Energy and momentum spectral function of coherent bremsstrahlung radiation

Daniel Zwanziger
Department of Physics, New York University, New York, New York 10003

(Received 19 March 1979)

We calculate nonperturbatively and for small Q the cross section $\sigma(Q)$ for a scattering process to occur with loss of four-momentum Q to unobserved photons, a quantity which may be measured by observing the net recoil to all other particles. The calculation proceeds by means of a threshold theorem which asserts that for small Q, $\sigma(Q) \sim \sigma_0 P(Q)$, where σ_0 is independent of Q, and $P(Q)$ is the spectral of the coherent state Ψ of bremsstrahlung photons defined by $a^{\mu}(k)\Psi = i(2\pi)^{-3/2}\sum_a u_a^{\mu}(u_a \cdot k)^{-1}\Psi$, where $a^{\mu}(k)$ is the annihilation operator for a photon of four-momentum k, and e_a and u_a are the charges and four-velocities of the scattered charged particles. Although Ψ is not in the Fock space, the evaluation of $P(Q) = \langle \Psi, \delta^4(Q-P_{op})\Psi\rangle$, where P_{op} is the operator of total electromagnetic four-momentum, is straightforward. The resulting function $P(Q)$ simplifies if Q is near the light cone, where the bulk of the probability is in fact located, $P(Q) \sim \theta(Q^0)\theta(Q^2)[\Gamma(1+B)]^{-1}B(Q^2/2)^{B-1}I_0(Q)\exp[F(Q)]$, where $I_0(Q) = -(2\pi)^{-3}[\sum_a e_a u_a (u_a \cdot Q)^{-1}]^2 > 0$, $B = \int d\hat{k}\, I_0(Q^0=1, \hat{Q}=\hat{k})$, and $F(Q)$ is given explicitly in the text, satisfies $F(\lambda Q) = F(Q) - B\ln\lambda$, and is a smooth function as the light cone is approached. The spectral function exhibits two scaling laws, one governing the approach to the origin along a ray $\lim_{\lambda\to 0}\lambda^{4-B}\sigma(\lambda Q) = \sigma_0 P(Q)$, the other governing the approach to the light cone at fixed energy $Q^0 = E$ and angle $\hat{k}$ $\lim_{|\vec{Q}|\to E}[(E-|\vec{Q}|)^{1-B}P(E,\hat{Q})] = \text{const}\times[\Gamma(1+B)]^{-1}BE^{B-1}I_0(k)\exp[F(k)]$, where $k=(E,E\hat{k})$. For e^+-e^- annihilation at 3 on 3 Gev, $\exp[F(k)] = E^{-B}\exp[F(\hat{k})]$ produces a 30% angular modulation.

I. INTRODUCTION

The recoil due to the four-momentum Q of unobserved photons emitted in a scattering process is a familiar phenomenon. The dependence of the cross section on Q may be measured experimentally giving the four-momentum spectral function $\sigma(Q)$. We calculate this quantity nonperturbatively for small values of Q, $Q\approx 0$. The first step is a threshold theorem which states that for small Q it is of the form

$$\sigma(Q)\sim\sigma_0 P(Q), \tag{1.1}$$

where σ_0 is independent of Q, and $P(Q)$ is a universal function of the charges e_a and four-velocities u_a of the charged scattered particles that is entirely independent of the nature of the scattering process. The theorem states, furthermore, that $P(Q)$ is the four-momentum spectral function

$$P(Q)=\langle\Psi,\delta^4(Q-P_{\rm op})\Psi\rangle \tag{1.2}$$

of the completely coherent state Ψ of bremsstrahlung photons defined by

$$a^{\mu}(k)\Psi=\frac{i}{(2\pi)^{3/2}}\sum_a \frac{e_a u_a^{\mu}}{u_a\cdot k}\Psi, \tag{1.3}$$

where $P_{\rm op}$ is the operator for the total four-momentum of the electromagnetic radiation, and $a^{\mu}(k)$ is the annihilation operator for a photon of momentum k. [Our sign convention for electric charge is such that charge conservation is expressed by $\sum_a e_a=0$, so that (1.3) is a transverse state $k\cdot a(k)\Psi=0$.] Although the state (1.3) is not normalizable owing to ultraviolet divergences, the projection (1.2) cuts off all integrals over photon momenta $\vec{k}$ at $|\vec{k}|=Q^0$, or less, so our concentration on the infrared phenomenon is undisturbed by ultraviolet difficulties.

The calculation of $P(Q)$, to which the bulk of this article is devoted, poses an interesting challenge to theory because the state (1.3) is not in the photon Fock space owing to infrared divergences. Our final expression for $P(Q)$ depends in an essential way on the non-Fock character of the state (1.3), which is intriguing for an experimentally measurable quantity. How the mathematical idealization involved here applies to the actual laboratory situation is discussed in Sec. VI.

To illustrate the non-Fock character of the state (1.3), suppose we write it as

$$\Psi=\exp\left[\int d^3k(2\omega)^{-1}a_{\mu}^{\dagger}(k)\sum_a e_a u_a^{\mu}(u_a\cdot k)^{-1}\right]\Omega,$$

where Ω is the vacuum state, and expand in powers of e, so the n-photon amplitude is of order e^n. Then in zeroth order one has $P^{(0)}(Q)=\delta^4(Q)$, and the naive first-order expression

$$P^{(1)}(Q)=(2Q^0)^{-1}\delta(Q^0-|\vec{Q}|)I_0(Q), \tag{1.4}$$

$$I_0(Q)=\frac{-1}{(2\pi)^3}\left(\sum_a\frac{e_a u_a}{u_a\cdot Q}\right)^2 \tag{1.5}$$

is recognized as the four-momentum spectral function of individual bremsstrahlung photons. [Our metric convention is $g_{\mu\nu}=\text{diag}(1,-1,-1,-1)$ so $I_0(Q)$ is positive because $\phi^{\mu}\equiv\sum_a e_a u_a^{\mu}(u_a\cdot Q)^{-1}$ is spacelike, being orthogonal to Q, $\phi\cdot Q=\sum_a e_a=0$

by charge conservation.] Now whereas individual photons are, of course, not detectable down to zero frequency, the quantity $\int_V P(Q)d^4Q$, which represents the probability that all photons together have total four-momentum Q in the volume V of four-space, must be finite even if V includes the origin. On the other hand, $\int P^{(1)}(Q)d^4Q$ displays the elementary infrared divergence of bremsstrahlung photons, $\int I_0(k)(2\omega)^{-1}d^3k$, so that (1.4) is not part of a correct expansion of the exact spectral function $P(Q)$.

The essential features of $P(Q)$ are contained in a simple asymptotic expression $P^{as}(Q)$ which is accurate near the mantle of the light cone, where the bulk of the probability is in fact concentrated,

$$P^{as}(Q)=N\theta(Q^0)\theta(Q^2)[\Gamma(1+B)]^{-1}B(\tfrac{1}{2}Q^2)^{B-1} \times I_0(Q)\exp[F(Q)]\,, \qquad (1.6)$$

where N is a normalization constant, and $F(Q)$, given in Eq. (5.28), is a smooth function as the light cone is approached, satisfying

$$F(\lambda Q)=F(Q)-B\ln\lambda\,. \qquad (1.7)$$

Here B represents a dimensional anomaly:

$$B\equiv\int d\hat{k}I_0(\hat{k})\,, \qquad (1.8a)$$

$$I_0(\hat{k})=\frac{-\frac{1}{2}}{(2\pi)^3}\left(\sum_a \frac{e_a u_a}{u_a^0-\vec{u}_a\cdot\hat{k}}\right)^2 \geq 0\,, \qquad (1.8b)$$

$$B=\frac{(-1)}{(2\pi)^2}\sum_{a,b}\frac{e_a e_b \psi_{a,b}}{\tanh\psi_{a,b}}\,, \qquad (1.8c)$$

where $\psi_{a,b}\geq 0$ is the hyperbolic angle between u_a and u_b, $\cosh\psi_{a,b}=u_a\cdot u_b$. It is proportional to e^2 and plays a role analogous to a running coupling constant. It may be thought of as the effective strength of the coupling to the low-frequency radiation field characteristic of the scattering process. It is a convenient measure of net charge scatter, and grows logarithmically with momentum transfer. Because $e^2\approx(137)^{-1}$ is small, B is small at laboratory energies, and the function $P(Q)\sim P^{as}(Q)$ is, at fixed energy Q^0, concentrated near the mantle of the light cone, and in the limit

$$\lim_{B\to 0}B(Q^2)^{B-1}\theta(Q^2)=\delta(Q^2)\,. \qquad (1.9)$$

Note that the left-hand side is nominally of order e^2, whereas the right-hand side is independent of e. The exact nonperturbative calculation replaces the no-photon spectral function $\delta^4(Q)$ and the one-photon spectral function proportional to $\delta(Q^2)$ by a radiative tail which extends inside the light cone that is of nominal order e^4. The energy spectral function

$$P_1(E)=\int P(Q^0=E,\vec{Q})\,d^3Q \qquad (1.10)$$

is of nominal order e^2, and the probability of energy loss less than E,

$$P(Q^0<E)=\int_0^E P_1(Q^0)\,dQ^0$$

is of leading order e^0.

For Q on the light cone, $Q^\mu=k^\mu=(E,E\hat{k})$, the angular modulation provided by $\exp[F(k)]$ $=E^{-B}\exp[F(\hat{k})]$ is of order 37% in e^+-e^- annihilation at 3 on 3 GeV energy, $\exp[F(0^\circ)]/\exp[F(90^\circ)]$ $=1.37$. The large size of this effect is due to a (logarithm)^2 dependence of $F(\hat{k})$ on the beam energy, Eq. (5.28). The sign of the effect produces a relative further enhancement of $I_0(k)$ where it is large to begin with, which manifests the positive correlation of bosons in the coherent state (1.3).

There are two distinct scaling laws revealed by $P(Q)$. The first,

$$P(\lambda Q)=\lambda^{B-4}P(Q)\,, \qquad (1.11)$$

governs the approach of Q to the origin along a ray. The second governs the approach of Q to the mantle of the light cone at fixed energy. With Q^μ $=(E,|\vec{Q}|\hat{k})$, it is conveniently expressed in terms of the light-cone variable $Q^-=E-|\vec{Q}|$:

$$\lim_{Q^-\to 0}[(Q^-)^{1-B}P(E,Q^-,\hat{k})] =N[\Gamma(1+B)]^{-1}BE^{B-1}I_0(k)\exp[F(k)] \qquad (1.12a)$$

$$=N[\Gamma(1+B)]^{-1}BE^{-3}I_0(\hat{k})\exp[F(\hat{k})]\,, \qquad (1.12b)$$

where $k^\mu=(E,E\hat{k})$, $I_0(\hat{k})=E^2I_0(k)$, $F(\hat{k})=F(k)+B\ln E$. The first scaling law provides a precise statement of the threshold theorem $\sigma(Q)\sim\sigma_0P(Q)$:

$$\lim_{\lambda\to 0}\lambda^{4-B}\sigma(\lambda Q)=\sigma_0P(Q)\,. \qquad (1.13)$$

The second provides a precise statement of the asymptotic relation near the light cone $P(Q)$ $\sim P^{as}(Q)$, namely, $P^{as}(Q)$ is an invariant function of Q and the u_a satisfying

$$\lim_{\lambda\to 0}\lambda^{1-B}P(E,\lambda Q^-,\hat{k})=\lim_{\lambda\to 0}\lambda^{1-B}P^{as}(E,\lambda Q^-,\hat{k})\,, \qquad (1.14)$$

where the limit is finite.

We also present a simple exact expression for the energy spectral function $P_1(E)=\int P(E,\vec{Q})\,d^3Q$. Because $P_1(E)$ is not an invariant function of the u_a, it is convenient to make the frame dependence explicit and define

$$P_1(E,\tau)\equiv\int P(Q)\delta(E-\tau\cdot Q)\,d^4Q\,, \qquad (1.15)$$

where τ is a unit future timelike four-vector, τ^2 $=1$, $\tau^0\geq 1$. This quantity is an invariant function of τ and the u_a that coincides with $P_1(E)$ for τ

$=(1,0,0,0)$. The frame dependence is measurable by making cuts in $P(Q)$ along different planes normal to different τ. In Sec. III, $P_1(E,\tau)$ is calculated, with the remarkably simple result

$$P_1(E,\tau)=N[\Gamma(1+B)]^{-1}B\theta(E)E^{B-1} \times \exp\left[\sum_{a,b} e_a e_b K(u_a,u_b,\tau)\right], \qquad (1.16)$$

where $K(u_a,u_b,\tau)$, Eq. (3.5), is the kernel for zero-frequency photons calculated explicitly in the preceding article,[1] and rederived by Hilbert-space methods in Appendix B. It is shown there that photon wave functions $\phi(k)$ with low-frequency limit

$$\lim_{\omega\to 0}\omega\phi^\mu(k)=i(2\pi)^{-3/2}\int \rho(u)\,u^\mu(u^0-\vec{u}\cdot k)^{-1}u_0^{-1}d^3u\,, \qquad (1.17)$$

where in practice $\rho(u)=\sum_a e_a\delta^3(\vec{u}-\vec{u}_a)u^0$, possess the finite Lorentz-invariant inner product

$$\langle\phi_1,\phi_2\rangle+\langle\rho_1,\rho_2\rangle=-\frac{1}{2}\int d\hat{k}\int_0^\infty d\omega\ln(a\omega)\frac{\partial}{\partial\omega}[\omega^2\phi_{1\mu}(k)(-g_{\mu\nu})\phi_{2\nu}(k)]+\int\frac{d^3u_1}{u_1^0}\frac{d^3u_2}{u_2^0}\rho_1(u_1)K(u_1,u_2,\tau)\rho_2(u_2)\,, \qquad (1.18)$$

where $\tau=(1,0,0,0)$. Although this one-photon inner product is indefinite, in second quantization the infrared coherent states, namely, with low-frequency limit (1.3), have a positive-definite inner product. The zero-frequency photon kernel also appears in the formula for $P(Q)$. In particular, $F(k)$ in Eq. (1.12) may be written

$$F(k)=\lim_{Q^-\to 0}\left[\sum_{a,b}e_a e_b K(u_a,u_b,\tau)-B\ln(Q^2)^{1/2}\right], \qquad (1.19)$$

where $Q^\mu=(E,|\vec{Q}|\hat{k})=(Q^2)^{1/2}\tau^\mu$, $k=(E,E\hat{k})$, $Q^-=E-|\vec{Q}|$. The zero-frequency photon kernel acquires a direct physical meaning from these formulas for the observable spectral functions of electromagnetic radiation near threshold, where it is the only ingredient after the power law and the one-photon formula. It appears in the exponent because the coherent state is an exponential in the creation operator.

The energy dependence of $P_1(E,\tau)$ given by $\theta(E)E^{B-1}$ is an old result of quantum electrodynamics, first conjectured by Schwinger[2] and subsequently derived by Yennie, Frautschi, and Suura.[3,4] The nonperturbative contribution to the radiative tail of the ψ particle was calculated by Yennie,[5] and radiative effects in high-energy scattering were reviewed by Tsai.[6] The problem of calculating $P(Q)$ was posed by Kulish and Faddeev[7] and had previously been considered by Kibble,[8] who obtained the scaling law $P(\lambda Q)=\lambda^{B-4}P(Q)$. Recently, Chahine[9] has calculated the energy spectral function $P_1(E,\tau)$ in the Breit frame for the scattering of a single charged particle, which corresponds to evaluating $K(u_a,u_b,\tau)$ for $\tau^\mu=(u_a+u_b)^\mu|u_a+u_b|^{-1}$. An advantage of considering the generic frame, besides being necessary for calculating $P(Q)$, is that $K(u_a,u_b,\tau)$ is a function of the three hyperbolic angles which form the sides of the hyperbolic triangle on the unit hyperboloid defined by u_a, u_b, and τ that almost possesses triangular symmetry, see Eq. (3.5d). On another occasion, the author[10] has presented an exact expression for $P(Q)$, which, however, was not notably explicit. It is obtained here by direct application of the method used in the preceding article.[1]

In Sec. II the threshold theorem is established. In Sec. III A, $P_1(E,\tau)$ is calculated, and in Sec. III B the radiative tail of a missing massive particle is expressed in terms of it. In Sec. IV an integral representation of $P(Q)$ is obtained, Eq. (4.23), which, however, requires analytic continuation for values of B less than three. [Recall that B is of order $(137)^{-1}$.] In Sec. V the asymptotic form of $P(Q)$ near the light cone is obtained, which is valid for small values of B. This section also contains a qualitative discussion of the energy-momentum spectral function and a numerical example. The concluding Sec. VI contains a discussion of how the idealization involved in the particular non-Fock representation (1.3) describes the actual laboratory situation, and a comparison with the photon mass and Hilbert-space methods. Asymptotic expressions for the zero-frequency photon kernel may be found in Appendix A. In Appendix B it is shown that traditional Hilbert-space methods lead to the same calculation.

II. THRESHOLD THEOREM

Let $S_n(p,k_1\cdots k_n)$ be the S-matrix element for emission of n unobserved photons, where p represents the set of momenta p_a of all the other particles. To satisfy the mass-shell constraints and the constraints owing to energy-momentum conservation, the p_a are expressed in terms of $Q\equiv k_1+\cdots+k_n$ and a remaining set α of kinematically independent variables, so S_n is a function of the α and the photon momenta $\vec{k}_1, \ldots, \vec{k}_n$:

$$S_n(p_a, k_1 \cdots k_n) = S_n(\alpha, k_1 \cdots k_n). \tag{2.1}$$

For convenience we shall suppress α. The desired cross section for fixed α and Q is the sum over all final photon states consistent with energy-momentum conservation:

$$\sigma(Q) = |S_0|^2 \delta^4(Q) + \sum_{n=1}^{\infty} (n!)^{-1} \int (dk_1)_a \cdots (dk_n)_a |S_n(k_1 \cdots k_n)|^2 \delta^4(k_1 + \cdots + k_n - Q), \tag{2.2}$$

$$\sigma(Q) = (2\pi)^{-4} \int d^4x\, e^{iQ\cdot x} \left[|S_0|^2 + \sum_{n=1}^{\infty} (n!)^{-1} \int (dk_1)_a \cdots (dk_n)_a S_n^*(k_1 \cdots k_n) S_{n,x}(k_1 \cdots k_n) \right], \tag{2.3a}$$

where

$$S_{n,x}(k_1 \cdots k_n) = S_n(k_1 \cdots k_n) e^{-i(k_1 + \cdots + k_n)\cdot x}. \tag{2.3b}$$

Here, for each variable $k_1, \ldots, k_n$, the expression

$$\int (dk)_a \phi_1^*(k) \phi_2(k) \equiv \langle \phi_1, \phi_2 \rangle \tag{2.4a}$$

symbolically represents the inner product, derived in Appendix B,

$$\int (dk)_a \phi_1^*(k)\phi_2(k) = (-\tfrac{1}{2}) \int d\hat{k} \int_0^\infty d\omega \ln a\omega \frac{\partial}{\partial\omega} [\omega^2 \phi_{1,\mu}^*(k)(-g^{\mu\nu})\phi_{2,\nu}(k)], \tag{2.4b}$$

Eq. (P5.29), which is the first term of Eq. (1.18), the second and zero-frequency term $\langle \rho, \rho \rangle$ being already exponentiated, Eq. (2.12). Here $k = (\omega, \vec{k}) = \omega(1, \hat{k})$ and a is a finite scale-breaking parameter. Note that the cross section has precisely the form of a four-momentum spectral function

$$P_\Psi(Q) = \langle \Psi, \delta^4(P_{\rm op} - Q)\Psi \rangle = (2\pi)^4 \int dx\, e^{iQ\cdot x} \langle \Psi, e^{-iP_{\rm op}\cdot x}\Psi \rangle, \tag{2.5}$$

where $P_{\rm op}$ is the total four-momentum operator, and the state Ψ is described by the wave function $\Psi = \{\Psi_n(k_1 \cdots k_n)\}$, $\Psi_n(k_1 \cdots k_n) = S_n(k_1 \cdots k_n)$.

Define the asymptotic spectral function by

$$\sigma^{\rm as}(Q) \equiv \lim_{\lambda \to 0} \lambda^{4-B} \sigma(\lambda Q), \tag{2.6}$$

where B remains to be determined, and change variables from x to $x' = \lambda x$, and from ω to $\omega' = \lambda^{-1}\omega$. Upon dropping primes, one obtains

$$\sigma^{\rm as}(Q) = \lim_{\lambda \to 0} \lambda^{-B} (2\pi)^{-4} \int d^4x\, e^{iQ\cdot x} \left[|S_0|^2 + \sum_{n=1}^{\infty} (n!)^{-1} \int (dk_1)_{\lambda a} \cdots (dk_n)_{\lambda a} \times \lambda^n S_n^*(\lambda k_1, \ldots, \lambda k_n) \lambda^n S_{n,\lambda^{-1}x}(\lambda k_1, \ldots, \lambda k_n) \right], \tag{2.7}$$

where $(dk)_{\lambda a}$ is the inner product (2.4b) with a replaced by λa. From the low-energy theorem,

$$\lim_{\lambda \to 0} \lambda^n S_n(\lambda k_1 \cdots \lambda k_n) = (-i)^n \phi(k_1) \cdots \phi(k_n) S_0, \tag{2.8a}$$

$$\phi^\mu(k) = \frac{i}{(2\pi)^{3/2}} \sum_a \frac{e_a u_a^\mu}{u_a \cdot k}, \tag{2.8b}$$

we obtain

$$\sigma^{\rm as}(Q) = |S_0|^2 \lim_{\lambda \to 0} \lambda^{-B} (2\pi)^{-4} \int d^4x\, e^{iQ\cdot x} \times \exp\left[\int (dk)_{\lambda a} \phi^*(k) \phi_x(k) \right], \tag{2.9}$$

where $\phi_x(k) = \phi(k) e^{-ik\cdot x}$. From Eq. (2.4) we have

$$\int (dk)_{\lambda a} \phi^*(k) \phi_x(k) = B \ln\lambda + \int (dk)_a \phi^*(k) \phi_x(k) \tag{2.10a}$$

$$= B \ln\lambda + \langle \phi, \phi_x \rangle, \tag{2.10b}$$

where B is now defined by Eqs. (1.8). With this identification of B, the limit $\lambda \to 0$ is finite and gives

$$\sigma^{\rm as}(Q) = |S_0|^2 (2\pi)^{-4} \int d^4x\, e^{iQ\cdot x} \exp(\langle \phi, \phi_x \rangle), \tag{2.11}$$

thereby establishing the threshold theorem: The threshold spectral function $\sigma^{as}(Q)$ is identical with the spectral function of the state $\Psi = \{\Psi_n\}$, $\Psi_n = (-i)^n \phi(k_1) \cdots \phi(k_n) S_0$ which is coherent at finite frequencies with coherence function $\phi(k)$, Eq. (2.8b), which is the bremsstrahlung amplitude.

According to Eqs. (P6.29) and (P6.30), or Appendix B, one has for the no-photon emission amplitude S_0,

$$S_0 = C \exp(\tfrac{1}{2}\langle \rho, \rho\rangle), \tag{2.12}$$

where C is an invariant function of the u_a, and $\langle \rho, \rho\rangle$ is a strictly zero-frequency contribution, represented by the second term of Eq. (1.18), and given explicitly below, Eq. (3.5). It is interpreted as an inner product, with kernel K, of zero-frequency photons.[4] With

$$\sigma^{as}(Q) \equiv |C|^2 P(Q) \equiv \sigma_0 P(Q) \tag{2.13}$$

this gives[11]

$$P(Q) = (2\pi)^{-4} \int d^4x\, e^{iQ\cdot x} \exp(\langle \rho, \rho\rangle + \langle \phi, \phi_x\rangle). \tag{2.14}$$

As discussed in the preceding article, the quantity $(\langle \rho, \rho\rangle + \langle \phi, \phi_x\rangle)$, Eq. (1.18), is invariant and finite. It replaces the naive expression

$$(2\pi)^{-3} \int d^3k (2\omega)^{-1} \phi^*_\mu(k)(-g^{\mu\nu})\phi_\nu(k) e^{-ik\cdot x},$$

which diverges at $\omega = 0$. From Eq. (2.4) we have

$$\langle \phi, \phi_x\rangle = -\frac{1}{2}\int d\hat{k} \int_0^\infty d\omega \ln a\omega \frac{\partial}{\partial\omega}[\omega^2 \phi^*_\mu(k)(-g^{\mu\nu})\phi_\nu(k)e^{-ik\cdot x}], \tag{2.15}$$

$$\langle \phi, \phi_x\rangle = \frac{-\frac{1}{2}}{(2\pi)^3}\int d\hat{k}\left(\sum_a \frac{e_a u_a}{u_a^0 - \vec{u}_a\cdot\hat{k}}\right)^2 \ln\left(\frac{ae^\gamma}{\epsilon + i(x^0 - \hat{k}\cdot\vec{x})}\right), \tag{2.16}$$

where $\gamma = -\int_0^\infty ds\, e^{-s}\ln s$ is Euler's constant. The spectral function $P(Q)$ is now expressed as the Fourier transform of a function which is analytic in the future tube, and hence it vanishes outside the future light cone.[12]

III. TWO SIMPLER PROBLEMS

A. Energy spectral function

It is easier to evaluate the energy spectral function

$$P_1(E, \tau) = \int P(Q)\delta(E - \tau\cdot Q)d^4Q$$

than the energy-momentum spectral function $P(Q)$. To perform this integration we choose coordinates with time axis aligned along τ. From Eq. (2.14) this gives

$$P_1(E, \tau) = (2\pi)^{-1}\int_{-\infty}^\infty dt\, e^{iEt}\exp(\langle \rho, \rho\rangle + \langle \phi, \phi_t\rangle), \tag{3.1}$$

where $\phi_t = \phi_{(x^0 = t, \vec{x} = 0)}$. From Eq. (2.16) we have

$$\langle \phi, \phi_t\rangle = B\ln[ae^\gamma(\epsilon + it)^{-1}], \tag{3.2}$$

so

$$P_1(E, \tau) = (ae^\gamma)^B \exp(\langle \rho, \rho\rangle)(2\pi)^{-1}\int_{-\infty}^\infty dt\, e^{iEt}(\epsilon + it)^B, \tag{3.3}$$

$$P_1(E, \tau) = (ae^\gamma)^B[\Gamma(1 + B)]^{-1}B\theta(E)E^{B-1}\exp(\langle \rho, \rho\rangle). \tag{3.4}$$

There remains only to substitute for

$$\langle \rho, \rho\rangle = \sum_{a,b} e_a e_b K(u_a, u_b), \tag{3.5a}$$

Eqs. (1.18) or (P6.30), or, making the frame dependence explicit,

$$T(\tau) \equiv \langle \rho, \rho\rangle = \sum_{a,b} e_a e_b K(u_a, u_b, \tau), \tag{3.5b}$$

where $K(u_a, u_b, \tau)$ is the kernel for zero-frequency photons.[4] As shown in the Appendix of the preceding article, $K(u_a, u_b, \tau)$ is an invariant function of u_a, u_b, τ:

$$K(u_a, u_b, \tau) = K(\psi_{a,b}, \psi_a, \psi_b), \tag{3.5c}$$

where $\psi_{a,b}$, ψ_a, and ψ_b are positive hyperbolic angles which form the three sides of the hyperbolic triangle whose vertices on the unit hyperboloid are given by u_a, u_b, and τ, so that $\cosh\psi_a = u_a \cdot \tau$, $\cosh\psi_b = u_b \cdot \tau$, $\cosh\psi_{a,b} = u_a \cdot u_b$, and, by Eq. (PA24),

$$K(\psi_{a,b}, \psi_a, \psi_b) = [(2\pi)^2 \tanh\psi_{a,b}]^{-1}[R(\tfrac{1}{2}(\psi_{a,b}+\psi_a+\psi_b)) - R(\tfrac{1}{2}(\psi_a+\psi_b-\psi_{a,b})) + R(\tfrac{1}{2}(\psi_{a,b}+\psi_a-\psi_b)) + R(\tfrac{1}{2}(\psi_{a,b}+\psi_b-\psi_a)) - R(\psi_{a,b})], \tag{3.5d}$$

where

$$R(\psi) = \int_0^\psi dx\, x/\tanh x \tag{3.5e}$$

is a Spence function. For $a=b$ this reduces to

$$K(0, \psi_a, \psi_a) = [(2\pi)^2 \tanh\psi_a]^{-1}\psi_a . \tag{3.5f}$$

We thus obtain for the energy spectral function in a generic frame the explicit expression

$$P_1(E,\tau) = (ae^\gamma)^B[\Gamma(1+B)]^{-1}B\theta(E)E^{B-1} \times \exp\Big[\sum_{a,b} e_a e_b K(u_a, u_b, \tau)\Big]. \tag{3.6}$$

The energy dependence of this expression is familiar, but the explicit frame dependence appears to be a new result.

B. Four-momentum of a missing particle

The preceding result may be applied to calculate the radiative tail of a particle observed in a missing-mass experiment. In such an experiment the four-momenta of all particles but one are measured precisely, apart from some infrared photons. Here we calculate the dependence of the cross section on the missing four-momentum L^μ in the neighborhood of the threshold for production of the missing particle, as in Ref. 5. Thus, if m is the mass of the missing particle, then $\eta \equiv (L^2)^{1/2} - m$ is small. The dependence on L^μ is given by[13]

$$\sigma_1(L) = \sigma_{1,0}\int P(L-p)(2E)^{-1}d^3p, \tag{3.7}$$

where $P(Q)$ is the energy-momentum spectral function of the photons, and $p^\mu = (E, \vec{p}) = [(m^2+\vec{p}^{\,2})^{1/2}, \vec{p}]$ is the four-momentum of the missing particle. Unless $L^2 > m^2$, $\sigma_1(L)$ will be zero. We evaluate the integral in the frame where L is at rest $L=(m+\eta, \vec{0})$ and η is small and calculate

$$R(L) \equiv \lim_{\eta\to 0}\eta^{1-B}\sigma_1(L^0 = m+\eta, \vec{L}=0), \tag{3.8}$$

$$R(L) = \sigma_{1,0}\lim_{\eta\to 0}\eta^{1-B}\int P(Q^0 = m+\eta-E, \vec{Q} = -\vec{p}) \times (2E)^{-1}d^3p. \tag{3.9}$$

Because the support of $P(Q)$ is inside the future light cone, the integrand vanishes unless

$$m+\eta \geq (\vec{p}^{\,2}+m^2)^{1/2} + |\vec{p}|, \tag{3.10}$$

so $\vec{p}$ is of order η. The change of variables $\vec{p} = \eta\vec{p}\,'$ gives, with

$$m+\eta-E = m+\eta-(m^2+\eta^2\vec{p}\,'^2)^{1/2} \to \eta,$$

$$R(L) = \sigma_{1,0}(2m)^{-1}\eta^{1-B}\int P(Q^0=\eta, \vec{Q} = -\eta\vec{p}\,')\eta^3 d^3p', \tag{3.11}$$

$$R(L) = \sigma_{1,0}(2m)^{-1}\eta^{1-B}\int P(Q^0=\eta, \vec{p})d^3p. \tag{3.12}$$

One recognizes that the integral is $P_1(\eta, \tau)$, which we evaluated in the rest frame of L, and hence by Eq. (3.6),

$$R(L) = \sigma_{1,0}(2m)^{-1}(ae^\gamma)^B[\Gamma(1+B)]^{-1}B \times \exp\Big[\sum_{a,b} e_a e_b K(u_a, u_b, \tau)\Big], \tag{3.13}$$

where $\tau = L/(L^2)^{1/2}$, which is independent of η. From Eq. (3.8) we conclude that for small η, $\sigma_1 = \eta^{B-1}R(L)$, and

$$\eta = [(L^2)^{1/2} - m] = (L^2-m^2)[(L^2)^{1/2}+m]^{-1} \approx (2m)^{-1}(L^2-m^2).$$

We thus obtain the threshold dependence of the cross section on the missing four-momentum

$$\sigma_1(L) = \sigma_{1,0}(2m)^{-1}(ae^\gamma)^B[\Gamma(1+B)]^{-1}B\theta(L^2-m^2)\theta(L^0)[(L^2-m^2)(2m)^{-1}]^B \exp\Big[\sum_{a,b} e_a e_b K(u_a, u_b, \tau)\Big], \tag{3.14}$$

where $\tau = L/(L^2)^{1/2}$.

IV. EVALUATION OF THE ENERGY-MOMENTUM SPECTRAL FUNCTION

We now turn to the main problem which is the evaluation of the energy-momentum spectral function $P(Q)$. For a first orientation, observe that the support property $P(Q) = \theta(Q^2)\theta(Q^0)P(Q)$ and scaling law $P(\lambda Q) = \lambda^{B-4}P(Q)$ established in Sec. II allow us to set

$$P(Q)=\theta(Q^2)\theta(Q^0)M^{B-4}P(\tau), \tag{4.1}$$

where $Q^\mu = M\tau^\mu$, and τ is a unit future timelike vector. If this expression is substituted into the definition of the frame-dependent energy spectral function we obtain

$$P_1(E,\tau_1)=\int M^{B-4}P(\tau)\delta(E-M\tau\cdot\tau_1)M^3dM\,d\mu(\tau), \tag{4.2}$$

where for $\tau^\mu=(\cosh\psi, \sinh\psi\hat{x})$,

$$d\mu(\tau)\equiv\sinh^2\psi d\psi\,d\hat{x}$$

is the invariant volume element on the unit timelike hyperboloid $\tau^2=1$. With $P_1(E,\tau_1)=E^{B-1}P_1(\tau_1)$, this gives

$$P_1(\tau_1)=\int\frac{P(\tau)}{(\tau\cdot\tau_1)^B}d\mu(\tau), \tag{4.3}$$

which expresses the frame dependence of the energy spectral function as an integral transform of the scaled energy-momentum spectral function. Since we have calculated $P_1(\tau_1)$ in Sec. III, our labors would be over if we could invert this integral transform. Our final expression for $P(\tau)$ will, in fact, be precisely this inversion which shows that knowledge of the energy spectral function in all frames is equivalent to knowledge of the energy-momentum spectral function.

We evaluate $P(Q)$, Eq. (2.14), in a frame where the time axis is aligned along Q, $Q=(M,0)$, $Q\cdot x = Mx^0$, and introduce spherical coordinates, $x^\mu = (x^0, r\hat{x})$. To exploit scaling we further transform from x^0 to $s=x^0/r$, so $\hat{x}$ and s are homogeneous variables and r is a scaling variable. With $\int d^4x = \int d\hat{x}\int_{-\infty}^{\infty}ds\int_0^\infty dr\,r^3$ we obtain from Eq. (2.16)

$$\langle\rho,\rho\rangle+\langle\phi,\phi_x\rangle=B\ln(ae^\gamma/r)+T_1(s,\hat{x}), \tag{4.4a}$$

$$T_1(s,\hat{x})=\langle\rho,\rho\rangle+\frac{\frac{1}{2}}{(2\pi)^3}\int d\hat{k}\left(\sum_a\frac{e_a u_a}{u_a^0-\vec{u}_a\cdot\hat{k}}\right)^2\ln[\epsilon+i(s-\hat{k}\cdot\hat{x})]\,. \tag{4.4b}$$

The integration over r in Eq. (2.14) may be performed for $B<4$ with the result

$$P(Q)=(ae^\gamma)^B M^{B-4}\Gamma(4-B)(2\pi)^{-4}\int d\hat{x}\int_{-\infty}^{\infty}ds(\epsilon-is)^{B-4}\exp[T_1(s,\hat{x})]\,. \tag{4.5}$$

By distorting the contour in various ways, various representations of $P(Q)$ are obtained. The one shown in Fig. 1 gives

$$P(Q)=(ae^\gamma)^B M^{B-4}(2\pi)^{-4}\Gamma(4-B)(-2i\sin\pi B)\int d\hat{x}\int_0^\infty ds\,s^{B-4}e^{i\pi B/2}\exp[T_1(s,\hat{x})]\,. \tag{4.6}$$

This expression represents $P(Q)$ only for $B>3$. We will later continue in e^2 to all positive B. Inspection of Eq. (4.4b) shows that the phase of $\exp[T_1(s,\hat{x})]$ is $\exp(-i\pi B/2)$ for $s>1$. Because $P(Q)$ is real, it is sufficient, after dropping $(-i)$, to take the imaginary part of the integral from $s=0$ to $s=1$. This gives, with $s=\tanh\nu$,

$$P(Q)=(ae^\gamma)^B M^{B-4}(-1)[\Gamma(B-3)]^{-1}(2\pi)^{-3}\,\mathrm{Im}\int d\hat{x}\int_0^\infty d\nu\cosh^2\nu(\sinh\nu)^{B-4}\exp[T_2(\nu,\hat{x})+(\tfrac{1}{2})i\pi B]\,, \tag{4.7a}$$

$$T_2(\nu,\hat{x})=\langle\rho,\rho\rangle+\frac{1}{2}\frac{1}{(2\pi)^3}\int d\hat{k}\left(\sum_a\frac{e_a u_a}{u_a^0-\vec{u}_a\cdot\hat{k}}\right)^2\ln[\epsilon+i(\sinh\nu-\cosh\nu\hat{x}\cdot\hat{k})]\,, \tag{4.7b}$$

where $\langle\rho,\rho\rangle$ is given in Eq. (3.5a), and we have used $\Gamma(4-B)\sin\pi B=-\pi/\Gamma(B-3)$.

The integral over $\hat{k}$ in the expression for T_2 will be obtained by analytic continuation of the kernel K, Eq. (3.5). Let σ^μ be the real unit spacelike four-vector

$$\sigma^\mu\equiv(\sinh\nu, \cosh\nu\hat{x})\,, \tag{4.8}$$

$\sigma^2=-1$, and let τ^μ be the complex unit timelike four-vector $\tau^\mu(g_{\mu\nu})\tau^\nu=1$,

$$\tau^\mu\equiv(\cosh(\nu+\tfrac{1}{2}i\pi-i\epsilon), \sinh(\nu+\tfrac{1}{2}i\pi-i\epsilon)\hat{x}) \tag{4.9}$$

or

$$\tau^\mu=\epsilon^\mu+i\sigma^\mu\,,$$

where ϵ^μ is the infinitesimal future four-vector $(\epsilon,\vec{0})$, so the logarithm appearing in Eq. (4.7b) may be written

$$\ln[\epsilon+i(\sigma^0-\vec{\sigma}\cdot\hat{k})]=\ln(\tau^0-\vec{\tau}\cdot\hat{k})\,.$$

According to Eqs. (P, A2) and (P, A3),

$$K(u_a, u_b) = K_i(\psi_{a,b}) + K_f(u_a, u_b), \tag{4.10}$$

$$K_f(u_a, u_b) = \frac{-\frac{1}{2}}{(2\pi)^3} \int d\hat{k} \, \frac{u_a \cdot u_b}{(u_a^0 - \vec{u}_a \cdot \hat{k})(u_b^0 - \vec{u}_b \cdot \hat{k})} \ln(u_a^0 - \vec{u}_a \cdot \hat{k}), \tag{4.11}$$

so with $\langle \rho, \rho \rangle = \sum_{a,b} e_a e_b K(u_a, u_b)$ we have for T_2, Eq. (4.7b),

$$T_2(\nu, \hat{x}) = \sum_{a,b} e_a e_b \left[K_i(\psi_{a,b}) - \frac{\frac{1}{2}}{(2\pi)^3} \int d\hat{k} \, \frac{u_a \cdot u_b}{(u_a^0 - \vec{u}_a \cdot \hat{k})(u_b^0 - \vec{u}_b \cdot \hat{k})} \ln \frac{u_a^0 - \vec{u}_a \cdot \hat{k}}{\tau^0 - \vec{\tau} \cdot \hat{k}} \right], \tag{4.12}$$

where τ^μ is given in Eq. (4.9). Comparison with Eq. (P, A14) shows that the second term is precisely $K_f(u_a, u_b, \tau)$, with $\tau^\mu = (\cosh\psi, \sinh\psi \hat{x})$ continued in ψ to $\psi = \nu + \frac{1}{2} i\pi - i\epsilon$, with ν real. Hence we have

$$T_2(\nu, x) = \sum_{a,b} e_a e_b K(u_a, u_b, \epsilon + i\sigma) = T(\epsilon + i\sigma), \tag{4.13}$$

in the notation of Eq. (3.5b). We thus obtain from Eq. (4.7a)

$$P(Q) = (ae^\gamma)^B M^{B-4} (-1)[(2\pi)^3 \Gamma(B-3)]^{-1} \operatorname{Im} \int d\hat{x} \int_0^\infty d\nu \cosh^2\nu (\sinh\nu)^{B-4} \exp[T(\epsilon + i\sigma) + \tfrac{1}{2} i\pi B], \tag{4.14}$$

where $\sigma^\mu = (\sinh\nu, \cosh\nu \hat{x})$. With $T(\tau) = \langle \rho, \rho \rangle$, the desired spectral function is exhibited here as an integral transform of the exponential of the zero-frequency photon kernel, analytically continued.

The continuation is easily effected. By Eq. (3.5), $K(u_a, u_b, \tau)$ is a function of the three invariants $\psi_{a,b}$, ψ_a, and ψ_b at values defined by $\cosh\psi_{a,b} = u_a \cdot u_b$, $\cosh\psi_a = u_a \cdot \tau = \cosh\psi u_a^0 - \sinh\psi \hat{x} \cdot \vec{u}_a$, and $a \to b$. As ψ is continued from real positive values to $\psi = \nu + \frac{1}{2} i\pi - i\epsilon$, the invariants ψ_a are continued to values determined by

$$\cosh\psi_a = u_a \cdot (\epsilon + i\sigma) = \epsilon + i u_a \cdot \sigma = \epsilon + i \sinh\nu_a$$
$$= \cosh[\nu_a + \tfrac{1}{2} i\pi - i\epsilon],$$

namely,

$$\psi_a = \nu_a + i\pi/2 - i\epsilon \quad \text{and } a \to b, \tag{4.15}$$

where $\nu_a = \nu_a(\nu, \hat{x})$, defined by

$$\sinh\nu_a = u_a \cdot \sigma = \sinh\nu u_a^0 - \cosh\nu \hat{x} \cdot \vec{u}_a, \tag{4.16}$$

is real. We thus have

$$K(u_a, u_b, \epsilon + i\sigma) = K(\psi_{a,b}, \nu_a + \tfrac{1}{2} i\pi, \nu_b + \tfrac{1}{2} i\pi). \tag{4.17}$$

From Eq. (3.5) one obtains explicitly

$$K(u_a, u_b, \epsilon + i\sigma) = K_R(u_a, u_b, \sigma) + i K_I(u_a, u_b, \sigma), \tag{4.18a}$$

$$K_R(u_a, u_b, \sigma) = (2\pi)^{-2} (\tanh\psi_{a,b})^{-1} [S(\tfrac{1}{2}(\nu_a + \nu_b + \psi_{a,b})) - S(\tfrac{1}{2}(\nu_a + \nu_b - \psi_{a,b}))$$
$$+ R(\tfrac{1}{2}(\psi_{a,b} + \nu_a - \nu_b)) + R(\tfrac{1}{2}(\psi_{a,b} + \nu_b - \nu_a)) - R(\psi_{a,b})], \tag{4.18b}$$

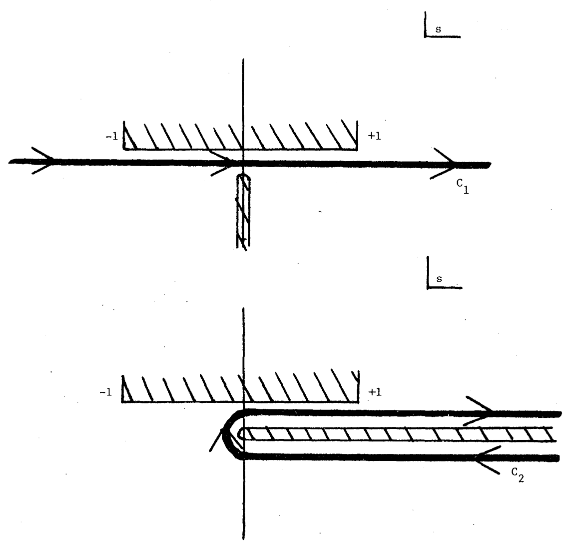

FIG. 1. Distortion of contour of integration from C_1 to C_2.

$$K_I(u_a, u_b, \sigma) = \frac{1}{(2\pi)^2} \frac{\frac{1}{2}\pi}{\tanh\psi_{a,b}} \times \ln \left\{ \frac{\cosh[\frac{1}{2}(\nu_a + \nu_b + \psi_{a,b})]}{\cosh[\frac{1}{2}(\nu_a + \nu_b - \psi_{a,b})]} \right\}, \tag{4.18c}$$

which, for $a = b$, reduces to

$$K_R(u_a, u_a, \sigma) = (2\pi)^{-2} \nu_a \tanh\nu_a, \tag{4.18d}$$

$$K_I(u_a, u_a, \sigma) = (2\pi)^{-2} (\pi/2) \tanh\nu_a, \tag{4.18e}$$

where

$$R(\psi) = \int_0^\psi dx\, x/\tanh x, \tag{4.18f}$$

$$S(\psi) = \int_0^\psi dx\, x \tanh x = \tfrac{1}{2} R(2\psi) - R(\psi), \tag{4.18g}$$

and we have used

$$R(\lambda_2+\tfrac{1}{2}i\pi)-R(\lambda_1+\tfrac{1}{2}i\pi)=\int_{\lambda_1+1/2i\pi}^{\lambda_2+1/2i\pi}du\,u/\tanh u$$
$$=\int_{\lambda_1}^{\lambda_2}dv(v+\tfrac{1}{2}i\pi)\tanh v$$
$$=S(\lambda_2)-S(\lambda_1)+\tfrac{1}{2}i\pi\ln\frac{\cosh\lambda_2}{\cosh\lambda_1}. \tag{4.18h}$$

Introducing the real and imaginary parts of $T(\epsilon+i\sigma)$

$$T(\epsilon+i\sigma)=T_R(\sigma)+i\,T_I(\sigma), \tag{4.19a}$$

$$T_{R,I}(\sigma)=\sum_{a,b}e_a e_b K_{R,I}(u_a,u_b,\sigma), \tag{4.19b}$$

one has, from Eq. (4.14),

$$P(Q)=(ae^{\gamma})^B M^{B-4}(-1)[(2\pi)^3\Gamma(B-3)]^{-1}$$
$$\times\int d\hat{x}\int_0^{\infty}d\nu\cosh^2\nu(\sinh\nu)^{B-4}$$
$$\times\sin[T_I(\sigma)+\tfrac{1}{2}\pi B]\exp[T_R(\sigma)]. \tag{4.20}$$

Recall that this integral represents $P(Q)$ in a frame with the time axis aligned along Q. To obtain it in a generic frame, note first that $T_R(\sigma)$ and $T_I(\sigma)$ are invariant functions of σ and the u_a, and that with $\sigma=(\sinh\nu,\cosh\nu\hat{x})$ the invariant $\sigma\cdot Q$ equals $M\sinh\nu$ in the special frame. Second, the unit spacelike hyperboloid possesses the Lorentz-invariant volume element

$$d\mu(\sigma)=d\hat{x}\,d\nu\cosh^2\nu,\quad -\infty<\nu<\infty. \tag{4.21}$$

Consequently, we have in any frame

$$P(Q)=(ae^{\gamma})^B(2\pi)^3\Gamma(B-3)^{-1}$$
$$\times(-1)\int d\mu(\sigma)\theta(\sigma\cdot Q)(\sigma\cdot Q)^{B-4}$$
$$\times\exp[T_R(\sigma)]\sin[T_I(\sigma)+\tfrac{1}{2}\pi B], \tag{4.22}$$

which is the desired expression for the energy-momentum spectral function, valid for $B>3$. In Sec. V an analytic continuation will be exhibited which is convenient for evaluating $P(Q)$ in the neighborhood of the light cone $Q^2=0$.

As a final remark, recall the integral transform (4.3),

$$P_1(\tau_1)=\int(\tau_1\cdot\tau)^{-B}P(\tau)d\mu(\tau).$$

With $\tau_1=(\cosh\psi,\sinh\psi\hat{x})$ it shows that $P_1(\tau_1)$ is analytic in the strip $-\pi/2<\mathrm{Im}\psi<\pi/2$, and consequently $P_1(\epsilon+i\sigma_1)$ is well defined for any $P(\tau)$. [In our case $P_1(\tau_1)$ is a particular function which, by Eq. (4.18f), is analytic in $-\pi<\mathrm{Im}\psi<\pi$.) Comparison of Eqs. (3.6) and (4.22) suggests that the inverse transform is

$$P(\tau)=-(2\pi)^{-3}(B-3)(B-2)(B-1)$$
$$\times\int d\mu(\sigma)\,\mathrm{sgn}(\sigma\cdot\tau)$$
$$\times(2i)^{-1}(\epsilon+i\sigma\cdot\tau)P_1(\epsilon+i\sigma). \tag{4.23}$$

It may be verified by substitution that this is true if $B>3$.

V. SPECTRAL FUNCTION NEAR THE LIGHT CONE

To describe the approach to the light cone $Q^2=0$ of the spectral function $P(Q)$, we introduce variables E, v, $\hat{k}$ defined by $Q^{\mu}=E(1,v\hat{k})$, so $P(Q)=P(E,v,\hat{k})$ and the light cone is approached as $v\to 1$. We shall calculate

$$P_2(k)\equiv B^{-1}\lim_{v\to 1}\{(Q^2)^{1-B}P(Q)\}|_{E,\hat{k}}, \tag{5.1a}$$

$$P_2(k)=B^{-1}\lim_{v\to 1}\{[E^2(1-v^2)]^{1-B}P(E,v,\hat{k})\}, \tag{5.1b}$$

where k is the four-vector $k^{\mu}=E(1,\hat{k})$ defined on the future light cone. We shall find that it is a finite Lorentz-invariant function of k and the u_a, so the approach to the light cone is described by a power law in $(1-v)$ or in $Q^-=E(1-v)$:

$$P(Q)\sim B(Q^-)^{B-1}(2E)^{B-1}P_2(k). \tag{5.2}$$

This power law in Q^- may be thought of as the radiative tail of a missing photon, analogous to the radiative tail of the missing massive particle discussed in Sec. III B.

In Eq. (4.22) for $P(Q)$ we introduce as a variable of integration $\alpha=\tanh\nu$, so that

$$\sigma^{\mu}=(\sinh\nu,\cosh\nu\hat{x})=\cosh\nu(\alpha,\hat{x}),$$
$$d\mu(\sigma)\theta(\sigma\cdot Q)=d\hat{x}\,d\alpha(\cosh\nu)^4\theta(\alpha-\vec{v}\cdot\hat{x}),$$

and we have

$$P(Q)=(ae^{\gamma})^B[(2\pi)\Gamma(B-3)]^{-1}E^{B-4}$$
$$\times(-1)\int d\hat{x}\int_{v\cos\theta}^{1}d\alpha(\alpha-v\cos\theta)^{B-4}H(\alpha,\hat{x}), \tag{5.3}$$

where the z axis of the angular integration has been aligned along $\hat{k}$, and

$$H(\alpha,\hat{x})\equiv(\cosh\nu)^B\exp[T_R(\sigma)]\sin[T_I(\sigma)+\tfrac{1}{2}\pi B]. \tag{5.4}$$

The basic technique for continuing from $B>3$ down to all positive B is by partial integrations. Assuming $B>3$, an integration by parts on α may be performed, with no contribution from the lower limit of integration. The asymptotic limits, as $\alpha\to 1$ or $\nu\to\infty$, of T_R and T_I are calculated in Appendix A, where it is shown that

$$\lim_{\alpha\to 1}[T_I(\sigma)+\tfrac{1}{2}\pi B]=0 \tag{5.5}$$

and that

$$F(\hat{x}) \equiv \lim_{\alpha\to 1} (T_R(\sigma) + B \ln\cosh\nu), \tag{5.6}$$

given in Eq. (A14), is finite, so there is no contribution from the upper limit either, and we have

$$P(Q) = (ae^{\gamma})^B[(2\pi)^3\Gamma(B-2)]^{-1}E^{B-4} \times \int d\hat{x} \int_{v\cos\theta}^{1} d\alpha(\alpha - v\cos\theta)^{B-3} \frac{\partial}{\partial\alpha} H(\alpha, \hat{x}). \tag{5.7}$$

This provides a continuation to all $B > 2$. A second integration by parts on α yields

$$P(Q) = (ae^{\gamma})^B[(2\pi)^3\Gamma(B-1)]^{-1}E^{B-4}J, \tag{5.8}$$

$$J = J_1 + J_{1,R}, \tag{5.9}$$

where J_1 is the contribution from the upper limit

$$J_1 = \int d\hat{x}(1 - v\cos\theta)^{B-2} \frac{\partial}{\partial\alpha} H(\alpha, \hat{x})\Big|_{\alpha=1} \tag{5.10}$$

and $J_{1,R}$ is given in Eq. (5.16). From Eqs. (5.5) and (5.6) we have

$$\frac{\partial}{\partial\alpha} H(\alpha, \hat{x})\Big|_{\alpha=1} = \exp[F(\hat{x})] \frac{\partial}{\partial\alpha} T_I(\alpha)\Big|_{\alpha=1}, \tag{5.11a}$$

which with Eq. (A10) gives

$$\frac{\partial}{\partial\alpha} H(\alpha, \hat{x})\Big|_{\alpha=1} = -2\pi^2 I_0(\hat{x}) \exp[F(\hat{x})], \tag{5.11b}$$

and so

$$J_1 = -(2\pi)^2 \int d\hat{x}(1 - v\cos\theta)^{B-2} I_0(\hat{x}) \exp[F(\hat{x})], \tag{5.12}$$

where $I_0(\hat{x})$ is defined in Eq. (1.8b). This expression needs no further continuation and represents J_1 for all B. After the change of variable

$$\cos\theta = 1 - (1-v)v^{-1}y,$$

or

$$1 - v\cos\theta = (1-v)(1+y),$$

so

$$J_1 = -2\pi^2(1-v)^{B-1}v^{-1} \int_0^{2\pi} d\phi \int_0^{2v(1-v)^{-1}} dy(1+y)^{B-2}(I_0e^F), \tag{5.13}$$

where I_0e^F is evaluated at $\cos\theta = 1 - (1-v)v^{-1}y$, we form

$$\lim_{v\to 1} (1-v)^{1-B}J_1 = -2\pi^2 \int_0^{2\pi} d\phi \int_0^{\infty} dy(1+y)^{B-2}(I_0e^F)(\cos\theta = 1, \phi). \tag{5.14}$$

For $0 < B < 1$ this integral is convergent, and recalling that the z axis is aligned along $\hat{k}$, we have

$$\lim_{v\to 1} (1-v)^{1-B}J_1 = 4\pi^3(B-1)^{-1}I_0(\hat{k}) \exp[F(\hat{k})]. \tag{5.15}$$

With $J = J_1 + J_{1R}$ there remains to consider

$$J_{1R} = -\int d\hat{x} \int_{v\cos\theta}^{1} d\alpha(\alpha - v\cos\theta)^{B-2}(\partial/\partial\alpha)^2 H(\alpha, \hat{x}). \tag{5.16}$$

An integration by parts on $\cos\theta$ is sufficient to continue this expression to all positive B. Calling J_2 the contribution at $\cos\theta = -1$, so that

$$J = J_1 + J_2 + J_{2R}, \tag{5.17}$$

and recalling that the z axis is aligned along $\hat{k}$, we have

$$J_2 = -2\pi[v(B-1)]^{-1} \int_{-v}^{1} d\alpha(\alpha+v)^{B-1}(\partial/\partial\alpha)^2 H(\alpha, -\hat{k}), \tag{5.18}$$

or, with $\alpha = -v + (1-v)z$ so $(\alpha + v) = (1-v)z$ and $1 + \alpha = (1-v)(1+z)$,

$$J_2 = -2\pi v^{-1}(B-1)^{-1}(1-v)^B \int_0^{(1+v)/(1-v)} dz\, z^{B-1}(\partial/\partial\alpha)^2 H(\alpha, -\hat{k})\Big|_{\alpha=-v+(1-v)z}. \tag{5.19}$$

In the limit $v \to 1$ of $(1-v)^{1-B}J_2(v)$ only the most singular term in the integrand at $\alpha = -1$ survives. With $H = \exp(X)\sin(T_I + \frac{1}{2}\pi B)$, where $X = T_R + B\ln\cosh\nu$, we find from Eqs. (A9), (A10), (A12), and (A16) that the leading singularity of $(\partial/\partial\alpha)^2 H(\alpha, -\hat{k})$ near $\alpha = -1$ is given by

$$\exp[F(\hat{k})] \sin\pi B(\partial/\partial\alpha)^2 X = \exp[F(\hat{k})] \sin\pi B(-2\pi)I_0(\hat{k})(1+\alpha)^{-1},$$

and thus

$$\lim_{v\to 1}(1-v)^{1-B}J_2=(2\pi)^2(B-1)^{-1}I_0(\hat{k})\exp[F(\hat{k})]\sin\pi B\int_0^\infty dz\,z^{B-1}(1+z)^{-1}.$$

The integral converges for $0<B<1$, giving

$$\lim_{v\to 1}(1-v)^{1-B}J_2=\tfrac{1}{2}(2\pi)^3(B-1)^{-1}I_0(\hat{k})\exp[F(\hat{k})]. \tag{5.20}$$

This equals the contribution from J_1, Eq. (5.15), and J_{2R} is annihilated in this limit. We thus obtain from Eq. (5.8)

$$\lim_{v\to 1}(1-v)^{1-B}P(E,v,\hat{k})=(ae^\gamma)^B[\Gamma(B)]^{-1}E^{B-1}I_0(\hat{k})\exp[F(\hat{k})]. \tag{5.21}$$

This establishes the finiteness of $P_2(k)=B^{-1}\lim_{v\to1}[E^2(1-v^2)]^{1-B}P(Q)$, with

$$P_2(k)=(ae^\gamma)^B[\Gamma(1+B)]^{-1}E^{-B-2}2^{1-B}I_0(\hat{k})\exp[F(\hat{k})]. \tag{5.22}$$

This may be expressed covariantly. With

$$I_0(k)=E^{-2}I_0(\hat{k})=\frac{-\frac{1}{2}}{(2\pi)^3}\left(\sum_a\frac{e_a u_a}{u_a\cdot k}\right)^2\geq 0 \tag{5.23}$$

and

$$F(k)\equiv F(\hat{k})-B\ln E, \tag{5.24a}$$

$$F(k)=\sum_{a,b}\frac{e_a e_b}{(2\pi)^2\tanh\psi_{ab}}\left[[\tfrac{1}{2}\ln(u_a\cdot k\;u_b\cdot k)+\ln 2]\psi_{a,b}+R\left(\tfrac{1}{2}\left(\psi_{a,b}+\ln\frac{u_a\cdot k}{u_b\cdot k}\right)\right)+R\left(\tfrac{1}{2}\left(\psi_{a,b}+\ln\frac{u_b\cdot k}{u_a\cdot k}\right)\right)-R(\psi_{a,b})\right]. \tag{5.24b}$$

by Eq. (A14),

$$P_2(k)=(ae^\gamma)^B 2^{1-B}[\Gamma(1+B)]^{-1}I_0(k)\exp[F(k)] \tag{5.25}$$

is a manifestly invariant function of k and the u_a defined on the future light cone.

We define an asymptotic spectral function $P^{\rm as}(Q)$ to be an invariant function of Q and the u_a which satisfies

$$\lim_{v\to 1}(Q^2)^{1-B}P^{\rm as}(Q)\Big|_{E,\hat{k}}=\lim_{v\to 1}(Q^2)^{1-B}P(Q)\Big|_{E,\hat{k}}, \tag{5.26a}$$

which we write as

$$P(Q)\sim P^{\rm as}(Q). \tag{5.26b}$$

This definition is not unique. We may regard $I(k)$ and $F(k)$ as limiting values on the light cone of functions $I(Q)$ and $F(Q)$ defined inside the light cone. Such an extension off the cone is also not unique, but it becomes so if we regard each term in the sums (5.23) and (5.24) as the value on the light cone of a function of the three independent invariants $u_a\cdot u_b$, $u_a\cdot Q$, and $u_b\cdot Q$ (*a priori* it could also depend on Q^2) namely

$$I(Q)=\frac{-\frac{1}{2}}{(2\pi)^3}\left(\sum_a\frac{e_a u_a}{u_a\cdot Q}\right)^2\geq 0, \tag{5.27}$$

$$F(Q)=\sum_{a,b}\frac{e_a e_b}{(2\pi)^2\tanh\psi_{a,b}}\left\{[\tfrac{1}{2}\ln(u_a\cdot Q\;u_b\cdot Q)+\ln 2]\psi_{a,b}+R\left[\tfrac{1}{2}\left(\psi_{a,b}+\ln\frac{u_a\cdot Q}{u_b\cdot Q}\right)\right]+R\left[\tfrac{1}{2}\left(\psi_{a,b}+\ln\frac{u_b\cdot Q}{u_a\cdot Q}\right)\right]-R(\psi_{a,b})\right\}. \tag{5.28}$$

This gives the simple expression

$$P^{\rm as}(Q)=(ae^\gamma)^B[\Gamma(1+B)]^{-1}\theta(Q^0)\theta(Q^2)B(\tfrac{1}{2}Q^2)^{B-1}I_0(Q)\exp[F(Q)]. \tag{5.29}$$

An alternative asymptotic spectral function $P'^{\rm as}(Q)$ features the zero-frequency photon kernel. Let $\tau^\mu=Q^\mu/(Q^2)^{1/2}$ and let $\psi_a\geq 0$ be the hyperbolic angle defined by $\cosh\psi_a=u_a\cdot\tau$, then

$$\lim_{v\to 1}[\psi_a+\ln(Q^2)^{1/2}]\Big|_{E,\hat{k}}=\ln(2u_a\cdot k). \tag{5.30}$$

From the asymptotic limit (A2) one has

$$F(k) = \lim_{v\to 1}\left\{\sum_{a,b} e_a e_b K(u_a, u_b, \tau) - B\ln(Q^2)^{1/2}\right\}, \tag{5.31}$$

where K is the zero-frequency photon kernel, Eq. (3.5), and thus

$$P'^{\mathrm{as}}(Q) = (ae^{\gamma})^B[\Gamma(1+B)]^{-1}\theta(Q^0)\theta(Q^2)2^{1-B}B[(Q^2)^{1/2}]^{B-4}I_0(\tau)\exp\left[\sum_{a,b} e_a e_b K(u_a, u_b, \tau)\right]. \tag{5.32}$$

To compare with low-order perturbation theory, suppose B is small. In this case $P^{\mathrm{as}}(Q)$, Eq. (5.29), is concentrated near the light cone. [And we presume the same is true for $P(Q)$.] It contains a factor $B(1-v)^{B-1}\theta(1-v)$, which for small B approximates the Dirac δ function

$$\lim_{B\to 0} B(1-v)^{B-1}\theta(1-v) = \delta(1-v), \tag{5.33}$$

$$P(Q) \approx \theta(E)(ae^{\gamma}E)^B E^{-4}\delta(1-v)I_0(\hat{k})\exp[F(\hat{k})], \tag{5.34}$$

for $B\approx 0$, which compares with the first-order perturbative expression

$$P^{(1)}(Q) = \theta(E)E^{-4}\delta(1-v)I_0(\hat{k}). \tag{5.35}$$

As $e^2\to 0$, then $\exp[F(\hat{k})]\to 1$, $B\to 0$, and the two expressions agree. However, whereas the first-order perturbative expression diverges at $E=0$ when integrated,

$$\int P(Q)d^4Q = \int P(Q)E^3 dE\, v^2 dv\, d\hat{k},$$

Eq. (5.34) gives a finite probability for energy loss less than E_0 into $d\hat{k}$, namely

$$\int_0^{E_0} P(Q)E^3 dE\, v^2 dv \approx (ae^{\gamma}E_0)^B I_0(\hat{k})\exp[F(\hat{k})]. \tag{5.36}$$

The relative probability for energy loss into $d\hat{k}$ is

$$I_0(\hat{k})\exp[F(\hat{k})], \tag{5.37}$$

which is the first-order perturbative expression modulated by $\exp[F(\hat{k})]$. As bosons in a coherent state, the photons are positively correlated, so we expect $\exp[F(\hat{k})]$ to enhance the one-photon distribution where it is large and reduce it where it is small. To confirm this and to estimate the magnitude of the effect, consider e^+-e^- annihilation into neutrals [or heavy charged particles, so factors of $(1-v\cos\theta)^{-1}$ are not important]. If the electron and positron have four-velocities $(1-v^2)^{-1/2}(1,0,0,v)$ and $(1-v^2)^{-1/2}(1,0,0,-v)$, so $I_0(\hat{k}) = I_0(\cos\theta)$, $F(\hat{k}) = F(\cos\theta)$, then

$$I_0(\cos\theta) = \frac{e^2}{(2\pi)^3}\,\frac{v^2(1-\cos^2\theta)}{(1-v^2\cos^2\theta)^2}, \tag{5.38}$$

$$F(\cos\theta) = \frac{e^2}{(2\pi)^2}\left\{\ln\left[\frac{4(1-v^2\cos^2\theta)}{1-v^2}\right] - \frac{1+v^2}{v}\left[\tfrac{1}{2}\ln\frac{4(1-v^2\cos^2\theta)}{1-v^2}\ln\frac{1+v}{1-v} + R\left(\tfrac{1}{2}\ln\frac{1+v}{1-v}+\tfrac{1}{2}\ln\frac{1-v\cos\theta}{1+v\cos\theta}\right)\right.\right.$$
$$\left.\left. + R\left(\tfrac{1}{2}\ln\frac{1+v}{1-v}+\tfrac{1}{2}\ln\frac{1+v\cos\theta}{1-v\cos\theta}\right) - R\left(\ln\frac{1+v}{1-v}\right)\right]\right\}. \tag{5.39}$$

One finds that for v relativistic, $I_0(\cos\theta)$ has its maximum at two lobes near the beam axis at $\cos\theta_{\max} = \pm[1-v^{-2}(1-v^2)]$, and a minimum at $\cos\theta = 0$. For 3 GeV on 3 GeV one obtains

$$\exp[F(\cos\theta_{\max})]/\exp[F(0)] = 1.37. \tag{5.40}$$

The relatively large 37% effect occurs because of the $\ln^2$ terms in (5.39) and implicit in $R(\psi)\sim\frac{1}{2}\psi^2$. In particular, one has

$$\tfrac{1}{2}\ln^2\frac{2}{1-v} \approx 2\ln^2\frac{2E}{m} \approx 176$$

at $E = 3$ GeV.

VI. CONCLUDING REMARKS

The threshold theorem for the energy-momentum spectral function was derived in Sec. II in the mathematical limit of strictly zero four-momentum. On the other hand, it is clear that our description of the scattering process is an idealization which breaks down at some sufficiently small frequency $\omega_{\min}$. For example, $\omega_{\min}$ may be the upper limit in frequency of incident coherent radiation which has been neglected, or it may be the very small width of a long-lived unstable particle that has been treated as stable. The formula which we derived for $P(Q)$ will be useful if there is a range of photon energies ω greater than $\omega_{\min}$ but smaller than some $\omega_{\max}$, determined by the momenta p_a of the charged particles, where the traditional bremsstrahlung spectrum holds. (In practice this range may extend over many decades.) To justify this, suppose that for $\omega < \omega_{\max}$ the outgoing radiation is coherent $a^\mu(k)\Psi = \phi^\mu(k)\Psi$ but that the coherence function $\phi^\mu(k)$ is given by

$$i(2\pi)^{-3/2}\sum_a e_a u_a^\mu (u_a\cdot k)^{-1}$$

only for $\omega > \omega_{\min}$. (The sum goes over incident and outgoing charged particles.) For $\omega < \omega_{\min}$ we assume only that $\lim_{\omega\to 0}\omega\phi^\mu(k)$ is finite. Inspection of Eq. (2.16) shows that in this case the contribution to the inner product $\langle\phi,\phi_x\rangle$ from $\omega<\omega_{\min}$ is of order $(\omega_{\min}x)\ln(\omega_{\min}x)$. Furthermore, as the scaling operations of Sec. II show, the significant values of x are of order $1/Q$, so the contribution to $P(Q)$ from $\omega<\omega_{\min}$ is of order $(\omega_{\min}/Q)\ln(\omega_{\min}/Q)$, which is negligible for $\omega_{\min}\ll Q$. Thus, although no experiment is performed at zero energy, and although only at mathematically zero energy are the Fock and non-Fock representations distinguished, as long as low-energy photons are well described by the traditional bremsstrahlung formula (1.3), the corresponding non-Fock representation will be a convenient mathematical idealization.

It may be illuminating to compare the present approach and the familiar photon mass method.[4] The reader will have observed that the spectral function $\sigma(Q)\sim\sigma_0 P(Q)$ was calculated by taking into account only the real bremsstrahlung photons, whereas the photon mass method requires a cancellation of real and virtual infrared divergences in the sum over final states. However, two significant features of the photon mass method do survive in the present treatment. Firstly, the coherent non-Fock state defined by the low-energy bremsstrahlung spectrum

$$\lim_{\lambda\to 0}\lambda a^\mu(\lambda k)\Psi = \frac{i}{(2\pi)^{3/2}}\sum_a \frac{e_a u_a^\mu}{u_a\cdot k}\Psi \qquad (6.1)$$

requires a sum over all photon numbers to give a positive probability because the one-photon inner product (1.18) is indefinite. This is the analog of the sum over final states. Secondly, the finite parameter a which appears in the inner product (1.18) reappears in $P(Q)$ as the factor a^B. The same factor of a^B will also occur in the normalization integral, so if the state Ψ is normalized, the spectral function $P(Q)$ will be independent of a. This is the analog of the cancellation of the photon mass λ out of cross sections. (Of course, the photon mass cancels out only in the limit $\lambda\to 0$, whereas the cancellation occurs here for finite a.) The analog is quite close if the state Ψ is of the form $\Psi = S\Psi_0$, where Ψ_0 is a Fock state and the S matrix is calculated according to the method of the preceding article,[1] namely the integral over virtual photon momenta, $\int d^4k\, g_{\mu\nu}(k^2+i\epsilon)^{-1}\cdots$, is replaced by

$$\int d^4k \ln[-(\tfrac{1}{2}a)^2k^2 - i\epsilon](-\tfrac{1}{2})(\partial/\partial k^\lambda)$$
$$\times[k^\lambda g_{\mu\nu}(k^2+i\epsilon)^{-1}\cdots],$$

which differs by a partial integration, and the resulting $|S|^2$ contains the factor a^{-B}.

The present article illustrates how the method of the preceding article may be used to calculate observable quantities. However, it is by no means restricted to states of the form (1.3) which are coherent at finite frequencies, but may also be used perturbatively. To do so simply requires the substitution just mentioned for virtual photons, and for real photons

$$d^3k(2\omega)^{-1}\cdots = \frac{1}{2}\int d\hat{k}\int_0^\infty d\omega\,\omega^{-1}\omega^2\cdots$$

similarly gets replaced by

$$-\frac{1}{2}\int d\hat{k}\int_0^\infty d\omega \ln(a\omega)\,\frac{\partial}{\partial\omega}[\omega^2\cdots].$$

[The additional zero-frequency photon contribution in Eq. (1.18) is calculated once and for all and exponentiates.]

We have seen that scaling laws with anomalous dimension dominate the energy-momentum spectral function of electromagnetic radiation $\sigma(Q)$ for small values of Q. For this the scale breaking mechanism, introduced in the preceding article[1] by means of the Hertz potential, is a convenient vehicle, and the resulting formalism has allowed the calculation of the threshold spectral function $P(Q)$ in a straightforward manner. It could also be calculated by other methods. In particular, the work referred to in Ref. 4 shows that the photon mass method and dimensional regularization leads to the same inner product (1.18) and hence the same calculation, and in Appendix B of the present article it is shown that this is also true of traditional Hilbert-space methods.[14] Whatever one's method of choice, it is significant that the kernel for zero-frequency photons, Eq. (3.5), is the only ingredient in the final expression for the threshold energy-momentum spectral function. Because this is the fundamental experimentally accessible quantity associated with the infrared problem, it appears that the basic infrared mechanism is laid bare in the one-photon inner product (1.18) which is characterized by this kernel.

ACKNOWLEDGMENTS

It is a pleasure to thank friends and colleagues for stimulating conversations, especially Professor Alberto Sirlin and Professor Eugene Speer. This research was supported in part by the National Science Foundation under Grant No. PHY74-22218A03.

APPENDIX A: ASYMPTOTIC PROPERTIES OF THE ZERO-FREQUENCY PHOTON KERNEL

Note first, from the definition of $R(\psi)$, Eq. (3.5),

$$R(-\psi) = -R(\psi)\,, \tag{A1}$$

$$\lim_{\psi\to\pm\infty} [R(\psi) \mp \tfrac{1}{2}\psi^2] = \pm\pi^2/12\,, \tag{A2}$$

and from Eq. (4.18g)

$$\lim_{\psi\to\pm\infty} [S(\psi) \mp \tfrac{1}{2}\psi^2] = \mp\pi^2/24\,. \tag{A3}$$

Note next that with $\sigma^\mu = (\sinh\nu, \cosh\nu\hat{x})$ and

$$\sinh\nu_a = \sigma\cdot u_a = \cosh\nu(\tanh\nu u_a^0 - \vec{u}_a\cdot\hat{x})$$

one has

$$\lim_{\nu\to\pm\infty} [\nu_a(\nu,\hat{x}) - \nu] = \pm\ln(u_a^0 \mp \vec{u}_a\cdot\hat{x})\,, \tag{A4}$$

$$\lim_{\nu\to\pm\infty} \frac{\partial\nu_a}{\partial\nu}(\nu,\hat{x}) = 1\,, \tag{A5}$$

$$\lim_{\nu\to\pm\infty} [\nu_a(\nu,\hat{x}) - \nu_b(\nu,\hat{x})] = \pm\ln\frac{u_a^0 \mp \vec{u}_a\cdot\hat{x}}{u_b^0 \mp \vec{u}_b\cdot\hat{x}}\,. \tag{A6}$$

Using these results and

$$K_I(-\sigma) = -K_I(\sigma)\,, \tag{A7}$$

one finds from Eq. (4.18c)

$$\lim_{\nu\to\pm\infty} K_I(u_a, u_b, \sigma) = \pm(2\pi)^{-2}(\tanh\psi_{a,b})^{-1}\tfrac{1}{2}\pi\psi_{a,b}\,, \tag{A8}$$

and hence from Eqs. (1.8) and (4.19b)

$$\lim_{\nu\to\pm\infty} T_I(\sigma) = \pm\tfrac{1}{2}\pi B \tag{A9}$$

and also, with $\alpha = \tanh\nu$,

$$\lim_{\alpha\to\pm1} \frac{\partial}{\partial\alpha} T_I(\alpha,\hat{x}) = -2\pi^2 I_0(\pm\hat{x})\,. \tag{A10}$$

Similarly, from Eq. (4.18b) one has

$$K_R(-\sigma) = K_R(\sigma)\,, \tag{A11}$$

$$\lim_{\nu\to\pm\infty} [T_R(\sigma) + B\ln\cosh\nu] = F(\pm\hat{x})\,, \tag{A12}$$

$$\lim_{\psi\to\pm\infty} [T(\tau) + B\ln\cosh\psi] = F(\pm\hat{x})\,, \tag{A13}$$

where $\tau^\mu = (\cosh\psi, \sinh\psi\hat{x})$, and

$$F(\hat{k}) = \sum_{a,b} e_a e_b[(2\pi)^2\tanh\psi_{ab}]^{-1}\Big\{[\tfrac{1}{2}\ln(u_a^0 - \vec{u}_a\cdot\hat{k}) + \tfrac{1}{2}\ln(u_b^0 - \vec{u}_b\cdot\hat{k}) + \ln 2]\psi_{a,b} + R\Big[\tfrac{1}{2}\Big(\psi_{a,b} + \ln\frac{u_a^0 - \vec{u}_a\cdot\hat{k}}{u_b^0 - \vec{u}_b\cdot\hat{k}}\Big)\Big] + R\Big[\tfrac{1}{2}\Big(\psi_{a,b} + \ln\frac{u_b^0 - \vec{u}_b\cdot\hat{k}}{u_a^0 - \vec{u}_a\cdot\hat{k}}\Big)\Big] - R(\psi_{a,b})\Big\}\,, \tag{A14}$$

where for $a=b$ the diagonal summand reduces to

$$e_a^2(2\pi)^{-2}\ln[2(u_a^0 - \vec{u}_a\cdot\hat{k})]\,.$$

Finally, we record that with $\alpha = \tanh\nu$ and

$$X(\alpha,\hat{x}) \equiv [T_R(\sigma) + B\ln\cosh\nu]\,, \tag{A15}$$

$$\frac{\partial X}{\partial\alpha}(\alpha,\hat{x}) = -2\pi I_0(\alpha\hat{x})\ln\frac{1+\alpha}{1-\alpha} + \text{reg}\,, \tag{A16}$$

where reg is a function which is analytic in α at $\alpha = \pm 1$.

APPENDIX B: HILBERT-SPACE DERIVATION

We wish to calculate the threshold spectral function in the Hilbert space where the photon annihilation operator $a^\mu(k)$ is represented by

$$a^\mu(k) = a_F^\mu(k) + \phi^\mu(k)\,, \tag{B1}$$

and the translation group by

$$a^\mu(k) \to a^\mu(k)e^{ik\cdot x}\,. \tag{B2}$$

Here $a_F^\mu(k)$ is a Fock representation of the canonical commutation relations

$$[a_F^\mu(k), a_F^\nu(k')] = -g_{\mu\nu}2\omega\delta(\vec{k} - \vec{k}')\,, \tag{B3}$$

so that there exists a normalized state, call it $|\phi\rangle$, with $\langle\phi|\phi\rangle = 1$, which is annihilated by $a_F^\mu(k)$:

$$a_F^\mu(k)|\phi\rangle = 0 \tag{B4}$$

or, by Eq. (B1),

$$a^\mu(k)|\phi\rangle = \phi^\mu(k)|\phi\rangle\,. \tag{B5}$$

In these expressions $\phi^\mu(k)$ is a transverse classical function $k\cdot\phi(k) = 0$ with low-frequency limit:

$$\lim_{\lambda\to0} \lambda\phi^\mu(\lambda k) = \frac{i}{(2\pi)^{3/2}}\sum_a \frac{e_a u_a{}^\mu}{u_a\cdot k}\,, \tag{B6}$$

where $\sum_a e_a = 0$ (see Introduction). We assume $\phi(k)$ to be quite regular, as required below, apart from this singularity at the origin. Because this wave function is not square integrable, $a^\mu(k)$ is a non-Fock (i.e., not equivalent to a Fock) representation of the canonical commutation relations.

The representation $a_F^\mu(k)$ is irreducible, so we may conclude that the generator P^μ of the translation group (B2) has the usual form

$$P^\mu = \int \frac{d^3k}{2\omega} a_\kappa^\dagger(k)(-g^{\kappa\lambda})a_\lambda(k)k^\mu , \tag{B7}$$

to within an additive constant which we take to be zero since we are not considering the momentum of other particles.

Consider next the matrix element of the translation operator in the coherent state $|\phi\rangle$,

$$F(x) \equiv \langle\phi|\exp(-iP\cdot x)|\phi\rangle , \tag{B8}$$

$$F(x) = \exp\left[\int \frac{d^3k}{2\omega}\phi_\mu^*(k)(-g^{\mu\nu})\phi_\nu(k)(e^{-ik\cdot x}-1)\right] . \tag{B9}$$

The characteristic appearance of the difference $(e^{-ik\cdot x}-1)$ renders this integral finite at $\omega=0$ despite the low-frequency limit (B6). Equation (B9) may be proved tediously by expanding in powers of x, or more briefly, as follows. We have

$$\partial_\mu F(x) = \langle\phi|\exp(-iP\cdot x)(-iP_\mu)|\phi\rangle$$
$$= \int \frac{d^3k}{2\omega}(-ik_\mu)G(k,x) ,$$

where

$$\begin{aligned} G(k,x) &= \langle\phi|\exp(-iP\cdot x)a_\kappa^\dagger(k)(-g^{\kappa\lambda})a_\lambda(k)|\phi\rangle \\ &= \langle\phi|\exp(-iP\cdot x)a_\kappa^\dagger(k)|\phi\rangle(-g^{\kappa\lambda})\phi_\lambda(k) \\ &= \langle\phi|a_\kappa^\dagger(k)e^{-ik\cdot x}\exp(-iP\cdot x)|\phi\rangle(-g^{\kappa\lambda})\phi_\lambda(k) \\ &= \langle\phi|\exp(-iP\cdot x)|\phi\rangle\phi_\kappa^*(k)(-g^{\kappa\lambda})\phi_\lambda(k)e^{-ik\cdot x} \\ &= F(x)\phi_\kappa^*(k)(-g^{\kappa\lambda})\phi_\lambda(k)e^{-ik\cdot x} . \end{aligned}$$

This gives

$$\partial_\mu F(x) = F(x)\int\frac{d^3k}{2\omega}(-ik_\mu)\phi_\kappa^*(k) \times(-g^{\kappa\lambda})\phi_\lambda(k)e^{-ik\cdot x}$$

or

$$\partial_\mu F(x) = F(x)\partial_\mu\int\frac{d^3k}{2\omega}\phi_\kappa^*(k) \times(-g^{\kappa\lambda})\phi_\lambda(k)(e^{-ik\cdot x}-1) .$$

The solution to this equation with $F(0)=1$ is given in Eq. (B9).

In order to calculate the spectral function in the state $|\phi\rangle$,

$$\sigma(Q) \equiv \langle\phi|\delta^4(Q-P)|\phi\rangle$$
$$= (2\pi)^{-4}\int d^4x\, e^{iQ\cdot x}\langle\phi|e^{-iP\cdot x}|\phi\rangle , \tag{B10}$$

$$\sigma(Q) = (2\pi)^{-4}\int d^4x\, e^{iQ\cdot x}F(x) , \tag{B11}$$

one would like to write the integral in Eq. (B9) as the difference

$$\int\frac{d^3k}{2\omega}\phi_\kappa^*(k)(-g^{\kappa\lambda})\phi_\lambda(k)e^{-ik\cdot x} - \int\frac{d^3k}{2\omega}\phi_\kappa^*(k)(-g^{\kappa\lambda})\phi_\lambda(k) .$$

This is not possible because of the infrared divergence owing to (B6), but an integration by parts on ω gives instead

$$F(x) = \exp(\langle\phi,\phi_x\rangle)\exp(-\langle\phi,\phi\rangle) , \tag{B12}$$

where

$$\langle\phi,\phi_x\rangle \equiv -\tfrac{1}{2}\int d\hat{k}\int_0^\infty d\omega\,\ln(a\omega) \times\frac{\partial}{\partial\omega}[\omega^2\phi_\mu^*(k)(-g^{\mu\nu})\phi_{\nu,x}(k)] , \tag{B13}$$

with $\phi_{\nu,x}(k)\equiv\phi_\nu(k)e^{-ik\cdot x}$, is recognized as the finite inner product (2.4) and a is an arbitrary constant. In order to perform the integration (B11), it is convenient to make a Lorentz-invariant factorization, which Eq. (B12) is not, and write instead[11]

$$F(x) = \exp[\langle\rho,\rho\rangle + \langle\phi,\phi_x\rangle]\exp[-\langle\rho,\rho\rangle - \langle\phi,\phi\rangle] , \tag{B14}$$

where the term in brackets is defined in Eq. (1.18). The Lorentz invariance of this inner product was proved in Ref. 1, and the evaluation of the spectral function in the state $|\phi\rangle$ now proceeds as in Sec. II, with the result given in the text.

A dense set of states is obtained from $|\phi\rangle$ by applying polynomials in the creation operator

$$a_F^\dagger(f) = \int\frac{d^2k}{2\omega}a_F^{\mu\dagger}(k)(-g_{\mu\nu})f^\nu(k) , \tag{B15}$$

where $f^\nu(k)$ is a transverse square integrable photon wave function. Because the translation (B2) may be written

$$a_F^\mu(k) \to a_F^\mu(k)e^{ik\cdot x} + \phi^\mu(k)(e^{ik\cdot x}-1) , \tag{B16}$$

it is easy to verify that for sufficiently regular $f^\nu(k)$ the threshold spectral function is unaffected by the above polynomials. (This is also true for off-diagonal matrix elements.) We have proved that the threshold spectral function has the stated form for a dense set of states in the Hilbert-space representation defined by (B1), (B2), and (B6).

[1]D. Zwanziger, Phys. Rev. D 19, 3614 (1979). Further references on the infrared problem may be found here. Equations in this article will be designated by a P preceding the equation number.

[2]J. Schwinger, Phys. Rev. 75, 898 (1949).

[3]D. Yennie, S. Frautschi, and H. Suura, Ann. Phys. (N. Y.) 13, 370 (1961).

[4]A lexicon of translation from the notation of Ref. 3 (YFS) may be helpful: $(\alpha A)_{\mathrm{YFS}}$ is designated B here whereas $(B)_{\mathrm{YFS}}$ is something else, and, as shown elsewhere, N. Papanicolaou and D. Zwanziger, Nucl. Phys. B101, 77 (1975), $[\alpha \tilde{B}(\epsilon)]_{\mathrm{YFS}}$ would be

$$\sum_{a,b} e_a e_b K(u_a, u_b, \tau) + B \ln\epsilon/\lambda = \langle \rho, \rho \rangle + B \ln\epsilon/\lambda = T(\tau) + B \ln\epsilon/\lambda,$$

here, for $\tau = (1,0,0,0)$, where λ is the photon mass. The reader may wonder what progress, if any (apart from making the frame dependence explicit) has been made over the 1961 formulation since the basic quantities of the present approach already appear there. In the opinion of the author, this circumstance is not embarrassing, but reassuring, rather, because any physical quantity in quantum electrodynamics (without magnetic monopoles) should be calculable using a finite photon mass which is set equal to zero at the end. Few would doubt, however, that the underlying zero mass theory is ultimately more economical.

[5]D. R. Yennie, Phys. Rev. Lett. 34, 239 (1975).

[6]Y. S. Tsai, Rev. Mod. Phys. 46, 815 (1974); 49, 421 (E) (1977).

[7]P. Kulish and L. Faddeev, Teor. Mat. Fiz. 4, 153 (1970) [Theor. Math. Phys. 4, 745 (1970)].

[8]T. Kibble, Phys. Rev. 175, 1624 (1968), Sec. 6.

[9]C. Chahine, Phys. Rev. D 18, 4617 (1978), Sec. 3B.

[10]D. Zwanziger, Phys. Rev. D 14, 2570 (1976), Appendix B.

[11]Because ρ is a state of strictly zero-frequency photons, it is translation invariant, $\rho_x = \rho$.

[12]R. Streater and A. Wightman, *PCT Spin and Statistics and All That* (Benjamin, N. Y., 1964), theorem (2.8).

[13]Instead of posing Eq. (3.7), a threshold theorem as in Sec. II could be proved to establish formula (3.12).

[14]For a recent study of the infrared problem in quantum electrodynamics by Hilbert-space methods, see J. Fröhlich, G. Morchio, and F. Strocchi, CERN Report No. TH 2544 (unpublished).