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Dynamics of a relativistic shell model for extended particles

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The relativistic harmonic-oscillator model for extended particles with SO(3,2) internal symmetry is used to compute the elastic form factor of the ground state and the radiative decay rate of the $s = 1 \rightarrow s = 0$ transition. The interactions considered do not test all possibilities of the model. The results are therefore only illustrations of effects due to the extended structure of the particles. The relativistic effects suppress the decay into final states with masses less than 17% of the initial mass, whereas the nonrelativistic decay rate is constant in this region. A representation of the excited states of the model by Hermite polynomials is given.

I. INTRODUCTION

In this paper we study elastic and inelastic interactions in a model of composite/extended particles based on the dynamical symmetry group[1] SO(3, 2). The kinematics of the model has been developed in Refs. 2–4; still the question about the existence and properties of interactions is left over.

The matter is not trivial as the model is related to Dirac's positive-energy equation,[5–7] which has no minimal-coupling interaction; this deprived it of any application.

However, recognizing the nonlocal nature of the model, it becomes clear that an overall minimal coupling, i.e., replacing *all* momentum operators P_μ by $P_\mu - eA_\mu$, generates a complicated array of interactions, compared to the local field theories.[8] Instead of a global study we take only the simplest terms of the interactions into account.

The "anti–de Sitter" SO(3, 2) model is a special case ($n = 2$) of a class of models governed by the real symplectic group $Sp(2n, R)$. One has local isomorphisms[2,9] for the lowest-dimensional symplectic groups $Sp(2,R) \sim SU(1,1)$, $Sp(4,R) \sim SO(3,2)$ and none for $Sp(8,R)$ and higher groups. Each of these groups when represented in terms of n boson operators and their conjugate operators[10–11] allows for a construction of a relativistic harmonic-oscillator (RHO) model for composite particles.

The calculations and procedures presented here extend to the entire class of $Sp(2n, R)$ models.

The essential idea in the construction of the RHO models is that under Lorentz transformations the boson operators themselves transform $a(0) \rightarrow a(\hat{p})$ and $a^\dagger(0) \rightarrow a^\dagger(\hat{p})$. The important distinction to the nonrelativistic harmonic-oscillator (HO) model of nuclear physics is the velocity dependence of the wave function.

In Refs. 12–16 wave functions are constructed by another method, without using explicitly dynamical groups. The main procedure there is to transform the arguments of the wave function, i.e., to express the wave function in terms of the center of mass and internal coordinates in configuration space and subject each coordinate to a Lorentz transformation $x'_\mu = L(\Lambda)_\mu^{\ \nu} x_\nu$.

The paper is organized as follows. In Sec. II, the kinematics are summarized as far as deviations from Ref. 4 make it necessary and new results are added. A representation of the internal wave function in terms of Hermite polynomials is given, and some symmetry properties of the theory are discussed. In Sec. III the quantum-mechanical model is extended into a quantum field theory, which is nonlocal. In Sec. IV different types of interactions are defined, a form factor and a decay rate are calculated and compared to the nonrelativistic "static" decay rate. The summary and discussion are contained in Sec. V.

II. KINEMATICS

In the $Sp(2n, R)$ theories there are n internal degrees of freedom in addition to the center-of-mass (orbital) coordinates. Thus in the case of $n = 2$ the composite particle is described by the center-of-mass coordinates in Minkowski space $x_\mu \in R^4$ and $p_\mu \in R^4$, $p_\mu{}^2 = M^2 [g_{\mu\nu} = \mathrm{diag}(+, -, -, -)$, $\mu = \nu = 0, 1, 2, 3]$, and the internal coordinates $\xi_k \in R^2$ ($k = 1, 2$). M is the rest mass of the particle.

To each direction ξ_k we associate boson creation $a_k^\dagger$ and annihilation a_k operators in the orthogonal representation

$$a_k^\dagger = 2^{-1/2}\left(\xi_k - \frac{\partial}{\partial \xi_k}\right), \tag{2.1}$$

$$a_k = 2^{-1/2}\left(\xi_k + \frac{\partial}{\partial \xi_k}\right) \tag{2.2}$$

with the commutation relations (CR's)

$$[a_k, a_l^\dagger] = \delta_{kl}, \tag{2.3}$$

$$[a_k, a_l] = 0, \quad [a_k^\dagger, a_l^\dagger] = 0 \quad (k, l = 1, 2). \tag{2.4}$$

A continuous representation of (2.1) and (2.2) in spherical coordinates leads to internal two-dimensional Kepler orbits. However, the harmonic-oscillator motion and the Kepler motion are isomorphic, and therefore can be translated into each other.[17]

The Lie algebra of Sp(4, R) differs from Ref. 4 in that the magnetic number is measured along the 03 axis[18-20]:

$$J_1 = S_{23} = -\tfrac{1}{2}(a_1^\dagger a_2 + a_2^\dagger a_1) , \tag{2.5}$$

$$J_2 = S_{31} = \tfrac{1}{2}i\,(a_1^\dagger a_2 - a_2^\dagger a_1) , \tag{2.6}$$

$$J_3 = S_{12} = \tfrac{1}{2}(a_1^\dagger a_1 - a_2^\dagger a_2) , \tag{2.7}$$

$$K_1 = S_{10} = \tfrac{1}{4}(a_1^{\dagger 2} + a_1^2 - a_2^{\dagger 2} - a_2^2) , \tag{2.8}$$

$$K_2 = S_{20} = -\tfrac{1}{4}i\,(a_1^{\dagger 2} - a_1^2 + a_2^{\dagger 2} - a_2^2) , \tag{2.9}$$

$$K_3 = S_{30} = \tfrac{1}{2}(a_1^\dagger a_2^\dagger + a_1 a_2) , \tag{2.10}$$

$$V_0 = S_{04} = \tfrac{1}{2}(a_1^\dagger a_1 + a_2^\dagger a_2 + 1) , \tag{2.11}$$

$$V_1 = S_{14} = \tfrac{1}{4}i\,(a_1^{\dagger 2} - a_1^2 - a_2^{\dagger 2} + a_2^2) , \tag{2.12}$$

$$V_2 = S_{24} = \tfrac{1}{4}(a_1^{\dagger 2} + a_1^2 + a_2^{\dagger 2} + a_2^2) , \tag{2.13}$$

$$V_3 = S_{34} = \tfrac{1}{2}i\,(a_1^\dagger a_2^\dagger - a_1 a_2) . \tag{2.14}$$

The S_{ab} operators satisfy the CR's of SO(3, 2)

$$[S_{ab}, S_{cd}] = -i\,(\eta_{ac} S_{bd} + \eta_{bd} S_{ac} - \eta_{ad} S_{bc} - \eta_{bc} S_{ad}) \tag{2.15}$$

with the metric $\eta_{ab} = \mathrm{diag}(+ - - - +)$, $a, b, c, d = 0, 1, 2, 3, 4$. Physically $\vec{J}$ and $\vec{K}$ are the generators of rotations and boosts, V_μ is a four-vector.

The subalgebra of $\{S_{\mu\nu}\}$ $(\mu, \nu = 0, 1, 2, 3)$ is the Lie algebra of the Lorentz group O(3, 1). For $a_2 = 0$, $\mathfrak{L}$Sp(4, R) reduces to $\mathfrak{L}$Sp(2, R), and by doubling $S_{\mu\nu}$ by another pair of boson operators a_3, a_4 we obtain the O(3, 1) part of Sp(8, R),

$$\mathfrak{L}\mathrm{Sp}(2,R) \subset \mathfrak{L}\mathrm{Sp}(4,R) \subset \mathfrak{L}\mathrm{Sp}(8,R) . \tag{2.16}$$

$S_{\mu\nu}(0)$ with $a_k^\dagger, a_k$ from Eqs. (2.1) are the Lorentz-group generators of the internal motion in the rest frame. The generators of the Poincaré group Π are

$$I_{\mu\nu} = L_{\mu\nu} + S_{\mu\nu}(0) , \tag{2.17}$$

and the generators of the translation group are

$$I_{\mu 4} = P_\mu , \tag{2.18}$$

where

$$L_{\mu\nu} = x_\mu P_\nu - x_\nu P_\mu \tag{2.19}$$

are the O(3, 1) generators for the orbital motion of the particle. There is no translation operator for the internal motion added to P_μ as the internal motion is translation invariant.

In general, $[L, S] = 0$, and the I satisfy the O(3, 1) CR

$$[I_{\mu\nu}, I_{\rho\sigma}] = -i\,(g_{\mu\rho} I_{\nu\sigma} + g_{\nu\sigma} I_{\mu\rho} - g_{\mu\sigma} I_{\nu\rho} - g_{\nu\rho} I_{\mu\sigma}) . \tag{2.20}$$

In an arbitrary reference frame the boson operators are defined by the following mapping. Consider the basis[3,7] $Q = \text{column}\,[a_1^\dagger, a_2^\dagger, a_2, -a_1]$. Then

$$Q(\Lambda) = e^{+i\Lambda_{\mu\nu} S_{\mu\nu}(0)}\, Q(0)\, e^{-i\Lambda_{\mu\nu} S_{\mu\nu}(0)}$$
$$= e^{+i\Lambda_{\mu\nu} \tilde{S}_{\mu\nu}(0)}\, Q(0) , \tag{2.21}$$

where $\tilde{S}_{\mu\nu}$ is a matrix defined by

$$[S_{\mu\nu}, Q_\alpha(0)] = (\tilde{S}_{\mu\nu})_{\alpha\beta} Q_\beta(0) . \tag{2.22}$$

For a boost with magnitude Λ and direction $\vec{e}$ in 3-dimensional space

$$\vec{\Lambda} = \Lambda(e_1, e_2, e_3)$$
$$= \Lambda(\cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta) , \tag{2.23}$$

where θ, ϕ are polar angles. The boosted boson operators are

$$a_1^\dagger(\Lambda) = \cosh(\tfrac{1}{2}\Lambda) a_1^\dagger(0)$$
$$+ i \sinh(\tfrac{1}{2}\Lambda)[(e_1 + ie_2) a_1(0) + e_3 a_2(0)] , \tag{2.24}$$

$$a_2^\dagger(\Lambda) = \cosh(\tfrac{1}{2}\Lambda) a_1^\dagger(0)$$
$$+ i \sinh(\tfrac{1}{2}\Lambda)[(-e_1 + ie_2) a_2(0) + e_3 a_1(0)] , \tag{2.25}$$

$$a_1(\Lambda) = \cosh(\tfrac{1}{2}\Lambda) a_1(0)$$
$$- i \sinh(\tfrac{1}{2}\Lambda)[(e_1 - ie_2) a_1^\dagger(0) + e_3 a_2^\dagger(0)] , \tag{2.26}$$

$$a_2(\Lambda) = \cosh(\tfrac{1}{2}\Lambda) a_2(0)$$
$$- i \sinh(\tfrac{1}{2}\Lambda)[-(e_1 + ie_2) a_2^\dagger(0) + e_3 a_1^\dagger(0)] . \tag{2.27}$$

In a frame rotated by an angle $\vec{\omega} = \omega(e_1, e_2, e_3)$ relative to the rest frame we have

$$a_1^\dagger(\omega) = [\cos(\tfrac{1}{2}\omega) + ie_3 \sin(\tfrac{1}{2}\omega)] a_1^\dagger(0)$$
$$- i(e_1 + ie_2) \sin(\tfrac{1}{2}\omega) a_2^\dagger(0) , \tag{2.28}$$

$$a_2^\dagger(\omega) = [\cos(\tfrac{1}{2}\omega) - ie_3 \sin(\tfrac{1}{2}\omega)] a_2^\dagger(0)$$
$$- i(e_1 - ie_2) \sin(\tfrac{1}{2}\omega) a_1^\dagger(0) , \tag{2.29}$$

$$a_1(\omega) = [\cos(\tfrac{1}{2}\omega) - ie_3 \sin(\tfrac{1}{2}\omega)] a_1(0)$$
$$+ i(e_1 - ie_2) \sin(\tfrac{1}{2}\omega) a_2(0) , \tag{2.30}$$

$$a_2(\omega)=[\cos(\tfrac{1}{2}\omega)+ie_3\sin(\tfrac{1}{2}\omega)]a_2(0)+i(e_1+ie_2)\sin(\tfrac{1}{2}\omega)a_1(0). \qquad (2.31)$$

In general, the boson operators in different frames of reference have the following commutation relations:

$$[a_k(\Lambda),a_l^\dagger(\Lambda')]=\delta_{kl}\cosh\tfrac{1}{2}(\Lambda-\Lambda'), \qquad (2.32)$$

$$[a_k(\Lambda),a_l(\Lambda')]=i[\delta_{k1}\delta_{l1}(e_1-ie_2)-\delta_{k2}\delta_{l2}(e_1+ie_2)+(\delta_{k1}\delta_{l2}+\delta_{k2}\delta_{l1})e_3]\sinh\tfrac{1}{2}(\Lambda-\Lambda'), \qquad (2.33)$$

$$[a_k^\dagger(\Lambda),a_l^\dagger(\Lambda')]=i[\delta_{k1}\delta_{l1}(e_1+ie_2)-\delta_{k2}\delta_{l2}(e_1-ie_2)+(\delta_{k1}\delta_{l2}+\delta_{k2}\delta_{l1})e_3]\sinh\tfrac{1}{2}(\Lambda-\Lambda'). \qquad (2.34)$$

Similarly for rotations:

$$[a_k(\omega),a_l^\dagger(\omega')]=\delta_{kl}\cos\tfrac{1}{2}(\omega-\omega')-ie_3(\delta_{k1}\delta_{l1}-\delta_{k2}\delta_{l2})\sin\tfrac{1}{2}(\omega-\omega')+i[\delta_{k1}\delta_{l2}(e_1-ie_2)-\delta_{k2}\delta_{l1}(e_1+ie_2)]\sin\tfrac{1}{2}(\omega-\omega'), \qquad (2.35)$$

$$[a_k(\omega),a_l(\omega')]\equiv 0, \qquad (2.36)$$

$$[a_k^\dagger(\omega),a_l^\dagger(\omega')]\equiv 0. \qquad (2.37)$$

These commutation relations reduce to (2.3) and (2.4), in identical reference frames ($\Lambda=\Lambda'$), and are, consequently, preserved by a Lorentz transformation.

The internal wave function $|0,0,\Lambda\rangle$ of the ground state (spin $s=0$ and magnetic number $m=0$) in a boosted reference frame is defined by the system of linear partial differential equations:

$$a_k(\Lambda)|0,0,\Lambda\rangle=0\quad(k=1,2). \qquad (2.38)$$

Looking for a complex Gaussian solution of (2.38) of the form

$$|0,0,\Lambda\rangle=\text{Norm}\exp\{-\tfrac{1}{2}[X_1(\xi_1^2+\xi_2^2)+iX_2(\xi_1^2-\xi_2^2)+2iX_3\xi_1\xi_2]\}, \qquad (2.39)$$

we obtain for X

$$X_1=(\cosh\Lambda+e_2\sinh\Lambda)^{-1}, \qquad (2.40)$$

$$X_2=-X_1e_1\sinh\Lambda, \qquad (2.41)$$

$$X_3=-X_1e_3\sinh\Lambda. \qquad (2.42)$$

The normalization

$$\text{Norm}=\pi^{-1/2}X_1^{1/2} \qquad (2.43)$$

is chosen such that

$$\int d^2\xi\langle 0,0,\Lambda|0,0,\Lambda\rangle=1. \qquad (2.44)$$

In the case

$$\cosh\Lambda=\hat{p}_0=(1-v^2)^{-1/2}, \qquad (2.45)$$

$$\sinh\Lambda=\hat{p}=v(1-v^2)^{-1/2}, \qquad (2.46)$$

the ground-state wave function has the same velocity dependence as in Refs. 3–5. With $\hat{p}_l=\hat{p}e_l$ $(l=1,2,3)$ we have

$$X_1=(\hat{p}_0+\hat{p}_2)^{-1/2}, \qquad (2.47)$$

$$X_2=-X_1\hat{p}_1, \qquad (2.48)$$

$$X_3=-X_1\hat{p}_3. \qquad (2.49)$$

In Eqs. (2.45) and (2.46) v is the velocity of the center of mass of the particle and $\hat{p}_0=p_0/M$, $\hat{p}=|\vec{p}|/M$, $\hat{p}_0^2-\hat{p}^2=1$. This choice for Λ must not be unique. Any other velocity dependence of Λ could be interpreted as a Lorentz boost for the internal motion of the composite system (e.g., $\cosh\frac{1}{2}\Lambda=\hat{p}_0$).[21] In Sec. IV we find that (2.45)–(2.46) alone leads to a relativistic invariant form factor.

Higher states with spin s and magnetic number m are obtained by the standard construction for integer and half-integer spins

$$|s,m,\Lambda\rangle=\frac{[a_1^\dagger(\Lambda)]^{s+m}[a_2^\dagger(\Lambda)]^{s-m}}{[(s-m)!(s+m)!]^{1/2}}|0,0,\Lambda\rangle, \qquad (2.50)$$

such that $|s,m,\Lambda\rangle$ are eigenfunctions of $\vec{J}^2$, J_3, V_0,

$$\vec{J}^2(\Lambda)|s,m,\Lambda\rangle=s(s+1)|s,m,\Lambda\rangle, \qquad (2.51)$$

$$J_3(\Lambda)|s,m,\Lambda\rangle,=m|s,m,\Lambda\rangle, \qquad (2.52)$$

$$V_0(\Lambda)|s,m,\Lambda\rangle=(s+\tfrac{1}{2})|s,m,\Lambda\rangle. \qquad (2.53)$$

The wave function is even under reflection of

$\vec{\xi} \to -\vec{\xi}$ for integer spin and odd for half-integer spins:

$$|s, m, \Lambda, -\vec{\xi}\rangle = (-1)^{2s}|s, m, \Lambda, \vec{\xi}\rangle . \quad (2.54)$$

In the theory of the nonrelativistic harmonic oscillator calculations are particularly simple because the wave functions are Hermite polynomials. We show that the RHO model is a true generalization of the HO model, as the wave functions can be written in terms of Hermite polynomials of complicated arguments.

Theorem 1. Given the creation operators $a_k^\dagger(\Lambda)$ [(2.24)–(2.25)] ($k=1, 2$) and the ground-state wave function $|0,0,\Lambda\rangle$ [Eq. (2.39)], in a boosted Lorentz frame Λ, then for each k

$$[a^\dagger(\Lambda)]^n|0,0,\Lambda\rangle = h_3^n H_n(h_1\xi_1 + h_2\xi_2)|0,0,\Lambda\rangle , \quad (2.55)$$

where

$$H_n = (-1)^n e^{(z^2/2)} \frac{d^n}{dz^n} e^{-(z^2/2)}$$

are Hermite polynomials of order n, and h_1, h_2, h_3 are complex numbers defined by

$$h_1h_3 = c_1 - c_3(X_1 + iX_2) - ic_4X_3 , \quad (2.56)$$

$$h_2h_3 = c_2 - c_4(X_1 - iX_2) - ic_3X_3 , \quad (2.57)$$

$$h_3 = -h_1c_3 - h_2c_4 . \quad (2.58)$$

X are the wave-function coefficients Eqs. (2.40)–(2.42) and c are the coefficients used to write the creation operators in the form

$$a^\dagger = c_1\xi_1 + c_2\xi_2 + c_3\frac{\partial}{\partial\xi_1} + c_4\frac{\partial}{\partial\xi_2} . \quad (2.59)$$

For the sake of clarity, the index k is suppressed in theorem 1 and the proof.

Proof. The proof follows by induction. The first two terms ($n=1, 2$) are used to define the constants h_1, h_2, h_3. The comparison of the coefficients of

$$a^\dagger(\Lambda)|0,0,\Lambda\rangle = \{[c_1 - c_3(X_1+iX_2) - ic_4X_3]\xi_1 + [c_2 - c_4(X_1 - iX_2) - ic_3X_3]\xi_2\}|0,0,\Lambda\rangle$$
$$= h_3(h_1\xi_1 + h_2\xi_2)|0,0,\Lambda\rangle \quad (2.60)$$

produces Eqs. (2.56) and (2.57). The second power $(a^\dagger)^2$ yields, by using only (2.56) and (2.57)

$$[a^\dagger(\Lambda)]^2|0,0,\Lambda\rangle = [h_3^2(h_1\xi_1 + h_2\xi_2)^2 + c_3h_1h_3 + c_4h_2h_3]|0,0,\Lambda\rangle , \quad (2.61)$$

which compared to

$$[a^\dagger(\Lambda)]^2|0,0,\Lambda\rangle = h_3^2[(h_1\xi_1 + h_2\xi_2)^2 - 1]|0,0,\Lambda\rangle \quad (2.62)$$

gives

$$c_3h_1h_3 + c_4h_2h_3 = -h_3^2 \quad (2.63)$$

or equivalently the last Eq. (2.58). We conclude the proof by showing that if the representation is valid for n it also holds for $n+1$. If

$$[a^\dagger(\Lambda)]^n|0,0,\Lambda\rangle = h_3^n H_n(h_1\xi_1 + h_2\xi_2)|0,0,\Lambda\rangle , \quad (2.64)$$

then ($z = h_1\xi_1 + h_2\xi_2$)

$$[a^\dagger(\Lambda)]^{n+1}|0,0,\Lambda\rangle = h_3^n\left[(c_1\xi_1 + c_2\xi_2)H_n(z) + c_3\frac{\partial}{\partial\xi_1}H_n(z) + c_4\frac{\partial}{\partial\xi_2}H_n(z) + c_3H_n(z)\frac{\partial}{\partial\xi_1} + c_4H_n(z)\frac{\partial}{\partial\xi_2}\right]|0,0;\Lambda\rangle \quad (2.65)$$

$$= h_3^{n+1}\left[zH_n(z) - \frac{\partial}{\partial z}H_n(z)\right]|0,0,\Lambda\rangle . \quad (2.66)$$

From $(\partial/\partial z)H_n(z) = nH_{n-1}(z)$ and the recursion relation $H_{n+1}(z) - zH_n(z) + nH_{n-1}(z) = 0$ follows

$$[a^\dagger(\Lambda)]^{n+1}|0,0,\Lambda\rangle = h_3^{n+1}H_{n+1}(z)|0,0,\Lambda\rangle . \text{ Q.E.D.} \quad (2.67)$$

In the nonrelativistic case, which is identical to the rest frame, $h_3 \to 1$ for both creation operators $a_1^\dagger$, $a_2^\dagger$ while $h_1 \to 2^{1/2}$, $h_2 \to 0$ for $a_1^\dagger$ and $h_1 \to 0$, $h_2 \to 2^{1/2}$ for $a_2^\dagger$. Therefore,

$$[a_k^\dagger(0)]^n|0,0,0\rangle = H_n(2^{1/2}\xi_k)|0,0,0\rangle . \quad (2.68)$$

Equations (2.28)–(2.31) may be used to show rotational symmetries of the wave function.

Theorem 2. The probability distribution in the rest frame defined by $\langle s, m, 0|s, m, 0\rangle$ is axial symmetric for $s > 0$ and spherically symmetric for $s = 0$.

Proof. The rotated boson operators in the rest frame, around the 03 axis $\vec{\omega} = \omega(0,0,1)$ are $a_1^\dagger(\omega) = e^{i\omega/2}a_1^\dagger(0)$ and $a_2^\dagger(\omega) = e^{-i\omega/2}a_2^\dagger(0)$. Then $|s, m, \omega\rangle = e^{im\omega}|s, m, 0\rangle$, which is axially symmetric. The ground state defined by $a_k(0)|0,0,0\rangle = 0$ is also a solution of $a_k(\omega)|0,0,0\rangle = 0$ for arbitrary rotation, and is thus rotationally symmetric.

The independence of the harmonic oscillator of any mass scale can be used to relate the total mass of the composite system to its spin. The on-mass-shell condition $P_\mu{}^2 = M^2$ can be made an operator equation by using the eigenvalue equation (2.53),

$$V_0(\Lambda)|\, s, m, \Lambda\rangle = (s+\tfrac{1}{2})|\, s, m, \Lambda\rangle ,$$

and a postulated functional relation between the rest mass and spin (e.g., a polynomial relation):

$$M^2(V_0) = P(V_0) = \alpha + \beta V_0 + \gamma V_0{}^2 + \cdots \quad (2.69)$$

The resulting wave equation

$$[P_\mu{}^2 - M^2(V_0)]\psi^{s,m}(x, \xi) = 0 , \quad (2.70)$$

$$s = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots;\ m = -s, -s+1, \ldots, s-1, s$$

relates the properties of the center-of-mass motion to the internal motion.

The total wave function factorizes into a center-of-mass and an internal wave function[22] as a consequence of (2.17):

$$\psi^{s,m}(x, \xi) = \int e^{ipx} |\, s, m, \hat{p}\rangle\, \delta(p^2 - M^2)\theta(p_0) d^4p,$$

$$\theta(\hat{p}_0) = \begin{cases} 1 \text{ if } \hat{p}_0 > 0, \\ 0 \text{ if } \hat{p}_0 < 0 \end{cases} . \quad (2.71)$$

The internal wave function is the same as in Eq. (2.38),

$$a_k(\Lambda)|\, 0, 0, \Lambda\rangle = 0 . \quad (2.72)$$

For a linear relation $M^2 = \alpha + \beta V_0(\hat{p})$ the mass-squared operator in configuration space is

$$M^2 = \alpha + \beta (P_\mu{}^2)^{-1/2} P_\mu V_\mu(0) , \quad (2.73)$$

where $V_\mu(0)$ is the rest-frame value, Eq. (2.11). Equation (2.73) for the free particle is then

$$[P_\mu{}^2 - \beta (P_\mu{}^2)^{-1/2} P_\mu V_\mu(0) - \alpha]\, \psi^{s,m}(x, \xi) = 0 . \quad (2.74)$$

This is a Majorana-type equation.[23, 24] Similar equations have been studied in the SO(4, 2) model in Ref. 18.

The set of two equations (2.38) and (2.73) determines the asymptotic states of the theory,[25] which can be used in an S-matrix formalism.

Viewed in totality, Eq. (2.70) is a complicated differential equation, mixing derivatives of x and ξ, especially when higher-order polynomials are used in (2.69).

III. NONLOCAL QUANTUM FIELD THEORY

We introduce a Fock space[26] for the total system, $\mathcal{H} = \mathcal{H}(R^4)\times\mathcal{H}(R^2)$, with the canonical creation and annihilation operators $B^\dagger$, B, such that

$$[B(s, m, p), B^\dagger(s', m', p')]_\pm = \delta_{ss'}\delta_{mm'}\,\delta(p - p') , \quad (3.1)$$

and

$$[B, B] = 0,\ [B^\dagger, B^\dagger] = 0 . \quad (3.2)$$

The set of multiparticle states

$$|\, N_{i_1}, \ldots, N_{i_n}\rangle = \prod_{k=1}^{n} |\, N_{i_k}\rangle \quad (N_{i_k} = 0, 1, 2, \ldots) \quad (3.3)$$

forms an orthonormal basis in $\mathcal{H}$. N_i are the occupation numbers for a state with quantum numbers s_i, m_i and momentum p_i. Then

$$B_i^\dagger |\, N_i\rangle = (N_i + 1)^{1/2} |N_i + 1\rangle \quad (3.4)$$

and

$$B_i |\, N_i\rangle = \sqrt{N_i}\, |\, N_i - 1\rangle . \quad (3.5)$$

The free field operators in second quantization are defined by

$$\phi_{s,m}(x, \xi) = (2\pi)^{-3/2} \int (dp)[B(s, m, p) e^{ipx} |\, s, m, \hat{p}\rangle + \text{H.c.}] , \quad (3.6)$$

where

$$(dp) = \delta(p^2 - M^2(s))\, \theta(p_0) d^4p . \quad (3.7)$$

Only the center-of-mass motion decomposes into positive and negative frequencies, whereas the internal wave function depends on the absolute value $|\, p_0|$.

A linear superposition such as

$$\phi(x, \xi) = \sum_{s,m} \alpha_{s,m}\, \phi_{s,m}(x, \xi) \quad (3.8)$$

has neither definite statistics nor definite mass. In Refs. 15 and 27, integer and half-integer components are subjected to Bose CR's. The unified treatment of integer and half-integer spin found its successful solution in the supersymmetry approach.

If we keep track[28] of the spin values and the statistics associated with them, the CR's are

$$[\phi_{s,m}(x, \xi), \phi^+_{s',m'}(y, \xi)]_\mp = (2\pi)^{-3} \int (dp)(e^{ip(x-y)} \mp e^{-ip(x-y)})\delta_{ss'}\delta_{mm'}\langle s, m, \hat{p}|\, s', m', \hat{p}\rangle , \quad (3.9)$$

where $-/+$ is used for (s, s') even/odd. Because of the extended structure the commutator (3.9) is nonlocal. The locality is regained when (3.9) is integrated over the internal coordinates:

$$\int_{-\infty}^{\infty} d^2\xi\, [\phi_{s,m}(x, \xi), \phi^+_{s',m'}(y, \xi)]_\mp = \delta_{ss'}\delta_{mm'} (2\pi)^{-3} \int_{-\infty}^{\infty} (dp)(e^{+ip(x-y)} \mp e^{-ip(x-y)}) . \quad (3.10)$$

In particular for bosons we obtain

$$\int_{-\infty}^{\infty} d^2\xi [\phi_{s,m}(x,\xi), \phi^+_{s',m'}(y,\xi)]_- = i\,\delta_{ss'}\,\delta_{mm'}\,D(x-y)\,, \quad (3.11)$$

where $D(x)$ is the Pauli-Jordan function.

The field theory for extended particles is expected to be nonlocal.[29] Also it has been shown that a field theory with infinite components and non-degenerate mass spectrum cannot be local.[30,31] Interesting consequences are implied by the form of the mass spectrum. For the Majorana theory[23] the mass spectrum is $M = M_0(s+\frac{1}{2})^{-1/2}$. This implies that the theory is TCP noninvariant[32] because it violates the compactness condition,[33] and consequently does not describe particles. On the other hand, in models with increasing mass as a function of spin, e.g., $M^2 = \alpha + \beta s$, $\alpha > 0$, the compactness condition is satisfied.

IV. INTERACTIONS

It is useful to distinguish between elastic and inelastic interactions.

Elastic interactions do not modify masses and spins of the particles involved, only the direction of motion of the center of mass is changed. The interaction does not couple directly to the internal motion.

In inelastic interactions masses and spins are changed and the interaction affects directly the internal motion.

Among the inelastic interactions, those containing an even number of boson operators preserve the boson and fermion characteristics of the particles, whereas those made of an odd number of boson operators change bosons into fermions and vice versa.

A. Elastic electromagnetic form factor

From Eq. (2.70) it follows that there exists an electric current density operator[34]

$$j_\mu(x,\xi) = e\sum_{s=0}^{\infty}\sum_{m=-s}^{s}\{[P_\mu\psi^{s,m\dagger}(x,\xi)]\psi^{s,m}(x,\xi) - \psi^{s,m\dagger}(x,\xi)P_\mu\psi^{s,m}(x,\xi)\} \quad (4.1)$$

whose matrix elements between equal-spin states are conserved

$$\partial_\mu\langle s|j_\mu(x,\xi)|s\rangle = 0\,. \quad (4.2)$$

We use the interaction Lagrangian

$$\mathcal{L}_I(x,\xi) = [j_\mu(x,\xi) + J_\mu(x)\delta(\xi)]A_\mu(x)\,, \quad (4.3)$$

where $J_\mu(x)$ is the current of a pointlike particle. The S matrix is given by

$$S = T\exp\left(-iN\int d^4x\int d^2\xi\,\mathcal{L}_I(x,\xi)\right). \quad (4.4)$$

The action implies a six-dimensional integral.

The scattering cross section of the extended particle with a pointlike particle, computed in first-order Born approximation, defines the elastic form factor $F_{s,m}$[35-37]

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma}{d\Omega}\bigg|_{\text{point}} F_{s,m}^2(p',p)\,. \quad (4.5)$$

$(d\sigma/d\Omega)$ point is the Rutherford cross section and (p',p) are the outgoing and incoming momenta.

In terms of internal wave functions

$$F_{s,m}^2(p',p) = \left|\int_{-\infty}^{\infty} d^2\xi\langle s,m,p'|s,m,p\rangle\right|^2. \quad (4.6)$$

In the special case of the ground state $s=0$, the form factor can be computed analytically. With Eq. (2.39) for $|0,0,p\rangle$ and the coefficients X given in (2.40)–(2.42) we have

$$F_0^2(p',p) = \left|\int d\xi\,\pi^{-1}[X_1(p')X_1(p)]^{+1/2}\exp\{-\tfrac{1}{2}[X_1(p)+X_1(p')](\xi_1^2+\xi_2^2) - \tfrac{1}{2}i[X_2(p)-X_2(p')](\xi_1^2-\xi_2^2) - i[X_3(p)-X_3(p')]\xi_1\xi_2\}\right|^2. \quad (4.7)$$

Using the integration formula (A7)

$$F_0(p',p) = 2[X_1(p')X_1(p)]^{+1/2}\{[X_1(p')+X_1(p)]^2 + [X_2(p)-X_2(p')]^2 + [X_3(p)-X_3(p')]^2\}^{-1/2}\,. \quad (4.8)$$

Let p, the incoming momentum, be oriented along the 03 axis $p=(p_0, 0, 0, p)$ and p', the outgoing momentum, be arbitrary $p'=(p_0, p_1', p_2', p_3')$. Then the form factor for the ground state is given by

$$F_0(p',p) = [1+\sinh^2\Lambda\,\sin^2(\tfrac{1}{2}\theta)]^{-1/2}\,. \quad (4.9)$$

In order to be relativistic invariant, the form factor must depend on the momentum transfer $\vec{q} = \vec{p}' - \vec{p}$ alone. This can be realized only with

$\sinh\Lambda = \hat{p}$, Eq. (2.46), when $F_0(p', p)$ becomes

$$F_0(p', p) = F_0(q) = [1 + p^2 \sin^2(\tfrac{1}{2}\theta)]^{-1/2}$$

$$= \left(1 + \frac{1}{4M^2}\vec{q}^2\right)^{-1/2}, \qquad (4.10)$$

with $\cosh\frac{1}{2}\Lambda = \hat{p}_0$, $\sinh\frac{1}{2}\Lambda = \hat{p}$, e.g., $F_0(p', p) = (1 + \hat{p}_0^2\, \vec{q}^2/4M^2)^{-1/2}$, which is not Lorentz invariant.

The charge distribution is[38]

$$\rho(r) = \int F(\vec{q}) e^{i\vec{q}\cdot\vec{x}}\, d^3q = \frac{4\pi}{r}\, 4M^2 K_1(2Mr), \qquad (4.11)$$

where K_1 is the Bessel function of pure imaginary argument of order one.[39] The mean square radius is $\langle r^2 \rangle = 1/4M^2$. This charge distribution can be compared to the Yukawa distribution

$$\rho(r) = \frac{2\pi^2}{r}\, 4M^2 e^{-2Mr} \qquad (4.12)$$

($\langle r^2 \rangle = 1/2M^2$), which is the Fourier transform of $(1 + \vec{q}^2/4M^2)^{-1}$, the square of the integrand in (4.11). Using the integral representation

$$K_1(x) = \int_0^\infty e^{-x\cosh t} \cosh t\, dt\ . \qquad (4.13)$$

One finds $K_1(x) > e^{-x}$ for $x < 2.1$, with $K_1(x) \to \infty$ for $x \to 0$, and $K_1(x) < e^{-x}$ for $x > 2.12$. Physically this means that a particle described by (4.11) has a much denser core and a thinner tail than the Yukawa distribution.

B. Decay

Perhaps the simplest inelastic interaction results from the Lagrangian

$$\mathcal{L}_I(x, \xi) = G \sum_{s, s'=0}^{\infty} \sum_{m, m'} \psi^\dagger_{s,m}(x, \xi) S_{\mu\nu}(0) \times F_{\mu\nu}(x) \psi_{s', m'}(x, \xi)\ . \qquad (4.14)$$

G is the coupling constant, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ the electromagnetic field tensor. $S_{\mu\nu}(0)$ are the internal generators of O(3,1). The interaction is automatically gauge invariant.

The S matrix is the same as in Eq. (4.4). The matrix element between two different states with spin s and s', such that $|s - s'| = 1$, describe the radiative decay or photoproduction of the excited state. Equation (4.14) is formally relativistic covariant.

The $S_{\mu\nu}$ operators do not commute with V_0, therefore, spins and masses are changed.

As an illustration consider the decay of a spin-1 state in the rest frame into a spin-0 particle and a photon.

The free parameters are the coupling constant G and the ratio between the masses of the initial and final state,

$$\epsilon = M^*(1)/M(0)\ , \qquad (4.15)$$

which determines completely the velocity of the decay product $v = (\epsilon^2 - 1)/(\epsilon^2 + 1)$ and the momentum of the photon k_μ.

The decay amplitude in momentum space factorizes into

$$T_m = G\epsilon^*_\mu(\lambda) T^\mu_m\ , \qquad (4.16)$$

with

$$T^\mu_m = k_\nu \langle 0, 0, \hat{p} |\, S_{\mu\nu}(0) |\, 1, m, 0 \rangle\ ; \qquad (4.17)$$

m is the magnetic number of the decaying state, $\epsilon_\mu(\lambda)$ is the helicity wave function of the photon.

After summing over helicities

$$\sum_\lambda \epsilon^*_\mu(\lambda) \epsilon_\nu(\lambda) = \delta_{\mu\nu}\ . \qquad (4.18)$$

The decay rate[40] is

$$\Gamma = \Gamma^* \tfrac{1}{3} \sum_{m=-1}^{1} \int d\Omega\, T^\mu_m T^{\nu *}_m \delta_{\mu\nu}\ , \qquad (4.19)$$

with

$$\Gamma^* = p^2/(32\pi^2 p_0 k_0 M^*)\ . \qquad (4.20)$$

Γ may be split into contributions from different states of polarization:

$$\Gamma = \sum_{m=-1}^{1} \Gamma_m\ . \qquad (4.21)$$

The decay rate is obtained by first computing T^μ_m, which involves an analytic integration over the internal coordinates, and then numerically integrating $T^\mu_m T^{\nu *}_m \delta_{\mu\nu}$ over the sphere.

Let $P^m_{\mu\nu}(\xi)$ be the eigenvalues of $S_{\mu\nu}(0)$ applied to the spin-1 wave function in the rest frame:

$$S_{\mu\nu}(0) |\, 1, m; 0 \rangle = P^m_{\mu\nu}(\xi) |\, 1, m; 0 \rangle\ . \qquad (4.22)$$

The mean value of $S_{\mu\nu}$ can be written as a complex Gaussian integral over the polynomials $P^m_{\mu\nu}$:

$$\langle 0, 0; \hat{p} |\, S_{\mu\nu}(0) |\, 1, m; 0 \rangle = \pi^{-1}[M/(p_0 + p_2)]^{1/2} [(1+m)!\,(1-m)!]^{-1/2}$$
$$\times \int_{-\infty}^{\infty} d^2\xi\, P^m_{\mu\nu}(\xi) \exp\{-\tfrac{1}{2}[1 + M/(p_0 + p_2)](\xi_1^2 + \xi_2^2) + ip_3(p_0 + p_2)^{-1}\xi_1\xi_2$$
$$- ip_1(p_0 + p_2)^{-1}(\xi_2^2 - \xi_1^2)\}\ . \qquad (4.23)$$

The eigenvalues are easily found by Theorem 1 after normal ordering of $S_{\mu\nu}(a_1^\dagger)^{1+m}(a_2^\dagger)^{1-m}$. The arguments of the Hermite polynomials $2^{1/2}\xi_k$ are denoted by the index $k=1,2$:

$$P_{23}^{1}=P_{23}^{-1}=-2^{-1/2}H_1(1)H_1(2)\ , \tag{4.24}$$

$$P_{23}^{0}=-2^{-1}[H_2(1)+H_2(2)]\ , \tag{4.25}$$

$$P_{31}^{1}=-P_{31}^{-1}=-2^{-1/2}iH_2(1), \tag{4.26}$$

$$P_{31}^{0}=2^{-1}i[H_2(1)-H_2(2)]\ , \tag{4.27}$$

$$P_{12}^{1}=2^{-1/2}H_2(1), \tag{4.28}$$

$$P_{12}^{0}=0\ , \tag{4.29}$$

$$P_{12}^{-1}=-2^{-1/2}H_2(2)\ , \tag{4.30}$$

$$P_{10}^{1}=2^{-5/2}[H_4(1)-H_2(1)H_2(2)+2]\ , \tag{4.31}$$

$$P_{10}^{0}=2^{-2}[H_3(1)H_1(2)-H_1(1)H_3(2)]\ , \tag{4.32}$$

$$P_{10}^{-1}=-2^{-5/2}[H_4(2)-H_2(1)H_2(2)+2]\ , \tag{4.33}$$

$$P_{20}^{1}=-2^{-5/2}i[H_4(1)+H_2(1)H_2(2)-2]\ , \tag{4.34}$$

$$P_{20}^{0}=-2^{-2}i[H_3(1)H_1(2)+H_1(1)H_3(2)]\ , \tag{4.35}$$

$$P_{20}^{-1}=-2^{-5/2}i[H_4(2)+H_2(1)H_2(2)-2]\ , \tag{4.36}$$

$$P_{30}^{1}=2^{-3/2}H_3(1)H_1(2)\ , \tag{4.37}$$

$$P_{30}^{0}=2^{-1}[H_2(1)H_2(2)+1]\ , \tag{4.38}$$

$$P_{30}^{-1}=2^{-3/2}H_1(1)H_3(2)\ . \tag{4.39}$$

With the integration formulas from Appendix A, the mean values $\langle 0,0,\hat{p}|\,S_{\mu\nu}(0)|\,1,m,0\rangle$ can be expressed as functions of $\hat{p}=(\cos\phi\sin\theta,\ \sin\phi\sin\theta,\ \cos\theta)$. The integrand in Eq. (4.19) is a smooth function in θ and ϕ such that the angular integral converges fast as a function of the mesh size.

The resulting decay rate $\Gamma_{1\to 0+\gamma}$ is shown in Fig. 1.

For comparison we study the "static limit" of the decay rate, which is obtained by replacing the relativistic Eq. (4.23) by the velocity-independent matrix element:

$$\langle 0,0,0|\,S_{\mu\nu}(0)|\,1,m,0\rangle \equiv E_m(S_{\mu\nu})\ . \tag{4.40}$$

The wave function for the emitted particle is assumed to be the same as in the rest frame. The only nonvanishing mean values are

$$E_1(S_{10})=2^{-3/2},\quad E_{-1}(S_{10})=-2^{-3/2}\ , \tag{4.41}$$

$$E_1(S_{20})=2^{-3/2}i\ ,\quad E_{-1}(S_{20})=2^{-3/2}i\ , \tag{4.42}$$

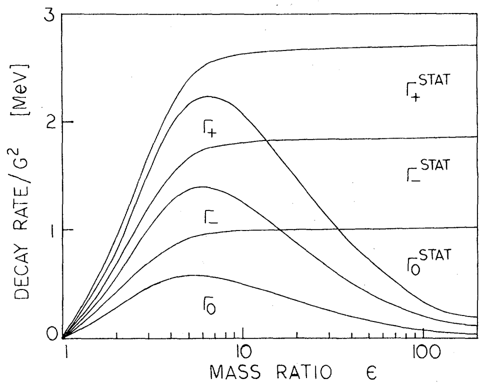

FIG. 1. Comparison of the relativistic decay rate $\Gamma=\Gamma_+ + \Gamma_- + \Gamma_0$ with its static limit for the decay $M^*(1)\to M(0)+\gamma$. The initial mass $M^*=770$ MeV is chosen to simulate the decay of the ρ meson into mesons with different masses. The velocity of the decay product is $v=(\epsilon^2-1)/(\epsilon^2+1)$. In the ultrarelativistic region ($\epsilon>200$) the decay is suppressed. The static limit reflects also the shape of the phase-space volume.

and

$$E_0(S_{30})=2^{-1}\ . \tag{4.43}$$

The amplitude squared is a simple expression:

$$\delta_{\mu\nu}T_1^{\mu}T_1^{\nu*}=\delta_{\mu\nu}T_{-1}^{\mu}T_{-1}^{\nu*}$$
$$=2^{-3}(k_1^{\,2}+k_2^{\,2})+2^{-2}k_0^{\,2}\ , \tag{4.44}$$

$$\delta_{\mu\nu}T_0^{\mu}T_0^{\nu*}=2^{-1}k_3^{\,2}+2^{-2}k_0^{\,2}\ . \tag{4.45}$$

The angular integral in (4.19) is straightforward. The result is

$$\Gamma_+^{\rm stat}=\Gamma_-^{\rm stat}=\tfrac{4}{13}\ \Gamma_{\rm total}^{\rm stat}\ , \tag{4.46}$$

$$\Gamma_0^{\rm stat}=\tfrac{5}{13}\ \Gamma_{\rm total}^{\rm stat}\ , \tag{4.47}$$

with

$$\Gamma_{\rm total}^{\rm stat}=\frac{13}{1152\pi}\ G^2M^*\ \frac{(\epsilon^2-1)^3}{\epsilon^4(\epsilon^2+1)}\ . \tag{4.48}$$

For $\epsilon\to 1$ ($M^*=M$), $\Gamma_{\rm total}^{\rm stat}\to 0$. For $\epsilon\to\infty$ ($M\to 0$), $\Gamma_{\rm total}^{\rm stat}\to(13/1152\pi)G^2M^*$, thus the decay into a zero-mass state is not suppressed. The static decay rate is shown in Fig. 1.

For the decay $\rho\to\pi+\gamma$ the mass ratio is $\epsilon=5.5$. The total relativistic decay rate (4.19) is $\Gamma=2.18G^2$ MeV, whereas $\Gamma^{\rm static}=2.23G^2$ MeV. For the choice $G^2=e^2=\frac{1}{137}$, the values are $\Gamma=15.93$ keV and $\Gamma^{\rm static}=16.32$ keV. The difference is 2.38% The known experimental value[41] is $\Gamma_{\rm exp}=35\pm 10$ keV.

This calculation did not take into account quark selection rules. It depends on the masses of the asymptotic states alone.

The decrease of the decay rate at high speed (large ϵ) can be understood from the overlap of the initial and final wave function, due to deformation. The measure for the overlap of the moving spin-0 wave function and the spin-1 wave function in the rest frame is the scalar product:

$$|\langle 0,0,\hat{p}\,|\,1,m,0\rangle|^2 = 4\epsilon(\epsilon-1)^2(\epsilon+1)^{-4} \times(\delta_{|m|,1}\sin^2\theta+\delta_{m,0}\cos^2\theta)\ . \tag{4.49}$$

At rest ($\epsilon=1$) the wave functions are orthogonal. At high speed ($\epsilon \gg 1$) the overlap decreases like ϵ^{-1}. The average over the sphere shows that the total overlap for $m=\pm1$ is twice the overlap for $m=0$:

$$(1/4\pi)\int d\Omega\,|\langle 0,0,\hat{p}\,|\,1,m,0\rangle|^2 = \tfrac{4}{3}\epsilon(\epsilon-1)^2(\epsilon+1)^{-4}(2\delta_{|m|,1}+\delta_{m,0})\ , \tag{4.50}$$

and reaches a maximum of about 15% at $\epsilon \cong 5.6$.

To compare the model with experimental values of the radiative decay rate of vector mesons, the SU(6) selection rules have to be included.

V. SUMMARY AND DISCUSSION

The aim of the paper is to explore the existence and properties of interactions in the SO(3, 2) model, in continuation of Refs. 3, 4 where the kinematics had been developed. The model is a relativistic harmonic-oscillator model, therefore, basically a shell model. Correspondingly the free wave functions are represented by Hermite polynomials of complex arguments in Theorem 1.

In a perturbative S-matrix calculation, with the asymptotic states defined by Eqs. (2.38), (2.50), and (2.70), we find the elastic form factor of the ground state and the decay rate for the $M^*(s=1)\to M(s=0)+\gamma$ transition.

It is remarkable that the relativistic invariance of the form factor of the ground state, which can be computed analytically, imposes a unique condition [Eqs. (2.45)–(2.46)] upon the transformation of the wave function under the Lorentz group.

The example of the radiative decay of a spin-1 particle into the ground state shows that the relativistic decay rate behaves qualitatively different from the static limit for large mass ratios ϵ.

The relativistic decay rate decreases monotonically for $\epsilon > 8$, whereas the static limit becomes asymptotically constant for $\epsilon > 20$. The static limit of the decay rate behaves essentially like the phase-space volume over the whole range of ϵ. In conclusion, this model favors radiative decays, or photoproduction, of particles having a ratio of initial to final masses (ϵ) in the range $3<\epsilon<15$, and strongly suppresses decays into very light particles $\epsilon > 200$.

The significance of the internal coordinates $\vec{\xi}$ was not mentioned. As in any computation of physical effects, one has to integrate over $\vec{\xi}$, it does not matter what they are. Often they are considered to be light-front variables, and the model is classified as a null-plane theory.[42] An interesting alternative is to consider $\vec{\xi}$ as stochastic parameters which describe the extended particles.[43]

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APPENDIX A: COMPLEX GAUSSIAN INTEGRATION IN TWO DIMENSIONS

Let

$$\langle x^m y^n\rangle = \int_{-\infty}^{\infty} dx\,dy\,x^m y^n e^A \tag{A1}$$

be a two-dimensional integral with the exponent

$$A = -a(x^2+y^2)+ib(x^2-y^2)+icxy,\quad a>0\ . \tag{A2}$$

Using the notation

$$f(n_1,n_2,n_3,n_4) = (a^2+b^2)^{n_1}[4(a^2+b^2)+c^2]^{n_2}\,c^{n_3} \times \exp\left(-in_4\arctan\frac{b}{a}\right) \tag{A3}$$

the following integration formulas are true:

$$\langle x^{2n}\rangle(a,b,c)=\langle y^{2n}\rangle(a,-b,c)\ , \tag{A4}$$

$$\langle x^{2n+m}y^m\rangle(a,b,c)=\langle x^m y^{2n+m}\rangle(a,-b,c)\ , \tag{A5}$$

$$\langle y^{2n}\rangle = \pi(2n-1)!!\,2^{n+1}f(\tfrac{1}{2}n,-\tfrac{1}{2}-n,0,n)\ . \tag{A6}$$

In particular,

$$\langle 1\rangle = 2\pi f(0,-\tfrac{1}{2},0,0)\ . \tag{A7}$$

A special case of (A7) for $c=0$ is given in Ref. 44.

$$\langle y^2\rangle = 4\pi f(\tfrac{1}{2},-\tfrac{3}{2},0,1), \tag{A8}$$

$$\langle y^4\rangle = 24\pi f(1,-\tfrac{5}{2},0,2), \tag{A9}$$

$$\langle x^2\rangle = 4\pi f(\tfrac{1}{2},-\tfrac{3}{2},0,-1)\ , \tag{A10}$$

$$\langle x^4\rangle = 24\pi f(1,-\tfrac{5}{2},0,-2)\ , \tag{A11}$$

$$\langle xy\rangle = 2\pi i f(0,-\tfrac{3}{2},1,0)\ , \tag{A12}$$

$$\langle xy^3\rangle = 12\pi i f(\tfrac{1}{2}, -\tfrac{5}{2}, 1, +1)\,, \tag{A13}$$

$$\langle x^3 y\rangle = 12\pi i f(\tfrac{1}{2}, -\tfrac{5}{2}, 1, -1)\,, \tag{A14}$$

$$\langle x^2y^2\rangle = 2\pi[f(0, -\tfrac{3}{2}, 0, 0) - 3f(0, -\tfrac{5}{2}, 2, 0)]\,. \tag{A15}$$

In the limit $a \to 1$, $b \to 0$, $c \to 0$ we obtain

$$\langle 1\rangle = \pi\,, \tag{A16}$$

$$\langle x^2\rangle = \langle y^2\rangle = \tfrac{1}{2}\pi\,, \tag{A17}$$

$$\langle x^4\rangle = \langle y^4\rangle = \tfrac{3}{4}\pi\,, \tag{A18}$$

$$\langle xy\rangle = \langle xy^3\rangle = \langle x^3y\rangle = 0\,. \tag{A19}$$

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[1]A. O. Barut, in *Seminar on High Energy Physics and Elementary Particles* (International Atomic Energy Agency, Vienna, 1965), p. 679.

[2]P. A. M. Dirac, J. Math. Phys. 4, 901 (1963).

[3]L. C. Biedenharn and H. van Dam, Phys. Rev. D 9, 471 (1974).

[4]H. van Dam and L. C. Biedenharn, Phys. Rev. D 14, 405 (1976).

[5]P. A. M. Dirac, Proc. R. Soc. London A322, 435 (1971); A328, 1 (1972).

[6]P. A. M. Dirac, *Directions in Physics* (Wiley, New York, 1978).

[7]N. N. Bogoliubov, A. A. Logunov, and I. T. Todorov, *Introduction to Axiomatic Quantum Field Theory* (Benjamin, Reading, Massachusetts, 1975).

[8]C. N. Yang and R. L. Mills, Phys. Rev. 96, 191 (1954).

[9]L. P. Staunton, J. Math. Phys. 19, 1471 (1978).

[10]A. O. Barut and R. Raczka, *Theory of Group Representations and Applications* (PWN-Polish Scientific Publishers, Warsaw, 1977).

[11]R. Gilmore, *Lie Groups, Lie Algebras, and Some of Their Applications* (Wiley, New York, 1974).

[12]A. L. Licht and A. Pagnamenta, Phys. Rev. D 2, 1150 (1970); 2, 1156 (1970).

[13]R. P. Feynman, M. Kislinger, and F. Ravndal, Phys. Rev. D 3, 2706 (1971).

[14]Y. S. Kim and M. E. Noz, Phys. Rev. D 12, 122 (1975); 12, 129 (1975); 15, 335 (1977).

[15]A. Z. Capri and C. C. Chiang, Nuovo Cimento 36A, 331 (1976).

[16]L. Maharana and S. P. Misra, Phys. Rev. D 18, 2530 (1978).

[17]A. O. Barut and I. H. Duru, Phys. Rev. D 10, 3448 (1974).

[18]A. O. Barut and H. Kleinert, Phys. Rev. 156, 1546 (1967). For a complete list of publications see Ref. 10.

[19]C. Itzykson, Commun. Math. Phys. 4, 92 (1967).

[20]D. Tz. Stoyanov and I. T. Todorov, J. Math. Phys. 9, 2146 (1968).

[21]I am grateful to H. Doebner for helpful comments on this point.

[22]H. Leutwyler and J. Stern, Phys. Lett. 69B, 207 (1977).

[23]E. Majorana, Nuovo Cimento 9, 335 (1932); D. M. Fradkin, Am. J. Phys. 34, 314 (1966).

[24]I. M. Gel'fand and A. M. Yaglom, Zh. Eksp. Teor. Fiz. 18, 703 (1948).

[25]Y. Nambu, Suppl. Progr. Theor. Phys. 37–38, 368 (1966).

[26]V. Fock, Z. Phys. 75, 622 (1932).

[27]G. Feldman and P. T. Mathews, Ann. Phys. (N.Y.) 40, 19 (1966); Phys. Rev. 151, 1176 (1966); 154, 1241 (1967).

[28]C. Fronsdal, Phys. Rev. 156, 1653 (1967).

[29]J. Rayski, Fortschr. Phys. 2, 165 (1954).

[30]I. T. Grodsky and R. F. Streater, Phys. Rev. Lett. 20, 695 (1968).

[31]A. I. Oksak and I. T. Todorov, Phys. Rev. D 1, 3511 (1970).

[32]A. I. Oksak and I. T. Todorov, Commun. Math. Phys. 11, 125 (1968).

[33]R. Haag and J. A. Swieca, Commun. Math. Phys. 1, 308 (1965).

[34]We use units with $\hbar = 1$, $c = 1$, $e^2 = \frac{1}{137}$.

[35]N. F. Mott and H. S. W. Massey, *The Theory of Atomic Collisions* (Clarendon, Oxford, 1965), 3rd edition.

[36]M. E. Rose, Phys. Rev. 73, 279 (1948); 82, 389 (1951).

[37]R. Herman and R. Hofstadter, *High-Energy Electron Scattering Tables* (Stanford Univ. Press, Stanford, Calif., 1960).

[38]I. S. Gradshteyn and I. W. Ryzhik, *Tables of Integrals, Series and Products* (Academic, New York, 1965).

[39]G. N. Watson, *A Treatise on the Theory of Bessel Functions* (Cambridge Univ. Press, Cambridge, England, 1944).

[40]H. Pilkuhn, *The Interactions of Hadrons* (North-Holland, Amsterdam, 1967).

[41]B. Gobbi *et al.*, Phys. Rev. Lett. 33, 1450 (1974).

[42]G. R. Bart and S. Fenster, Phys. Rev. D 16, 3554 (1977).

[43]Z. Haba and J. Lukierski, Nuovo Cimento 41A, 470 (1977).

[44]Fr. Iseli, Ann. Math. Pura Appl. 27, 1 (1910).