in Fig. 10. We suppose that the p_1 line is a fermion and the p_2 line is an antifermion. Define $u(p_1)$ such that $[\gamma\cdot p_1 + (p_1^2)^{1/2}]u(p_1)=0$ and define $v(p_2)$ such that $[\gamma\cdot p_2 - (p_2^2)^{1/2}]v(p_2)=0$. We suppose that p_1 and p_2 are timelike although our factorized expression will not depend on that assumption. If $\Gamma_\mu(p_1,p_2)$ is the vertex function shown in Fig. 10 then, for large values of $q^2=(p_1+p_2)^2$ and fixed values of p_1^2, p_2^2, define $\Gamma(p_1,p_2)$ by

$$\tilde{u}(p_1)\Gamma_\mu(p_1,p_2)v(p_2) = \tilde{u}(p_1)\gamma_\mu v(p_2)\Gamma(p_1,p_2)\,. \tag{25}$$

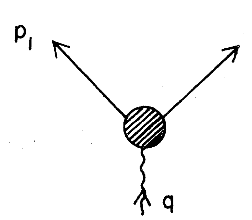

FIG. 10. The form factor.

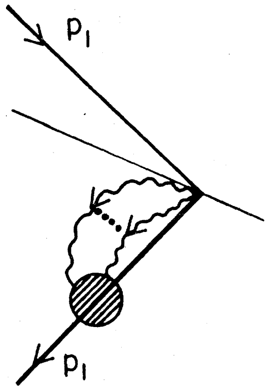

FIG. 11. A cut vertex giving $v(p_1^2)$.

Then $\Gamma(p_1,p_2)=\Gamma(p_1^2,q^2,p_2^2)$. The factorization is given by

$$\begin{aligned}\Gamma(p_1^2,q^2,p_2^2)&=v(p_1^2)\Gamma(0,q^2,0)v(p_2^2)\,,\\ \Gamma(p_1^2,q^2,p_2^2)&=v(p_1^2)\Gamma(q^2)v(p_2^2)\end{aligned} \tag{26}$$

and holds so long as

$$\frac{p_1^2p_2^2}{\mu^2q^2}\ll 1\,. \tag{27}$$

[If p_1^2 or p_2^2 go to zero the criterion (27) changes. In case $p_1^2\to 0$ (27) is replaced by $m^2p_2^2/\mu^2q^2\ll 1$ and $p_2^2/q^2\ll 1$.] $v(p^2)$, not to be confused with the Dirac spinor $v(p)$, is given by

$$v(p^2)=\frac{p_-^{\ \sigma}}{4}\,\mathrm{tr}\gamma_+\sum_{n=0}^{\infty}\Gamma_{\sigma,n}(p)\,, \tag{28}$$

in case $p_\mu=(E,0,0,-p)$. In (28) $\Gamma_{\sigma,n}(p)$ is the particular cut vertex shown in Fig. 11 and $\Gamma_{\sigma,0}(p)=\gamma_-p_-^{\ -\sigma}$. The detailed definitions and renormalization of the Γ_σ are discussed in Ref. 7. The σ label is redundant in this example and appears only in order to make connection with renormalization programs necessary to given meaning to $v(p^2)$. If the p_1 line were an incoming line a similar expression for $v(p_1^2)$ would exist in terms of the incoming cut vertices such as appear in deeply inelastic electron scattering. Two points are crucial: (i) The vertices appearing in (26) are renormalized. (ii) $\Gamma(q^2)$ does not obey a Callan-Symanzik equation because the mass inserted vertex functions do not renormalize according to dimensional counting. This last point is discussed in Ref. 5 in some detail.

The arguments for the validity of (26) have been previously discussed. There is no rigorous proof of this result; however the heuristic arguments given in Ref. 5 make it a rather plausible result. The factorization includes all logarithms and is not simply a leading logarithm result. In the Appendix the one-loop vertex graph is discussed as an example of (26).

IV. THE SUDAKOV FORM FACTOR

In this section we shall put together the results of Secs. II and III and derive an expression for the Sudakov form factor. This expression will include all factors of $\ln^n q^2$ but will neglect inverse powers of q^2.

Begin by considering the graphs shown in Fig. 12. These graphs are to be considered as a contribution to νW_2. Only the vertex function $\Gamma_\mu(p,p+q)$ is included with no self-energy corrections on the p or $p+q$ lines. Now, choosing $p_+=\underline{p}=0$ we have

$$\int_1^\infty d\omega\,\omega^{-\sigma}\nu W_2=E_\sigma(q^2)\,. \tag{29}$$

For large q^2 this E_σ obeys the Callan-Symanzik equation

$$\left(-q^2\frac{\partial}{\partial q^2}+\beta\frac{\partial}{\partial g}+\delta\frac{\partial}{\partial\alpha}-\gamma_\sigma'\right)E_\sigma(q^2)=0\,. \tag{30}$$

γ_σ' is that contribution to γ_σ coming from graphs where photons appear in an arbitrary way at the composite vertex but end on only one of the external fermions in the mass inserted vertex function. α is the covariant gauge parameter.

Let us first suppose that vacuum-polarization graphs are neglected. Then $\beta=0$ and

$$E_\sigma(q^2)=e_\sigma\exp[-\gamma_\sigma'(g)\ln(-q^2/\mu^2)]\,. \tag{31}$$

For large σ, $(p+q)^2/q^2\ll 1$ in the integral (29) so one may use (26) to write

$$E_\sigma(q^2)=-\frac{\Gamma(q^2)}{\pi}\int_{1-\epsilon'}^\infty d\omega\,\omega^{-\sigma}\,\mathrm{Im}\left[\frac{v(-q^2(\omega-1+i\epsilon))}{\omega-1+i\epsilon}\right], \tag{32}$$

where $\epsilon\to 0$, $\epsilon'\to 0$, $\epsilon/\epsilon'\to 0$. We have written (32) for spacelike q^2. q^2 timelike can be obtained by

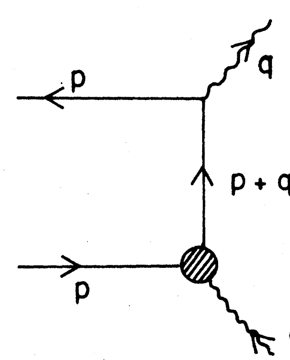

FIG. 12. A particular contribution to νW_2.

continuation. By a change of variables we can write (32) as

$$E_\sigma(q^2) = -\frac{\Gamma(q^2)}{\pi}\int_{-\epsilon'}^{\infty} d\lambda e^{-\lambda}\,\mathrm{Im}\left[\frac{v((-q^2/\sigma)\lambda+i\epsilon)}{\lambda+i\epsilon}\right]. \tag{33}$$

Write (33) as

$$E_\sigma(q^2) = \Gamma(q^2)f\left(\frac{q^2}{\sigma}\right). \tag{34}$$

Thus

$$e_\sigma \exp[-\gamma'_\sigma(g)\ln(-q^2/\mu^2)] = \Gamma(q^2)f\left(\frac{q^2}{\sigma}\right). \tag{35}$$

For large values of σ, $\gamma'_\sigma \to \frac{1}{2}\gamma_\sigma$ so $\gamma'_\sigma(g) \to (g^2/8\pi^2)\ln\sigma + c(g)$. Defining

$$e'_\sigma = \exp\left(-\frac{g^2}{16\pi^2}\ln^2\sigma\right)e_\sigma$$

and

$$l\left(\frac{q^2}{\sigma}\right) = \frac{\exp\left[\frac{g^2}{16\pi^2}\ln^2\left(-\frac{q^2}{\mu^2\sigma}\right)\right]}{f(q^2/\mu^2\sigma)}$$

one finds

$$\Gamma(q^2) = l\left(\frac{q^2}{\sigma}\right)e'_\sigma \exp\left[-\frac{g^2}{16\pi^2}\ln^2\left(-\frac{q^2}{\mu^2}\right) - c(g)\ln\left(-\frac{q^2}{\mu^2}\right)\right]. \tag{36}$$

Equation (36) means that

$$e'_\sigma = \sigma^\nu, \quad l(x) = x^\nu,$$

with ν depending on g^2 and μ^2/m^2. Thus, it is always possible to write

$$\Gamma(q^2) = \Gamma_0 \exp\left[-\frac{g^2}{16\pi^2}\ln^2\left(-\frac{q^2}{M^2}\right)\right], \tag{37}$$

where $M^2 = M^2((\mu^2/m^2), g)$. This is the famous double-logarithm form of the Sudakov form factor.

Now include vacuum-polarization graphs. Then

$$E_\sigma(q^2) = e_\sigma \exp\left\{-\int_{\mu^2}^{-q^2}\frac{dQ^2}{Q^2}\gamma'_\sigma[g(Q^2,g)]\right\}, \tag{38}$$

where e_σ depends on σ and $g^2(q^2,g)$. In case there is a fixed point at $g = g_\infty$ one easily finds

$$\Gamma(q^2) \underset{q^2\to-\infty}{\sim} \Gamma_0 \exp\left[-\frac{g_\infty^{\;2}}{16\pi^2}h(g_\infty)\ln^2\left(-\frac{q^2}{M^2}\right)\right], \tag{39}$$

where M^2 depends on μ^2/m^2, g, and g_∞.

We can get an expression closer to that which might emerge in a non-Abelian gauge theory if we choose $g^2 < 0$ in the Abelian theory. In this case the theory is asymptotically free though it is not a unitary theory. In this case one finds

$$E_\sigma(q^2) = e_\sigma[g(q^2,g),g]\exp\{-[\ln\sigma\gamma_1(q^2) + \gamma_2(q^2)]\} = \Gamma(q^2)f\left(\frac{q^2}{\sigma}\right). \tag{40}$$

In (40)

$$\gamma_1(q^2) = \tfrac{3}{2}\ln\ln\left(-\frac{q^2}{\mu^2}\right) + O\left(\frac{\ln\ln(-q^2)}{\ln(-q^2)}\right),$$

$$\gamma_2(q^2) = \tfrac{3}{2}(C_E - 1)\ln\ln\left(-\frac{q^2}{\mu^2}\right) + O\left(\frac{\ln\ln(-q^2)}{\ln(-q^2)}\right),$$

and we have specialized to the Landau gauge. If $\ln\sigma \gg 1$ but $[\ln\sigma/\ln(-q^2)]\ln\ln(-q^2) \ll 1$ we may write

$$e_\sigma\left[-\tfrac{3}{2}\ln\sigma\ln\ln\left(-\frac{q^2}{\mu^2}\right) - \tfrac{3}{2}(C_E-1)\ln\ln\left(-\frac{q^2}{\mu^2}\right)\right] = \Gamma(q^2)f\left(\frac{q^2}{\sigma}\right). \tag{41}$$

If we further choose $\ln^2\sigma/(\ln - q^2) \ll 1$, e_σ only depends on g and σ and we may write

$$\Gamma(q^2) = \exp\left[-\tfrac{3}{2}\ln\left(-\frac{q^2}{\mu^2}\right)\ln\ln\left(-\frac{q^2}{\mu^2}\right) - \tfrac{3}{2}(C_E-1)\ln\ln\left(-\frac{q^2}{\mu^2}\right)\right]\tilde{e}_\sigma l\left(\frac{q^2}{\mu^2\sigma}\right), \tag{42}$$

where

$$\tilde{e}_\sigma = e^{(3/2)\ln\sigma}e_\sigma,$$

$$l\left(\frac{q^2}{\mu^2\sigma}\right) = \frac{\exp\left[\frac{3}{2}\ln\left(-\frac{q^2}{\mu^2\sigma}\right)\ln\ln\left(-\frac{q^2}{\mu^2\sigma}\right)\right]}{f\left(\frac{q^2}{\mu^2\sigma}\right)}.$$

Now if we were free to vary σ widely it could be concluded that $\tilde{e}l = (-q^2/\mu^2)^\gamma$; however, we are restricted to the region $\ln^2\sigma/\ln(-q^2) \ll 1$ and this restriction allows l to have dependences other than a pure power. For example, l can have a term such as $[\ln(-q^2/\mu^2\sigma)]^\gamma$ and to the order we are working such a term has no σ dependence. Our conclusion is that the strongest behavior of l allowed is $(-q^2/\mu^2)^\gamma$ but that many weaker q^2 dependences are allowed. We can state this precisely by writing

$$\ln\Gamma(q^2) \underset{q^2\to-\infty}{\sim} -\tfrac{3}{2}\ln\left(-\frac{q^2}{\mu^2}\right)\ln\ln\left(-\frac{q^2}{\mu^2}\right) + O\left(\ln\left(-\frac{q^2}{\mu^2}\right)\right). \tag{43}$$

We suspect that the correction to the dominant term in (43) is in fact of order $\ln\ln(-q^2/\mu^2)$, but this has not been established. If one now supposes that non-Abelian theories work much the same as Abelian theories, one would arrive at the result

$$\ln\Gamma(q^2) \underset{q^2\to-\infty}{\sim} \frac{8}{11C_2(G)-4T(R)}\ln\left(-\frac{q^2}{\mu^2}\right) \times \ln\ln\left(-\frac{q^2}{\mu^2}\right) \quad (44)$$

where the notation is standard.

ACKNOWLEDGMENT

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APPENDIX

In this appendix it will be shown how the factorized expression (26) reproduces the one-loop form factor in massive QED. We begin with a calculation of the order g^2 expression for $v(p^2)$. The relevant graph is shown in Fig. 13 and we choose $\underline{p}=(p_1,p_2)=0$ for simplicity. Then

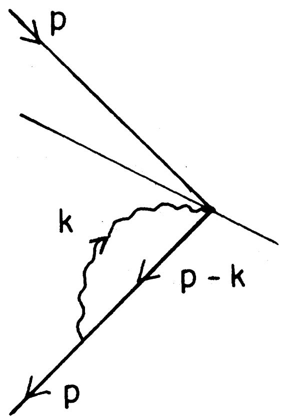

FIG. 13. The lowest-order contribution to $v(p^2)$.

$$v(p^2)=\frac{ip_-^{\sigma}}{4}\frac{g^2}{(2\pi)^4}\left\{\int\frac{d^4k\,\mathrm{tr}\gamma_+\gamma_-\gamma\cdot(p-k)\gamma_-}{(k^2-\mu^2+i\epsilon)[(p-k)^2-m^2+i\epsilon]}\frac{p_-^{-\sigma}}{k_-}-\int\frac{d^4k\,\mathrm{tr}\gamma_+\gamma_-(\hat{p}-k)\gamma_-}{(k^2-\mu^2+i\epsilon)[(\hat{p}-k)^2-m^2+i\epsilon]}\frac{p_-^{-\sigma}}{k_-}\right\}. \quad (A1)$$

Thus

$$v(p^2)=\frac{2ig^2}{(2\pi)^4}\int\frac{d^4k}{k_-}\frac{(p-k)_-}{(k^2-\mu^2+i\epsilon)}\left[\frac{1}{(p-k)^2-m^2+i\epsilon}-\frac{1}{(\hat{p}-k)^2-m^2+i\epsilon}\right],$$

where $\hat{p}_+=0$, $\hat{p}_-=p_-$. If the k_+ contour is distorted so as to enclose the pole at $k_+=[(\underline{k}^2+\mu^2)/2k_-)]-i\epsilon$, one finds

$$v(p^2)=-\frac{g^2}{8\pi^3}\int_0^{p_-}\frac{dk_-}{k_-}d^2\underline{k}\,(p-k)_-\left[\frac{1}{p_-(\underline{k}^2+\mu^2)+k_-(m^2-\mu^2)-2(p-k)_-k_-p_+}-\frac{1}{p_-(\underline{k}^2+\mu^2)+k_-(m^2-\mu^2)}\right].$$

Then

$$v(p^2)=\frac{g^2}{8\pi^2}\int_0^1\frac{dx}{x}(1-x)\ln\left[\frac{\mu^2+x(m^2-\mu^2)-x(1-x)p^2}{\mu^2+x(m^2-\mu^2)}\right]. \quad (A2)$$

For large p^2

$$v(p^2)=\frac{g^2}{16\pi^2}\ln^2\left(-\frac{p^2}{\mu^2}\right). \quad (A3)$$

Using the result that $\Gamma(q^2)=-g^2/16\pi^2\ln^2(-q^2/\mu^2)$ in lowest order leads to the result

$$\Gamma(p_1,p_2)=-\frac{g^2}{16\pi^2}\left[\ln^2\left(-\frac{q^2}{\mu^2}\right)-1\ln^2\left(-\frac{p_1^2}{\mu^2}\right)-1\ln^2\left(-\frac{p_2^2}{\mu^2}\right)\right] \quad (A4)$$

which is the result for the off-shell vertex function so long as $p_1^2p_2^2/\mu^2Q^2\ll 1$.

The reader may note that it is not permissible to set $\mu^2=0$ in the above expression for $v(p^2)$. The reason for this is that in the zero-mass-photon limit one must exhibit the wee parton cancellation before renormalized cut vertices exist. In inclusive processes the $\mu^2\to 0$ singularity in $v(p^2)$ is canceled by a $\mu^2\to 0$ singularity in a different cut vertex. However, in the form factor there is no wee-parton cancellation. In our context this simply means that we are not able to discuss the $p_1^2p_2^2/\mu^2Q^2\geq O(1)$ region of the Sudakov form factor.

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