Covariance problem in two-dimensional quantum chromodynamics

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The problem of covariance in the field theory of a two-dimensional non-Abelian gauge field is considered. Since earlier work has shown that covariance fails (in charged sectors) for the Schwinger model, particular attention is given to an evaluation of the role played by the non-Abelian nature of the fields. In contrast to all earlier attempts at this problem, it is found that the potential covariance-breaking terms are identical to those found in the Abelian theory provided that one expresses them in terms of the total (i.e., conserved) current operator. The question of covariance is thus seen to reduce in all cases to a determination as to whether there exists a conserved global charge in the theory. Since the charge operator in the Schwinger model is conserved only in neutral sectors, one is thereby led to infer a probable failure of covariance in the non-Abelian theory, but one which is identical to that found for the U(1) case.

I. INTRODUCTION

Non-Abelian gauge fields have of late been increasingly looked upon as one of the more promising approaches to a theory of the strong interactions. This trend has motivated a number of investigations into two-dimensional versions of such models in the hope that some insight might be gained into the general features of their four-dimensional counterparts. However, because of the more singular behavior of the inverse Laplacian in one spatial dimension, it has generally been recognized that the covariance proofs which are presumably valid in four dimensions do not apply in the two-dimensional case. In fact, the first suggestion that something might be amiss for two-dimensional gauge theories seems to be Zumino's suggestion that the Coulomb gauge formulation of the Schwinger model[1,2] is not covariant. He observed[3] that the equation

$$(-\partial^2 + e^2/\pi)j^\mu(x) = 0$$

of the Schwinger model implies that the charge operator

$$Q = \int j^0(x)dx^1$$

satisfies the equation

$$({\partial_0}^2 + e^2/\pi)Q = 0\,.$$

Thus the global charge operator Q is not conserved unless it vanishes. Since one cannot require Q to vanish identically if the canonical formalism is to be retained, Zumino concluded that "the Coulomb gauge formulation of the theory is not truly covariant unless one is willing to restrict oneself to states of zero charge."

Although Zumino's conclusion is valid, it has become usual to base assertions of covariance or noncovariance upon an examination of the Dirac-Schwinger commutator condition[4]

$$-i[T^{00}(x), T^{00}(x')] = -[T^{01}(x) + T^{01}(x')]\partial_1\delta(x_1 - x_1')\,.$$

A proof following this route was in fact supplied by the author[5] with the result being that the covariance-breaking terms were found to be proportional to

$$\lim_{L\to\infty}[j^1(L) - j^1(-L)]\,. \tag{1.1}$$

Since Brown's solution[2] shows that this does not vanish, one establishes noncovariance in a very satisfying way—namely the proof depends on the nonvanishing of the same combination (1.1) as that which leads to the conclusion that $\partial_0 Q \neq 0$ even though $\partial_\mu j^\mu = 0$. Thus Zumino's comment is easily understood within the context of Ref. 5.

Although one would expect intuitively that the covariance question would be answered similarly for the non-Abelian case as for the Abelian, recent work on this problem has asserted otherwise. In particular, Li and Willemsen[6] have claimed that the Poincaré algebra is satisfied (a result which may be shown using Ref. 5 not to be valid). Bars and Green[7] have asserted on the other hand that the problem is intimately associated with the non-Abelian nature of the fields and have concluded that the theory is covariant only in the color-singlet sector. In this paper it is shown that in fact the covariance question has essentially nothing to do with the non-Abelian aspect of the theory. The covariance-breaking term in the Poincaré algebra is found to be identical to that of Ref. 5, provided that for the current one takes the conserved total current operator rather than just the spin-$\frac{1}{2}$ contribution.

In the following section the results obtained earlier[5] in the Schwinger model are summarized and also strengthened by explicit consideration of the asymptotic properties of the current operator and energy-momentum tensor. Section IV presents the non-Abelian version of this theory and is followed in Sec. V by a discussion of Lorentz transformation properties leading to the claims summarized above. A brief conclusion presents some consequences of the results obtained.

II. THE SCHWINGER MODEL

The Schwinger model can be described by the Lagrangian

$$\mathcal{L} = \tfrac{1}{2} i\psi\alpha^\mu\partial_\mu\psi + \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} - \tfrac{1}{2}F^{\mu\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu) + ej^\mu A_\mu ,$$

where the current operator is formally defined by the limit

$$j^\mu(x) = \lim_{\mathbf{x}' \to \mathbf{x}} \tfrac{1}{2}\psi(x)\alpha^\mu q\psi(x')$$

and the charge matrix q is given as

$$q = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

In the Coulomb gauge

$$A_1 = 0 ,$$

and one is led in the usual way to consider the equation

$$-\nabla^2 A^0 = ej^0 . \tag{2.1}$$

As the inversion of the Laplacian in one spatial dimension gives a function which does not vanish for large values of its argument, it was proposed in Ref. 5 that one handle this complication by placing the system in the one-dimensional box $|x| \le L$ and solving (2.1) using Dirichlet boundary conditions. By further requiring that the Hamiltonian

$$H = \int_{-L}^{L} T^{00} dx_1 ,$$

where

$$T^{00} = -\tfrac{1}{2}i\psi\alpha_1\partial_1\psi + \tfrac{1}{2}F_{01}{}^2 ,$$

yield equations of motion which are identical to the Lagrangian equations, there results as the solution to (2.1)

$$A^0(x) = -\tfrac{1}{2}e\int_{-L}^{L} |x - x'|j^0(x')dx_1' + \tfrac{1}{2}LeQ ,$$

where

$$Q = \int_{-L}^{L} j^0(x)dx_1 .$$

Corresponding to this, one has the electric field

$$F^{01} = \tfrac{1}{2}e\int_{-L}^{L} \epsilon(x - x')j^0(x')dx_1' .$$

Using the fact that

$$\partial_\nu T^{\mu\nu} = 0$$

one can easily compute

$$[K, T^{00}(x)] = i(x^0\partial^1 - x^1\partial^0)T^{00} - 2iT^{00} - i(\tfrac{1}{2}eL)F^{01}\partial_0 Q ,$$

where the Lorentz generator K is defined to be

$$K = x^0 P - \int x^1 T^{00} dx_1$$

and the momentum operator P is given by

$$P = \int dx\, T^{01} dx_1$$

with

$$T^{01} = -\tfrac{1}{2}i\psi\partial_1\psi .$$

This easily yields

$$[K, H] = -iP + i(\tfrac{1}{2}e^2 L)D\partial_0 Q , \tag{2.2}$$

where

$$D = \int x_1 j^0(x)dx_1$$

is the dipole operator. It is to be noted that the anomalous term in (2.2) cannot vanish except in $Q = 0$ sectors because of the equation

$$(\partial_0{}^2 + e^2/\pi)Q = 0 ,$$

and one consequently infers that the model cannot be covariant except in those neutral sectors.

Despite these formal operator results there exist at least three technical points which deserve consideration in view of the existence of Brown's explicit solution of the model. These have to do with the asymptotic behavior of (i) $j^1(x)$, (ii) $T^{01}(x)$, and (iii) $T^{00}(x)$ and $T^{11}(x)$. We consider each of these in turn.

As seen from Eq. (1.1) the combination $j^1(L) - j^1(-L)$ cannot vanish in all sectors if the nonconservation of the global charge operator is valid. Using Brown's Eq. (3.7) (taking out a factor of e to correspond to a trivial current redefinition)

$$\langle 0|(j^0(x)\psi(x_1)\cdots\psi(x_{2n}))_+|0\rangle\epsilon(x_1,\ldots,x_{2n}) = \sum_{a=1}^{2n}(q\tilde{\alpha}\partial)_a\Delta(x - x_a)G(x_1,\ldots,x_{2n}) ,$$

one obtains by straightforward manipulations that

$$\langle 0|[j^1(\pm L, x^0)\psi(x_1)\cdots\psi(x_{2n})]_+|0\rangle\epsilon(x_1,\ldots,x_{2n}) = \pm i\left(\frac{e^2}{\pi}\right)^{1/2}\sum_{a=1}^{2n} q_a \exp\left[-i\left(\frac{e^2}{\pi}\right)^{1/2}(x^0 - x_a^0)\right] \times G(x_1,\ldots,x_{2n}) ,$$

thus confirming the asserted results

$$j^1(L)+j^1(-L)=0\,,$$

$$j^1(L)-j^1(-L)\neq 0\,.$$

The asymptotic behavior of T^{01} is of interest in that the presence of the box itself is expected to break translational invariance. This is not apparent so long as one commutes P with $\psi(x)$ with $|x_1|<L$ but could conceivably affect such things as $[H,P]$. Probably the easiest way to handle this is to write

$$P=\int_{-\infty}^{\infty}T^{01}dx_1-\int_L^{\infty}T^{01}dx_1-\int_{-\infty}^{-L}T^{01}dx_1\,,$$

and to note that translational invariance is restored for $L\to\infty$ provided that $T^{01}(\pm L)$ vanishes sufficiently rapidly. This is verified by merely observing from Brown's solution that the coupling term is irrelevant to the computation of T^{01} and that translational invariance is thus restored for $L\to\infty$ exactly as it would be for a free field in such a box.

Finally one considers T^{00} and T^{11} where the latter is given by

$$T^{11}=-\tfrac{1}{2}i\psi\alpha_1\partial_1\psi-\tfrac{1}{2}F_{01}{}^2\,.$$

Since one can verify by inspection of Brown's solution that the first term in each of the operators goes to zero for large values of its spatial argument, one requires only a calculation of matrix elements of $(F_{01})^2$ to complete the demonstration of consistency between the formal operator approach and the exact solution.

The technique employed is that of Brown with the result that his Eq. (3.7) generalizes to

$$\langle 0|(j^0(x)j^0(x')\psi(x_1)\cdots\psi(x_{2n}))_+|0\rangle\epsilon(x_1,\ldots,x_{2n})=\sum_{a,b=1}^{2n}(q\tilde{\alpha}\partial)_a(q\tilde{\alpha}\partial)_b\Delta(x-x_a)\Delta(x-x_b)G(x_1,\ldots,x_{2n})$$
$$+\langle 0|(j^0(x)j^0(x'))_+|0\rangle G(x_1,\ldots,x_{2n})\,. \tag{2.3}$$

From (2.3) one obtains after subtracting the vacuum expectation value of $(F_{01})^2$ the result

$$\langle 0|(\{[F^{01}(\pm L,x^0)]^2-\langle 0|[F^{01}(\pm L,x^0)]^2|0\rangle\}\psi(x_1)\cdots\psi(x_{2n}))_+|0\rangle\epsilon(x_1,\ldots x_{2n})$$
$$=e^2\sum_{a,b=1}^{2n}q_aq_b\tfrac{1}{16}\epsilon(x^0-x_a^0)\epsilon(x^0-x_b^0)\exp\left[-i\left(\frac{e^2}{\pi}\right)^{1/2}(|x^0-x_a^0|+|x^0-x_b^0|)\right].$$

This coincides with what is obtained from a computation of the matrix element of $\frac{1}{4}Q^2$, i.e., with the asymptotic operator values of $(F_{01})^2$. One thus confirms the expected nonexistence of H as a well-defined operator in the limit $L\to\infty$ for charged sectors. On the other hand, the principal aim of this calculation, namely, the display of the close correspondence between the operator manipulations of Ref. 5 and Brown's solution, has been successful and serves to make credible the subsequent extension to the non-Abelian case. This contrasts with the approach of Bars and Green,[7] whose assumptions concerning the asymptotic behavior of $j^\mu(\pm L)$ are demonstrably incorrect for the U(1) model and certainly unsatisfactory as an assumption in the non-Abelian case.

III. THE NON-ABELIAN MODEL

In carrying out the extension to a non-Abelian gauge field we choose a description of the theory in terms of the Lagrangian[8]

$$\mathcal{L}=\tfrac{1}{2}i\psi\alpha^\mu\partial_\mu\psi-\tfrac{1}{2}m\psi\beta\psi+\tfrac{1}{4}F_a^{\mu\nu}F_{\mu\nu}^a$$
$$-\tfrac{1}{2}F_a^{\mu\nu}(\partial_\mu A_\nu^a-\partial_\nu A_\mu^a+ieA_\mu t^a A_\nu)+ek_a^\mu A_\mu^a\,,$$

where for convenience all fields are taken to be Hermitian and k_a^μ is the current operator

$$k_a^\mu=\tfrac{1}{2}\psi\alpha^\mu T_a\psi\,.$$

As usual the matrices t^a and T^a are respectively the regular representation and any convenient representation of the group. Since we elect to work in the Coulomb gauge,

$$A_a^1=0\,,$$

and there follows in the usual way the constraint equation

$$-\nabla^2A_a^0=ek_a^0\,. \tag{3.1}$$

Thus one has a formulation in which the gauge field is expressed in terms of the spinor field so that there are no true dynamical degrees of freedom associated with the former.

At this point one is confronted with the problem of the inversion of the one-dimensional Laplacian, the unsatisfactory handling of which seems to be the root cause of the failure of some of the earlier attempts at this problem. Solving Eq. (3.1) in the interval $(-L,+L)$ we use the Green's function[5]

$$G(x,x')=-\tfrac{1}{2}|x_1-x_1'|+\frac{L}{2}\left(1-\frac{x_1x_1'}{L^2}\right),$$

appropriate to Dirichlet boundary conditions. This allows one to write[9]

$$A_a^0 = -\tfrac{1}{2}e\int_{-L}^{L} |x_1 - x_1'|k_a^0(x')dx_1' + \tfrac{1}{2}LeQ_a$$
$$-\frac{x_1 e}{2L}D_a + \tfrac{1}{2}(A_{a+}^0 + A_{a-}^0) + \frac{x_1}{2L}(A_{a+}^0 - A_{a-}^0) \qquad (3.2)$$

without loss of generality. In writing (3.2) use has been made of the definitions

$$A_{a\pm}^0 \equiv A_a^0(\pm L)$$

and of the charge operators

$$Q_a \equiv \int_{-L}^{L} k_a^0(x)dx_1$$

and dipole operators

$$D_a \equiv \int_{-L}^{L} x_1 k_a^0(x)dx_1 .$$

As pointed out in the Abelian case in Ref. 5 one must now infer the form of the operators $A_{a\pm}^0$ by requiring that the Hamiltonian H generate the time development of ψ as required by the Lagrange equation

$$\left[\alpha^\mu\left(\frac{1}{i}\partial_\mu - eT_a A_\mu^a\right) + m\beta\right]\psi = 0 .$$

Using the forms

$$T^{00} = -\tfrac{1}{2}i\psi\alpha_1\partial_1\psi + \tfrac{1}{2}m\psi\beta\psi + \tfrac{1}{2}(F_a^{01})^2 ,$$
$$T^{01} = -\tfrac{1}{2}i\psi\partial_1\psi \qquad (3.3)$$

as prescribed by the conserved canonical energy-momentum tensor, one finds that consistency requires

$$A_{a+}^0 + A_{a-}^0 = 0 ,$$
$$A_{a+}^0 - A_{a-}^0 = eD_a ,$$

thereby yielding

$$A_a^0 = -\tfrac{1}{2}e\int_{-L}^{L} |x_1 - x_1'|k_a^0(x')dx_1' + \tfrac{1}{2}LeQ_a$$

and

$$F_a^{01} = \tfrac{1}{2}e\int_{-L}^{L} \epsilon(x_1 - x_1')k_a^0(x')dx_1' .$$

It is of interest at this point to complete the calculation of all the commutators of H and the momentum operator P with ψ, k^μ, and $F^{\mu\nu}$. The validity of

$$[P, \chi] = i\partial_1\chi$$

for all χ is established by inspection provided that

$$\lim_{L\to\infty} k_a^0(\pm L) = 0 ,$$

a result which is reasonable and also expected in the light of Brown's solution of the Schwinger model [as opposed to the corresponding result for $k_a^1(\pm L)$]. Similarly one obtains from $[H, k_a^\mu]$ the result

$$\partial_\mu(k_a^\mu - iA_\nu t_a F^{\mu\nu}) = 0 , \qquad (3.4)$$

as required by the Lagrange equation

$$\partial_\nu F_a^{\mu\nu} = ek_a^\mu - ieA_\nu t_a F^{\mu\nu} . \qquad (3.5)$$

The conservation law (3.4) is now expressible as

$$\partial_\mu j_a^\mu = 0 ,$$

where j_a^μ is the total current operator

$$j_a^\mu = k_a^\mu - iA_\nu t_a F^{\mu\nu} .$$

The commutator $[H, F_a^{01}]$ is somewhat more interesting. Here one obtains

$$-i[H, F_a^{01}] = ej_a^1 - \tfrac{1}{2}e[j_a^1(L) + j_a^1(-L)] . \qquad (3.6)$$

Since, however, (3.5) evaluated for $\mu = 1$ at $x = \pm L$ together with the result

$$F_a^{01}(\pm L) = \pm e\tfrac{1}{2}Q_a$$

implies the necessary vanishing of the symmetrical combination $j_a^1(L) + j_a^1(-L)$, the consistency of (3.6) with the Lagrangian equations is ensured. It is also of interest to note that the condition $j^1(L) + j^1(-L) = 0$ has been shown here to hold for the Abelian case by direct reference to Brown's solution. With this then the consistency question becomes a matter of determining the behavior of the theory under pure Lorentz transformations, the subject to which we now turn our attention.

IV. LORENTZ-TRANSFORMATION PROPERTIES

Using the usual formal expression for K the generator of Lorentz transformations with T^{00} given by (3.4), one derives by straightforward calculation the result

$$[K, \psi] = i(x^0\partial^1 - x^1\partial^0)\psi - \tfrac{1}{2}i\alpha_1\psi + T^a\Lambda_a\psi , \qquad (4.1)$$

where

$$\Lambda_a = \tfrac{1}{2}e\int_{-L}^{L} |x_1 - x_1'|F_a^{01}(x')dx_1' .$$

The fact that ψ undergoes a gauge transformation under the operations of the Lorentz group is not, of course, an unexpected phenomenon. It is familiar from quantum electrodynamics as well as the case of four-dimensional non-Abelian gauge theories. As ψ is the only independent dynamical variable in the model under consideration, the transformation properties of k_a^μ and $F_a^{\mu\nu}$ can readily be determined from (4.1). Thus one obtains

$$[K, k_a^\mu] = i(x^0\partial^1 - x^1\partial^0)k_a^\mu - i\epsilon^{\mu\nu}k_\nu^a + t_{abc}\Lambda_b k_c^\mu$$

while for $F^{\mu\nu}$ there follows

$$[K, F^{\mu\nu}] = i(x^0\partial^1 - x^1\partial^0)F^{\mu\nu} + t_{abc}\Lambda_b F_c^{\mu\nu} + i\tfrac{1}{2}eL[j_a^1(L) - j_a^1(-L)]. \qquad (4.2)$$

As will shortly become apparent, it is only the presence of the last term in (4.2) which stands in the way of proving covariance. Thus despite the obvious convenience of setting $j_a^1(L) - j_a^1(-L)$ equal to zero, the example of the Schwinger model (where that term is definitely not zero in nonsinglet sectors) indicates the prudence of not neglecting it. Furthermore, since

$$\partial_0 Q_a = -[j_a^1(L) - j_a^1(-L)],$$

the physical content of the last term in (4.2) will be most transparent if one writes it as

$$-i\tfrac{1}{2}eL\partial_0 Q_a.$$

The central problem of the verification of the Poincaré algebra can now be undertaken. This task is simplified considerably by the fact that no modification of the canonical energy-momentum tensor has been allowed in the present approach. Thus $T^{\mu\nu}$ satisfies the equations

$$\partial_\mu T^{\mu\nu} = 0$$

so long as the Lagrangian equations of motion are valid. Keeping this fact in mind one computes the commutator of K with T^{00} using (4.1) and (4.2) thereby obtaining

$$\begin{aligned}[K, T^{00}] &= i(x^0\partial^1 - x^1\partial_0)[-\tfrac{1}{2}i\psi\alpha^1\partial_1\psi + \tfrac{1}{2}m\psi\beta\psi + \tfrac{1}{2}(F_a^{01})^2] \\ &\quad -\tfrac{1}{2}\psi\alpha_1\partial^0\psi - \tfrac{1}{2}\psi\partial_1\psi - \tfrac{1}{2}i\psi\alpha_1 T^a\psi\partial_1\Lambda_a \\ &\quad -i\tfrac{1}{2}eLF_a^{01}\partial_0 Q_a \\ &= \tfrac{1}{2}i(x^0\partial^1 - x^1\partial^0)T^{00} - 2iT^{01} - i\tfrac{1}{2}eLF_a^{01}\partial_0 Q_a.\end{aligned} \qquad (4.3)$$

Upon integration over the remaining spatial coordinate Eq. (4.3) becomes

$$\begin{aligned}[K, H] &= -iP + i\tfrac{1}{2}eL(A_{a+}^0 - A_{a-}^0)\partial_0 Q_a \\ &= -iP + i\tfrac{1}{2}e^2 LD_a\partial_0 Q_a.\end{aligned} \qquad (4.4)$$

It is striking that except for the presence of the gauge group index the extra term in (4.4) which breaks Poincaré invariance is identical to that which follows from the last equation of Ref. 5 for the Abelian case.

V. CONCLUSION

In this paper the results obtained earlier for the case of the Schwinger model have been extended to the non-Abelian theories. It has been seen that the results are identical in every way provided that the Poincaré-breaking terms in Ref. 5 are replaced by corresponding expressions involving the total conserved-current operator. This remarkable result is at variance with all previous approaches to the covariance problem in that these other attempts have claimed that special effects are associated with the non-Abelian nature of the field.

There is on the other hand the one disadvantage here relative to the Schwinger model that one does not have available an exact solution. If, however, one were to find nonconservation of charge either in all sectors or in nonsinglet sectors, then the failure of covariance would clearly follow. Results which could establish the existence or nonexistence of conserved global charge operators would thus be of considerable interest in the realm of two-dimensional quantum chromodynamics.[10]

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[9]It may be of interest to note that to this point Ref. 7 is compatible with our development.
[10]It should be noted that a recent effort in this direction by N. K. Pak, Nuovo Cimento 46A, 440 (1978) is not a success since among other things it is at variance with results which have been explicitly confirmed by reference to Brown's solution of the Schwinger model.