Instantaneous Coulomb interaction for a long-range background field in quantum chromodynamics

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The instantaneous Coulomb interaction is studied in the SU(2) Yang-Mills theory. The Coulomb Green's function and instantaneous Coulomb potential of a static quark-antiquark pair are evaluated for a background gauge field $A_a^i(\vec{x})$ that is spherically symmetric and of long range, i.e., that is of order $|\vec{x}|^{-1}$ as $|\vec{x}|\to\infty$. The field $A_a^i(\vec{x})$ is of the same form as the Wu-Yang magnetic-monopole field. Expansion of the Coulomb Green's function in vector spherical harmonic functions reduces the problem to a radial problem. It is shown that the background field changes the asymptotic form of the instantaneous Coulomb interaction; specifically for the monopole field the correction term is of the same magnitude as the ordinary Coulomb interaction at large distances. In addition, the instanton contribution to the $q\bar{q}$ potential energy is calculated in the temporal-gauge formulation of the theory, and compared to the instantaneous Coulomb interaction. This calculation illustrates the interpretation of instantons as tunneling field configurations. The possibility that long-range field fluctuations with the magnetic-monopole form occur in an ionized meron phase of quantum chromodynamics is discussed.

I. INTRODUCTION

This paper is a continuation of the study of the instantaneous Coulomb interaction in the pure Yang-Mills theory with gauge group SU(2) that was begun in Ref. 1 (which will be referred to as I in this paper). In I the Coulomb Green's function and the instantaneous Coulomb potential for a static color-singlet quark-antiquark pair are described for a background gauge field $A_a^i(\vec{x})$ that is spherically symmetric and of short range. In the present paper these quantities will be examined for a background field with the same angular form that is of long range. A gauge field $A_a^i(\vec{x})$ is said to be of long range if it is asymptotically of order $|\vec{x}|^{-1}$ as $|\vec{x}|\to\infty$. This asymptotic behavior is associated with magnetic monopoles; the field to be studied here is similar to the Wu-Yang magnetic-monopole field.[2] There have been suggestions that long-range gauge fields are relevant to the problem of quark confinement in quantum chromodynamics (QCD).[3-5]

The background gauge fields considered in I and in this paper are large gauge fields in the sense that they are of order $1/g$ where g is the coupling constant. Similar large gauge fields occur in calculations of the $q\bar{q}$ static potential that use semiclassical techniques, e.g., of the instanton contribution to the $q\bar{q}$ potential.[6] Therefore in order to compare the instantaneous Coulomb potential to the semiclassical potential, the instanton contribution to the potential will be calculated in this paper as well. The derivation presented here differs from that of Ref. 6 in that it is done in the temporal gauge and it uses a generalization of the Wilson-loop formula[7] that avoids the introduction of a singular string operator to approximate the $q\bar{q}$ state.[8]

The instantaneous Coulomb potential (ICP) for a $q\bar{q}$ pair in an SU(2) color-singlet combination $I_{\text{color}}=0$ in the background field $A_a^i(\vec{x})$ is defined as[1,4,6]

$$u(\vec{r},\vec{r}\,')=-\tfrac{1}{4}g^2\int d^3x\,\partial_i G_{ab}(\vec{x},\vec{r})\partial_i G_{ab}(\vec{x},\vec{r}\,'), \tag{1.1}$$

where $\vec{r}$ and $\vec{r}\,'$ are the positions of the quark and antiquark, and $G_{ab}(\vec{x},\vec{x}\,')$ is the Coulomb Green's function defined by the equation[9]

$$-D_{abi}(\vec{x})\partial_i G_{bc}(\vec{x},\vec{x}\,')=\delta_{ac}\,\delta^3(\vec{x}-\vec{x}\,'), \tag{1.2}$$

where the covariant derivative $D_{abi}(\vec{x})$ is

$$D_{abi}(\vec{x})=\delta_{ab}\partial_i-g\epsilon_{abc}A_{ci}(\vec{x}). \tag{1.3}$$

If the background field $A_a^i(\vec{x})$ is of long range, then the field-dependent term in $D_{abi}(\vec{x})$ and the derivative term both decrease as $|\vec{x}|^{-1}$ as $|\vec{x}|\to\infty$. Therefore a long-range gauge field has a greater effect on the large-distance behavior of $u(\vec{r},\vec{r}\,')$ than a short-range field does. It is shown in I that a short-range background field does not affect the asymptotic behavior of the $q\bar{q}$ potential; it is argued that the effect of short-range fields is equivalent to a charge renormalization.

The background field to be considered is of the form

$$A_a^i(\vec{x})=\frac{2}{g}\epsilon_{aij}\frac{x^j}{x^2}b(x), \tag{1.4}$$

where $x=|\vec{x}|$ and the dimensionless radial function $b(x)$ approaches a nonzero constant b_0 as $x\to\infty$,

$$b(x) \to b_0 \text{ as } x \to \infty ; \tag{1.5}$$

thus $A_a^i(\vec{x})$ is of long range. The value of the constant b_0 will be left unspecified except that it will be assumed to lie in the range $0 \leq b_0 \leq 1$. It will be argued that the value $b_0 = \frac{1}{2}$ is of particular interest. For $b_0 = \frac{1}{2}$ the field $A_a^i(\vec{x})$ is precisely the Wu-Yang monopole field.[2] The field $A_a^i(\vec{x})$ is transverse, $\partial_i A_a^i(\vec{x}) = 0$.

The effect of the long-range gauge field $A_a^i(\vec{x})$ on the ICP of a static $q\bar{q}$ pair has been discussed before.[4,5,10] The aim of the present paper is to give a more complete analysis of this effect than was given before. To be specific, it is shown in I that the spherically symmetric field in Eq. (1.4) allows separation of angular and radial variables in Eq. (1.2) for the Coulomb Green's function $G_{ab}(\vec{x}, \vec{x}')$. Then if the radial function $b(x)$ has a simple form so that the radial problem can be solved, $G_{ab}(\vec{x}, \vec{x}')$ and the ICP can be analyzed in detail.

The ICP $u(\vec{r}, \vec{r}')$ is not of fundamental significance because it is not gauge invariant. Nevertheless, it is possible that a qualitatively correct *model* of the exact gauge-invariant $q\bar{q}$ static potential would be provided by the potential $V(\vec{r} - \vec{r}')$ defined by[1]

$$V(\vec{r} - \vec{r}') = \langle \Omega | u(\vec{r}, \vec{r}') | \Omega \rangle , \tag{1.6}$$

where $|\Omega\rangle$ is the vacuum state of the Coulomb-gauge theory. There are several reasons for studying the potential $V(\vec{r} - \vec{r}')$ as a model of the $q\bar{q}$ static potential. First, it is a generalization of the ordinary Coulomb potential of an Abelian gauge theory. To be specific, there exists a gauge-invariant $q\bar{q}$ state for which $V(\vec{r} - \vec{r}')$ is the potential energy of the quarks, and the state is defined in such a way that if the gauge group is Abelian it is the exact $q\bar{q}$ state. Since the exact state is the gauge-invariant state with lowest energy, $V(\vec{r} - \vec{r}')$ can thus be thought of as an estimate of the $q\bar{q}$ potential in the sense of the variational principle. These remarks will be verified later in this section.

Second, $V(\vec{r} - \vec{r}')$ partially takes into account the nonlinearity of the non-Abelian gauge theory by the dependence of $u(\vec{r}, \vec{r}')$ on the gauge field $A_a^i(\vec{x})$. Thus studying $V(\vec{r} - \vec{r}')$ is at least a start toward understanding the difference between quark interactions in non-Abelian and Abelian theories. Third, in Coulomb-gauge perturbation theory, the instantaneous Coulomb interaction is precisely the term in the Hamiltonian that produces the antiscreening of color charge, which is the origin of asymptotic freedom in QCD.[11] If asymptotic freedom and infrared slavery have the same origin, then $V(\vec{r} - \vec{r}')$ may itself show some sign of quark confinement. Fourth and last, it might be argued that although more complex effects due to the self-couplings of the gauge fields would change the potential, they could only lower the energy and would therefore tend to oppose quark confinement; this is the case, for example, in Coulomb-gauge perturbation theory in which transverse gluon fluctuations partially shield the color charges in the same way that fluctuations of the electron-positron field shield electric charges in quantum electrodynamics.[11] So in summary, even if $V(\vec{r} - \vec{r}')$ ignores some important effects of the gluon self-couplings, it may still show some sign of the essential difference between the $q\bar{q}$ potential in QCD and the ordinary Abelian Coulomb potential.

The meaning of the background field $A_a^i(\vec{x})$ can be seen from Eq. (1.6) for $V(\vec{r} - \vec{r}')$. The ICP $u(\vec{r}, \vec{r}')$ is a functional of the gauge field $A_a^i(\vec{x})$; the potential $V(\vec{r} - \vec{r}')$ is the vacuum expectation value of $u(\vec{r}, \vec{r}')$. The background field $A_a^i(\vec{x})$ can thus be thought of as a vacuum fluctuation of the gauge field operator. This is made more apparent by introduction of the vacuum wave functional $\langle A_a^i | \Omega \rangle$, the probability amplitude for vacuum fluctuations $A_a^i(\vec{x})$.[1,5] The potential $V(\vec{r} - \vec{r}')$ can be written in terms of $\langle A_a^i | \Omega \rangle$ as

$$V(\vec{r} - \vec{r}') = \int dA_a^i(\vec{x}) |\langle A_a^i | \Omega \rangle|^2 u(\vec{r}, \vec{r}') ; \tag{1.7}$$

it is the average of $u(\vec{r}, \vec{r}')$ over all vacuum fluctuations $A_a^i(\vec{x})$ weighted by the probability distribution $|\langle A_a^i | \Omega \rangle|^2$. The vacuum functional $|\langle A_a^i | \Omega \rangle|^2$ is not known, but it can be expressed formally as a path integral[1,12]

$$|\langle A_a^i | \Omega \rangle|^2 = \frac{1}{Z} \int dA'^i_a(x) e^{-S(A')} \times \prod_x \delta(A'^i_a(\vec{x}, 0) - A_a^i(\vec{x})) , \tag{1.8}$$

where the variable x refers to a four-dimensional Euclidean space, $S(A)$ is the Euclidean action of the theory, and integration over Faddeev-Popov ghost field variables is left understood; Z is a normalization factor. To justify the use of a long-range field $A_a^i(\vec{x})$ of the form (1.4) it might be remarked that the contribution of an isolated meron solution[6] to the path integral in Eq. (1.8) creates a vacuum fluctuation of precisely this form where $b(x) \to \frac{1}{2}$ as $x \to \infty$.[12] If there is an ionized meron phase,[6,12] typical vacuum fluctuations would be superpositions of the field in Eq. (1.4). This paper describes an attempt to understand the effect of such vacuum fluctuations on the quark-antiquark potential energy.

The outline of the paper is as follows. In the remainder of Sec. I the potential $V(\vec{r} - \vec{r}')$ will be

derived as a variational estimate of the $q\bar{q}$ static potential in the temporal-gauge version of the theory.[1,5] In Sec. II the Coulomb Green's function $G_{ab}(\vec{x},\vec{x}')$ and ICP $u(\vec{r},\vec{r}')$ for the background field in Eq. (1.4) will be analyzed. Angular and radial variables will be separated as in I by expanding $G_{ab}(\vec{x},\vec{x}')$ in vector spherical harmonic functions. Two angular modes of special significance will be examined in some detail. In Sec. III the instanton contribution to the $q\bar{q}$ potential energy will be derived in a temporal-gauge formulation, and the result compared to the ICP calculated here and in I. In Sec. IV the calculations will be summarized. Finally the Appendix provides a simple general proof of a theorem[4] regarding the distribution of negative eigenvalues of the differential operator $-D_{abi}(\vec{x})\partial_i$.

To verify that the potential $V(\vec{r}-\vec{r}')$ is a variational estimate of the exact $q\bar{q}$ static potential, consider the temporal-gauge version of the pure Yang-Mills theory, i.e., the gauge in which the time components of the gauge fields are required to vanish $A_a^0=0$. The canonical fields are the gauge field A_a^i and the color-electric field E_a^i; the canonical commutation relation is

$$[E_a^i(\vec{x}),\ A_b^j(\vec{y})]=-i\,\delta_{ab}\,\delta_{ij}\,\delta^3(\vec{x}-\vec{y})\ . \tag{1.9}$$

Gauss's law, $D_{abi}E_{bi}=j_a^0$ where j_a^0 is the color charge density of the quarks, is imposed in this gauge as a constraint on the physical states; then physical states are invariant under time-independent local gauge transformations.[5,13] A general gauge-invariant state containing a static quark at $\vec{r}$ and antiquark at $\vec{r}'$ can be written[5]

$$|s\rangle=q^\dagger(\vec{r})K(A)\bar{q}^\dagger(\vec{r}')|\,\Omega\rangle\ , \tag{1.10}$$

where $q(\vec{r})$ and $\bar{q}(\vec{r}')$ are nonrelativistic quark and antiquark field operators and $|\,\Omega\rangle$ is the vacuum state of the pure gauge theory (i.e., without quarks). The operator $K(A)$, which is a functional of the gauge fields $A_a^i(\vec{x})$ and a matrix that acts in the color space of the static quarks, is chosen such that the operator $q^\dagger(\vec{r})K(A)\,\bar{q}^\dagger(\vec{r}')$ is gauge invariant. Under a gauge transformation $g(\vec{x})$ the gauge field transforms as

$$A_i^g(\vec{x})=g(\vec{x})[A_i(\vec{x})+\partial_i]\,g^\dagger(\vec{x})\,, \tag{1.11}$$

where $A_i(\vec{x})=-ig\frac{1}{2}\sigma_a A_{ai}(\vec{x})$. The function $K(A)$ must transform according to[5]

$$K(A')=g(\vec{r})K(A)\,g^\dagger(\vec{r}') \tag{1.12}$$

to ensure that $q^\dagger K\bar{q}^\dagger$ is gauge invariant. The physical $q\bar{q}$ state is the state $|s\rangle$ with minimum energy; the energy of that state is the exact $q\bar{q}$ static potential. Therefore the problem is to find the functional $K(A)$ that minimizes the energy and obeys the condition (1.12). Also the operator $K(A)$ is taken to be unitary, $K^\dagger(A)K(A)=1$.

When the operator $K(A)$ acts on the vacuum state $|\Omega\rangle$ it creates a coherent state of the color-electric field $E_a^i(\vec{x})$. Since $K(A)$ is unitary $K(A)$ shifts E_a^i by

$$K^\dagger(A)E_a^i(\vec{x})K(A)=\bar{E}_a^i(\vec{x})+E_a^i(\vec{x})\ , \tag{1.13}$$

where

$$\bar{E}_a^i(\vec{x})=K^\dagger(A)[E_a^i(\vec{x}),\ K(A)]$$

$$=K^\dagger(A)\,\frac{\delta K(A)}{i\delta A_a^i(\vec{x})}\ ; \tag{1.14}$$

the second line follows from the commutation relation (1.9). The mean field $\bar{E}_a^i(\vec{x})$ is a functional of A_a^i and a matrix operator in the quark color space. The energy of the state $|\,s\rangle$ can be shown to be[5]

$$H=\tfrac{1}{2}\int d^3x\,\langle s|\,E_{ai}{}^2+B_{ai}{}^2\,|\,s\rangle/\langle s|\,s\rangle$$

$$=\tfrac{1}{4}\int d^3x\,\mathrm{tr}\langle\Omega\,|\bar{E}_a^i(\vec{x})\bar{E}_a^i(\vec{x})|\Omega\rangle\,, \tag{1.15}$$

where tr means trace over color indices.

There are presumably many functionals $K(A)$ that satisfy Eq. (1.12). One of them is the string operator $K_\Gamma(A)$ defined by

$$K_\Gamma(A)=P\exp\Big(-\int_\Gamma d\vec{s}\cdot\vec{A}\Big), \tag{1.16}$$

where Γ is a path in $\vec{x}$ space from $\vec{r}'$ to $\vec{r}$, and P indicates path ordering. The mean field $\bar{E}_a^i(\vec{x})$ associated with $K_\Gamma(A)$ is nonzero only along Γ. To be precise, let $\vec{x}(s)$ be a parametrization of Γ, with $0\leq s\leq 1$; then $\bar{E}_a^i(\vec{x})$ can be written

$$\bar{E}_a^i(\vec{x})=g\int_0^1 ds\,\frac{dx^i}{ds}\,\delta^3(\vec{x}-\vec{x}(s))$$
$$\times K^\dagger_{\Gamma(s)}(A)\tfrac{1}{2}\sigma_a K_{\Gamma(s)}(A), \tag{1.17}$$

where $\Gamma(s)$ is the portion of the curve Γ from $\vec{r}'$ to $\vec{x}(s)$. Because $\bar{E}_a^i(\vec{x})$ is singular the energy of the state constructed with $K_\Gamma(A)$ is divergent.

The choice of $K(A)$ for which the energy is $V(\vec{r}-\vec{r}')$ derives from the process of Coulomb gauge fixing. Let $U(\vec{x};A)$ be the gauge function that transforms $A_i(\vec{x})$ to the Coulomb gauge[14]; that is, the transverse field $\tilde{A}_i(\vec{x})$ gauge equivalent to $A_i(\vec{x})$ is

$$\tilde{A}_i(\vec{x})=U^\dagger(\vec{x};A)[A_i(\vec{x})+\partial_i]\,U(\vec{x};A)\ ,$$
$$\partial_i\tilde{A}_i(\vec{x})=0\ . \tag{1.18}$$

The gauge function $U(\vec{x};A)$ transforms under the

gauge transformation of $A_i(\vec{x})$ in Eq. (1.11) according to

$$U(\vec{x};A') = g(\vec{x})U(\vec{x};A) . \tag{1.19}$$

Therefore a functional $K(A)$ that obeys the condition (1.12) is

$$K(A) = U(\vec{r};A)U^\dagger(\vec{r}';A) . \tag{1.20}$$

The mean field $\bar{E}^i_a(\vec{x})$ corresponding to this choice of $K(A)$ can be shown to be

$$\bar{E}^i_a(\vec{x}) = g V_{ab}(\vec{x})[\partial_i G_{bc}(\vec{x},\vec{r}) - \partial_i G_{bc}(\vec{x},\vec{r}')] \times U(\vec{r}')\tfrac{1}{2}\sigma_c U^\dagger(\vec{r}') , \tag{1.21}$$

where $G_{bc}(\vec{x},\vec{x}')$ is the Coulomb Green's function defined in Eq. (1.4) and $V_{ab}(\vec{x})$ is

$$V_{ab}(\vec{x}) = \tfrac{1}{2}\,\mathrm{Tr}\sigma_a U(\vec{x})\sigma_b U^\dagger(\vec{x}) . \tag{1.22}$$

The background field that defines $G_{ab}(\vec{x},\vec{x}')$ is $\tilde{A}_i(\vec{x})$ [Eq. (1.18)]. The energy H of the state $|s\rangle$ defined by $K(A)$ is

$$H = \tfrac{1}{8}g^2 \int d^3x \langle\Omega|\,[\,\partial_i G_{ab}(\vec{x},\vec{r}) - \partial_i G_{ab}(\vec{x},\vec{r}')\,]^2|\,\Omega\rangle$$

$$= \langle\Omega|\,-\tfrac{1}{2}u(\vec{r},\vec{r}) - \tfrac{1}{2}u(\vec{r}',\vec{r}') + u(\vec{r},\vec{r}')\,|\,\Omega\rangle , \tag{1.23}$$

where $u(\vec{r},\vec{r}')$ is the ICP defined in Eq. (1.1). The first two terms in Eq. (1.23) are quark self-energy terms[15]; the final term is precisely the potential $V(\vec{r}-\vec{r}')$ of Eq. (1.6).

Furthermore, the trial state $|s\rangle$ defined by this choice of $K(A)$ is the *exact* $q\bar{q}$ state if the theory is an Abelian gauge theory, for in that case the operator $K(A)$ creates precisely the Coulomb field of the static charges at $\vec{r}$ and $\vec{r}'$. To be specific, in an Abelian theory with gauge group U(1) the gauge function $U(\vec{x};A)$ is

$$U(\vec{x};A) = e^{i\omega(\vec{x})} , \tag{1.24}$$

where $(1/g)\partial_i\omega(\vec{x})$ is the longitudinal part of $A_i(\vec{x})$:

$$A_i(\vec{x}) = \tilde{A}_i(\vec{x}) + \frac{1}{g}\partial_i\omega(\vec{x}) . \tag{1.25}$$

Then the functional $K(A)$ is simply

$$K(A) = \exp\left(-i\int d^3x\,\vec{A}\cdot\nabla\varphi\right) , \tag{1.26}$$

where $\varphi(\vec{x})$ is the ordinary scalar potential

$$\varphi(\vec{x}) = \frac{g}{4\pi|\vec{x}-\vec{r}|} - \frac{g}{4\pi|\vec{x}-\vec{r}'|} , \tag{1.27}$$

and the mean field $\bar{E}_i(\vec{x})$ is the Coulomb field

$$\bar{E}_i(\vec{x}) = -\partial_i\varphi(\vec{x}) . \tag{1.28}$$

Thus the potential $V(\vec{r}-\vec{r}')$ of Eq. (1.6) is an estimate of the $q\bar{q}$ static potential in the sense of the variational principle, for a trial state that is exact for an Abelian theory. These remarks are intended as a justification for studying this potential.

Finally it might be pointed out that the formula (1.1) for the ICP $u(\vec{r},\vec{r}')$ has a simple interpretation in the language of electrostatics. Integration by parts in Eq. (1.1) plus use of the definition (1.2) of $G_{ab}(\vec{x},\vec{r}')$ leads to the expression for $u(\vec{r},\vec{r}')$,

$$u(\vec{r},\vec{r}') = -\tfrac{1}{4}g^2\int d^3x\, G_{ab}(\vec{x},\vec{r})\rho_{ab}(\vec{x},\vec{r}') , \tag{1.29}$$

where

$$\rho_{ab}(\vec{x},\vec{r}') = \delta_{ab}\,\delta^3(\vec{x}-\vec{r}') - g\epsilon_{acd}A^i_d(\vec{x})\partial_i G_{cb}(\vec{x},\vec{r}') . \tag{1.30}$$

The Green's function $G_{ab}(\vec{x},\vec{r})$ can be interpreted as the scalar potential at the point $\vec{x}$ due to a static quark at $\vec{r}$, and $\rho_{ab}(\vec{x},\vec{r}')$ as the charge density at $\vec{x}$ due to the static antiquark at $\vec{r}'$. Then the potential energy $u(\vec{r},\vec{r}')$ is the integral of the product of the scalar potential and the charge density, as in electrostatics. The charge density $\rho_{ab}(\vec{x},\vec{r}')$ is the sum of two terms. The term $\delta_{ab}\,\delta^3(\vec{x}-\vec{r}')$ is the charge density of the antiquark itself and the term $-g\epsilon_{acd}A^i_d(\vec{x})\partial_i G_{cb}(\vec{x},\vec{r}')$ is the charge density of the gluon fields that is induced by the antiquark, since $-\partial_i G_{cb}(\vec{x},\vec{r}')$ can be interpreted as the electric field at $\vec{x}$ due to the antiquark. This *induced* color charge density, which owes its existence to the presence of vacuum fluctuations of $A^i_a(\vec{x})$ and the color-electric field $-\partial_i G_{ab}(\vec{x},\vec{r}')$ of the antiquark, has the effect of changing the point charge of the antiquark into a charge distribution of nonzero spatial extent. The presence of this induced charge density is the physical origin of the asymptotic freedom of QCD; it antiscreens the point charge.

II. THE INSTANTANEOUS COULOMB POTENTIAL FOR A LONG-RANGE BACKGROUND FIELD

In this section the Coulomb Green's function $G_{ab}(\vec{x},\vec{x}')$ and instantaneous Coulomb potential (ICP) $u(\vec{r},\vec{r}')$ will be examined for the background field $A^i_a(\vec{x})$ of the form (1.4), (1.5). The most interesting question concerns the effect of $A^i_a(\vec{x})$ on the large-distance behavior of these quantities.

It is shown in I that Eq. (1.2) for $G_{ab}(\vec{x},\vec{x}')$ can be reduced to a radial equation if $G_{ab}(\vec{x},\vec{x}')$ is expanded in vector spherical harmonic (VSH) functions.[1,16] As in I the VSH functions will be denoted

$\vec{D}^{\sigma}_{nm}(\theta,\varphi)$ where the indices σ, n, m take the values

$$\begin{aligned}&\sigma=-1,0,1,\\&n=0,1,2,\ldots \text{ if } \sigma=+1,\\&n=1,2,3,\ldots \text{ if } \sigma=0,-1,\\&m=0,\pm1,\pm2,\ldots,\pm(n+\sigma).\end{aligned}\tag{2.1}$$

The index n is an orbital angular momentum quantum number, and $(n+\sigma)$ and m are total angular momentum quantum numbers. The VSH functions $\vec{D}^{\sigma}_{nm}$ are defined by[1]

$$\begin{aligned}\vec{D}^{1}_{nm}(\theta,\varphi)&=x\nabla Y^{m}_{n+1}(\theta,\varphi)+(n+1)\hat{r}Y^{m}_{n+1}(\theta,\varphi),\\\vec{D}^{0}_{nm}(\theta,\varphi)&=\nabla\times[\vec{x}Y^{m}_{n}(\theta,\varphi)],\\\vec{D}^{-1}_{nm}(\theta,\varphi)&=x\nabla Y^{m}_{n-1}(\theta,\varphi)-n\hat{r}Y^{m}_{n-1}(\theta,\varphi),\end{aligned}\tag{2.2}$$

where Y^{m}_{n} are the scalar spherical harmonics and (x,θ,φ) are the polar coordinates of $\vec{x}$.

The Coulomb Green's function may be expanded in VSH functions as

$$G_{ab}(\vec{x},\vec{x}')=\sum_{\sigma,n,m}K^{\sigma}_{nm}D^{\sigma}_{nma}(\theta,\varphi)g^{\sigma}_{n}(x,x')\times[D^{\sigma}_{nmb}(\theta',\varphi')]^{*},\tag{2.3}$$

where (x,θ,φ) are the polar coordinates of $\vec{x}$, and the normalization constant K^{σ}_{nm} is

$$K^{\sigma}_{nm}=\frac{1}{4\pi}\frac{(n+\sigma-m)!}{(n+\sigma+m)!}\frac{1}{n(n+1)}k^{\sigma}_{n},\qquad k^{1}_{n}=n,\quad k^{0}_{n}=2n+1,\quad k^{-1}_{n}=n+1.\tag{2.4}$$

It is shown in I that the radial Green's function $g^{\sigma}_{n}(x,x')$ obeys

$$\frac{1}{x^2}\delta(x-x')=\left[-\frac{d^2}{dx^2}-\frac{2}{x}\frac{d}{dx}+\frac{n(n+1)}{x^2}-\frac{2b(x)}{x^2}f^{\sigma}_{n}\right]g^{\sigma}_{n}(x,x'),\tag{2.5}$$

where the constant f^{σ}_{n} is

$$f^{1}_{n}=-n,\quad f^{0}_{n}=1,\quad f^{-1}_{n}=n+1,\tag{2.6}$$

and $b(x)$ is the radial function that determines $A^{i}_{a}(\vec{x})$. Also the ICP $u(\vec{r},\vec{r}')$ is shown to be a sum over contributions from modes with quantum numbers (n,σ):

$$u(\vec{r},\vec{r}')=-\tfrac{1}{4}g^2\sum_{n,\sigma}\frac{1}{4\pi}[2(n+\sigma)+1]\times P_{n}(\hat{r}\cdot\hat{r}')X^{\sigma}_{n}(r,r'),\tag{2.7}$$

where P_n is the Legendre polynomial of degree n and the radial function $X^{\sigma}_{n}(r,r')$ is

$$X^{\sigma}_{n}(r,r')=g^{\sigma}_{n}(r,r')+\int_{0}^{\infty}dx\,g^{\sigma}_{n}(x,r)2b(x)f^{\sigma}_{n}g^{\sigma}_{n}(x,r').\tag{2.8}$$

The two terms in this formula for $u(\vec{r},\vec{r}')$ can be identified with the contributions from the two terms in the charge density $\rho_{ab}(\vec{x},\vec{r}')$ in the electrostatic analogy [Eqs. (1.29), (1.30)].

In I the functions $g^{\sigma}_{n}(x,x')$ and $X^{\sigma}_{n}(r,r')$ are calculated for a short-range field for which $b(x)$ is zero for x greater than a scale parameter ρ. For such a field the radial Green's function $g^{\sigma}_{n}(x,x')$ is asymptotically proportional to x^{-n-1} as $x\to\infty$. The resulting ICP $u(\vec{r},\vec{r}')$ is of the form

$$u(\vec{r},\vec{r}')=-\frac{3g^2}{16\pi}\frac{1}{|\vec{r}-\vec{r}'|}+\Delta u(\vec{r},\vec{r}');\tag{2.9}$$

the first term in this formula is the ordinary Coulomb potential, and the correction term $\Delta u(\vec{r},\vec{r}')$ contains the dependence on the background field. The potential $\Delta u(\vec{r},\vec{r}')$ decreases to zero as $|\vec{r}|^{-2}$ as $|\vec{r}|\to\infty$ with $\vec{r}'$ fixed and is symmetric with respect to interchange of $\vec{r}$ and $\vec{r}'$. For r and r' much larger than the scale ρ, $\Delta u(\vec{r},\vec{r}')$ is approximately

$$\Delta u(\vec{r},\vec{r}')\sim-\frac{g^2}{16\pi}M\rho^3\frac{\hat{r}\cdot\hat{r}'}{r^2(r')^2},\tag{2.10}$$

where M is a dimensionless constant that depends on the magnitude of the background field. At sufficiently large distances $\Delta u(\vec{r},\vec{r}')$ is negligible compared to $|\vec{r}-\vec{r}'|^{-1}$. Translation invariance is restored in I in a heuristic way; the translationally invariant potential $v(\vec{r}-\vec{r}')$ considered there is

$$v(\vec{r}-\vec{r}')=-\frac{3g^2}{16\pi}\frac{1}{|\vec{r}-\vec{r}'|}+\xi\int d^3c\,\Delta u(\vec{r}-\vec{c},\ \vec{r}'-\vec{c}),\tag{2.11}$$

where ξ is a parameter with units (length)$^{-3}$ that can be interpreted as the density of field fluctuations of the given form in the vacuum.[1] The asymptotic behavior (2.10) of $\Delta u(\vec{r},\vec{r}')$ implies that the correction term in Eq. (2.11) is proportional to $|\vec{r}-\vec{r}'|^{-1}$ at large distances and thus

$$v(\vec{r}-\vec{r}')\sim -\frac{3g^2}{16\pi}\frac{1}{|\vec{r}-\vec{r}'|}(1+\tfrac{4}{3}\pi\rho^3\xi M)\ . \quad (2.12)$$

Thus the effect of short-range vacuum fluctuations of the gauge field on the large-distance behavior of $V(\vec{r}-\vec{r}')$ is equivalent to a charge renormalization. The constant M is positive for the background field considered in I.

In this paper the ICP $u(\vec{r},\vec{r}')$ will be studied for the long-range field in Eq. (1.4). Only two of the angular modes will be examined in detail, namely those with quantum numbers $(n,\sigma)=(0,1)$ and with $(n,\sigma)=(1,-1)$. These modes are the most interesting ones for reasons that will be explained. The analysis of other modes would be a straightforward generalization.

First consider the case $(n,\sigma)=(0,1)$. The VSH functions $\vec{D}^1_{0m}$ are

$$\vec{D}^1_{00}=\hat{e}_3\ ,\qquad \vec{D}^1_{0,\pm1}=\hat{e}_1\pm i\hat{e}_2\ . \quad (2.13)$$

By Eq. (2.6) the constant f^1_0 is equal to zero so the equation for the radial Green's function $g^1_0(x,x')$ is independent of $b(x)$. These modes are not affected by $A^i_a(\vec{x})$. Thus the radial Green's function is just

$$g^1_0(x,x')=\frac{1}{x}\,\theta(x-x')+\frac{1}{x'}\,\theta(x'-x)\ , \quad (2.14)$$

and the contribution to $G_{ab}(\vec{x},\vec{x}')$ is

$$G^{(0,1)}_{ab}(\vec{x},\vec{x}')=\frac{1}{4\pi}\,\delta_{ab}\left[\frac{1}{x}\,\theta(x-x')+\frac{1}{x'}\,\theta(x'-x)\right]. \quad (2.15)$$

Finally the contribution to the ICP $u(\vec{r},\vec{r}')$ from modes with $(n,\sigma)=(0,1)$ is

$$u^{(0,1)}(\vec{r},\vec{r}')=-\frac{3g^2}{16\pi}\left[\theta(r-r')\frac{1}{r}+\theta(r'-r)\frac{1}{r'}\right]. \quad (2.16)$$

The importance of the mode $(n,\sigma)=(0,1)$ is that its contribution to $G_{ab}(\vec{x},\vec{x}')$ and $u(\vec{r},\vec{r}')$ is the one that decreases most slowly at large distances, as $|\vec{x}|^{-1}$ and $|\vec{r}|^{-1}$, respectively. Then when $r\to\infty$ the ICP $u(\vec{r},\vec{r}')$ approaches $u^{(0,1)}(\vec{r},\vec{r}')$, which is just the ordinary Coulomb potential for large r. The contribution from this mode responsible for the asymptotic Coulomb form of $G_{ab}(\vec{x},\vec{x}')$ and $u(\vec{r},\vec{r}')$ is unchanged by the presence of the background field $A^i_a(\vec{x})$.

Next consider the modes with $(n,\sigma)=(1,-1)$. The VSH function $\vec{D}^{-1}_{10}$ is

$$\vec{D}^{-1}_{10}=\hat{r}\ . \quad (2.17)$$

This particular angular mode has been discussed before[4,5] for the background field $A^i_a(\vec{x})$. Here a simple definite choice of the radial function $b(x)$ will be made and the corresponding radial Green's function $g^{-1}_1(x,x')$ calculated explicitly. This allows a more complete discussion of the nature of the contribution from this mode.

The equation for the radial Green's function $g^{-1}_1(x,x')$ is

$$\frac{1}{x^2}\,\delta(x-x')=\left[-\frac{d^2}{dx^2}-\frac{2}{x}\,\frac{d}{dx}+\frac{2}{x^2}-\frac{4b(x)}{x^2}\right]\times g^{-1}_1(x,x') \quad (2.18)$$

where $b(x)\to b_0$ as $x\to\infty$. In order to illustrate the behavior of $g^{-1}_1(x,x')$, let $b(x)$ be simply

$$b(x)=\begin{cases} b_0 & \text{if } x\geq\rho \\ b_0\dfrac{x^2}{\rho^2} & \text{if } x\leq\rho\ , \end{cases} \quad (2.19)$$

where the scale ρ is arbitrary. Since $b(x)\sim x^2$ as $x\to 0$, the field $A^i_a(\vec{x})$ is nonsingular at $\vec{x}=0$. The discontinuity of $b'(x)$ at $x=\rho$ is unphysical but does not introduce any divergences in the quantities to be discussed. Large-distance effects should be relatively insensitive to the precise behavior of $b(x)$ for $x\lesssim\rho$.

The equation for $g^{-1}_1(x,x')$ can be solved by introducing two zero-mode functions[1] $\varphi_\pm(x)$ that satisfy

$$0=\left[-\frac{d^2}{dx^2}-\frac{2}{x}\,\frac{d}{dx}+\frac{2}{x^2}-\frac{4b(x)}{x^2}\right]\varphi_\pm(x)\ ; \quad (2.20)$$

and in addition the functions $\varphi_+(x)$ ($\varphi_-(x)$) are specified to be regular at $x=\infty$ ($x=0$). The radial Green's function is given in terms of $\varphi_\pm(x)$ by

$$g^{-1}_1(x,x')=\kappa[\theta(x-x')\varphi_+(x)\varphi_-(x')+\theta(x'-x)\varphi_-(x)\varphi_+(x')], \quad (2.21)$$

where the constant κ is determined by the Wronskian

$$\frac{1}{\kappa}=x^2W(\varphi_+,\varphi_-)=x^2(\varphi_+\varphi_-'-\varphi_-\varphi_+')\ . \quad (2.22)$$

The solution of Eq. (2.20) for $\varphi_\pm(x)$ is straightforward if $b(x)$ is of the form (2.19). The interior solutions ($x\leq\rho$) are

$$\varphi_+(x)=c_1j_1(qx)+c_2y_1(qx),\qquad \varphi_-(x)=cj_1(qx) \quad (x\leq\rho), \quad (2.23)$$

where

$$q^2=4b_0/\rho^2\ ; \quad (2.24)$$

j_1 and y_1 are spherical Bessel functions[17] and c_1, c_2, and c are constants. The exterior solutions ($x \geq \rho$) are

$$\varphi_+(x) = \left(\frac{\rho}{x}\right)^{\nu+1}$$
$$\varphi_-(x) = \alpha\left(\frac{x}{\rho}\right)^{\nu} + (1-\alpha)\left(\frac{\rho}{x}\right)^{\nu+1} \qquad (x \geq \rho)\,, \tag{2.25}$$

where

$$\nu = -\tfrac{1}{2} + (\tfrac{9}{4} - 4b_0)^{1/2}\,, \tag{2.26}$$

and α is a constant. Note that $\varphi_+(\varphi_-)$ is regular at $x=\infty$ ($x=0$). The exterior solution in Eq. (2.25) is valid for $b_0 \leq \frac{9}{16}$. If $b_0 > \frac{9}{16}$ then ν is complex and the expressions in Eq. (2.25) are no longer real. For the moment it will be assumed that $b_0 \leq \frac{9}{16}$. This restriction does not rule out the most interesting case $b_0 = \frac{1}{2}$ in which $A_a^i(\vec{x})$ is the Wu-Yang monopole field.

The parameter α that determines the exterior part of $\varphi_-(x)$ is found by matching interior and exterior solutions of $\varphi_-(x)$ and $\varphi_-'(x)$ at $x=\rho$; the result is

$$\alpha = \frac{1}{2\nu+1}\left[\nu - 1 + \frac{z\, j_0(z)}{j_1(z)}\right], \tag{2.27}$$

where $z = q\rho = 2b_0^{1/2}$. The Wronskian κ is easily computed to be

$$\frac{1}{\kappa} = (2\nu+1)\alpha\rho\,. \tag{2.28}$$

The constants c_1 and c_2 will not be needed in what follows.

The asymptotic form of $\varphi_+(x)$ at large x (i.e., $x \geq \rho$) depends on the value of b_0. In contrast, for a short-range field[1] $\varphi_+(x) \sim x^{-2}$ as $x \to \infty$, independent of the field. This change in the asymptotic form of $\varphi_+(x)$, and thus of the radial Green's function $g_1^{-1}(x,x')$, is the significant difference between the cases of a long-range background field and a short-range field. The origin of the difference is that in the covariant derivative $D_{abi}(\vec{x})$ [Eq. (1.3)] the derivative term $\delta_{ab}\partial_i$ and the field-dependent term $-g\epsilon_{abc}A_{ci}(\vec{x})$ are both of order $|\vec{x}|^{-1}$ as $|\vec{x}| \to \infty$ if $A_a^i(\vec{x})$ is of long range.

The radial Green's function $g_1^{-1}(x,x')$ is given in terms of $\varphi_\pm(x)$ in Eq. (2.21). The interesting question is the nature of the large-distance behavior of $g_1^{-1}(x,x')$. To be definite consider the value of $g_1^{-1}(x,x')$ for $x > x' > \rho$:

$$g_1^{-1}(x,x') = \kappa\left(\frac{\rho}{x}\right)^{\nu+1}\left[\alpha\left(\frac{x'}{\rho}\right)^{\nu} + (1-\alpha)\left(\frac{\rho}{x'}\right)^{\nu+1}\right]. \tag{2.29}$$

Then $g_1^{-1}(x,x')$ is proportional to $x^{-\nu-1}$. For $b_0 = 0$, $\nu = 1$ so $g_1^{-1}(x,x')$ is proportional to x^{-2} as $x \to \infty$. For $b_0 > 0$, $\nu < 1$ so $g_1^{-1}(x,x')$ decreases more slowly than x^{-2} as $x \to \infty$. In particular for $b_0 = \frac{1}{2}$ the exponent ν vanishes and $g_1^{-1}(x,x')$ decreases only as x^{-1} as $x \to \infty$; this is the same asymptotic decrease as that of the radial Green's function of the long-range mode $(n,\sigma) = (0,1)$. Finally for $b_0 > \frac{1}{2}$, $\nu < 0$, and $g_1^{-1}(x,x')$ decreases even more slowly than $g_0^1(x,x')$.

If x and x' are both much greater than ρ, the second term in Eq. (2.29) can be neglected. Thus at large distances $x, x' \gg \rho$ the contribution of the mode $(n,\sigma) = (1,-1)$ to the Coulomb Green's function $G_{ab}(\vec{x},\vec{x}')$ is

$$G_{ab}^{(1,-1)}(\vec{x},\vec{x}') \simeq \frac{1}{4\pi}\hat{x}_a\hat{x}_b'\,\frac{1}{2\nu+1} \times\left[\theta(x-x')\,\frac{(x')^{\nu}}{x^{\nu+1}} + \theta(x'-x)\,\frac{x^{\nu}}{(x')^{\nu+1}}\right]. \tag{2.30}$$

When $b_0 = 0$, $\nu = 1$, and $G_{ab}^{(1,-1)}(\vec{x},\vec{x}')$ is asymptotically (i.e., as $|\vec{x}| \to \infty$) negligible compared to $G_{ab}^{(0,1)}(\vec{x},\vec{x}')$. But for $b_0 = \frac{1}{2}$, $\nu = 0$, and $G_{ab}^{(1,-1)}(\vec{x},\vec{x}')$ is comparable to $G_{ab}^{(0,1)}(\vec{x},\vec{x}')$. The asymptotic behavior is important because $G_{ab}(\vec{x},\vec{x}')$ is the analog of the electrostatic scalar potential at the point $\vec{x}$ due to a charge at $\vec{x}'$. A slow decrease of $G_{ab}(\vec{x},\vec{x}')$ at large distances implies that the influence of the charge at $\vec{x}'$ is large at large distances. The mode $(n,\sigma) = (1,-1)$ is analogous to the dipole contribution to the scalar potential, and thus is expected to decrease as $|\vec{x}|^{-2}$; but the background gauge field changes the asymptotic behavior to $|\vec{x}|^{-\nu-1}$ which decreases more slowly.

Finally the contribution of the angular modes with $(n,\sigma) = (1,-1)$ to the ICP $u(\vec{r},\vec{r}')$ is, by Eqs. (2.7), (2.8),

$$u^{(1,-1)}(\vec{r},\vec{r}') = -\frac{g^2}{16\pi}\hat{r}\cdot\hat{r}'X_1^{-1}(r,r')\,, \tag{2.31}$$

where

$$X_1^{-1}(r,r') = g_1^{-1}(r,r') + \int_0^{\infty} dx\, g_1^{-1}(x,r)4b(x)g_1^{-1}(x,r')\,. \tag{2.32}$$

Again the interesting question is the nature of $u^{(1,-1)}(\vec{r},\vec{r}')$ for large distances. Consider the case $r > r' \gg \rho$. It can be shown that the radial function $X_1^{-1}(r,r')$ is approximately

$$X_1^{-1}(r,r') \simeq \frac{1}{2\nu+1}\,\frac{(r')^{\nu}}{r^{\nu+1}}\left[1 + \frac{8b_0}{(2\nu+1)^2} + \frac{4b_0}{2\nu+1}\ln\frac{r}{r'}\right]; \tag{2.33}$$

this result can be derived directly from the exact solution for $g_1^{-1}(x,x')$ or more simply from the approximate form in Eq. (2.30). Again the exponent ν determines the large-distance behavior of $u^{(1,-1)}(\vec{r},\vec{r}')$. For $b_0=0$ the radial function $X_1^{-1}(r,r')$ has the free form $X_1^{-1}(r,r')\simeq\frac{1}{3}r'/r^2$. But for $b_0>0$, $X_1^{-1}(r,r')$ decreases more slowly than r^{-2} as $r\to\infty$. The logarithmic factor $\ln(r/r')$ comes from the interaction of the quark at $\vec{r}$ with the gluon charge density induced by the presence of the antiquark at $\vec{r}'$ [see Eq. (1.30)].

For the magnetic-monopole field $b_0=\frac{1}{2}$ the exponent ν vanishes, so asymptotically (with $r>r'$)

$$u^{(1,-1)}(\vec{r},\vec{r}')\simeq-\frac{g^2}{16\pi}\hat{r}\cdot\hat{r}'\frac{1}{r}\left[5+2\ln\frac{r}{r'}\right]. \tag{2.34}$$

In this case the magnitude of $u^{(1,-1)}(\vec{r},\vec{r}')$ is comparable at large distances to $u^{(0,1)}(\vec{r},\vec{r}')$. However, the directions of the forces implied by these potentials are quite different because of the angular factor $\hat{r}\cdot\hat{r}'$ in $u^{(1,-1)}(\vec{r},\vec{r}')$. The force associated with the potential $u^{(0,1)}(\vec{r},\vec{r}')$ is, for large r

$$\vec{F}^{(0,1)}=-\nabla u^{(0,1)}(\vec{r},\vec{r}')=-\frac{3g^2}{16\pi}\frac{1}{r^2}\hat{r}\,; \tag{2.35}$$

the quark is attracted toward the origin. But the force associated with $u^{(1,-1)}(\vec{r},\vec{r}')$ is, for large r

$$\vec{F}^{(1,-1)}=-\frac{g^2}{16\pi}\frac{1}{r^2}\left[\hat{r}(\hat{r}\cdot\hat{r}')\left(8+4\ln\frac{r}{r'}\right)-\hat{r}'\left(5+2\ln\frac{r}{r'}\right)\right] \tag{2.36}$$

which has a complicated angular dependence. For $\hat{r}\cdot\hat{r}'=+1$, i.e., both quark and antiquark on the same side of the monopole, $\vec{F}^{(1,-1)}$ is in the direction $-\hat{r}$. For $\hat{r}\cdot\hat{r}'=0$, $\vec{F}^{(1,-1)}$ is in the direction of $+\hat{r}'$, i.e., perpendicular to the direction toward the origin; and for $\hat{r}\cdot\hat{r}'=-1$, i.e., quark and antiquark on opposite sides of the monopole, $\vec{F}^{(1,-1)}$ actually points in the direction $+\hat{r}$, away from the origin. This angular dependence is perfectly sensible when $b_0=0$: The term $\vec{F}^{(1,-1)}$ is a small dipole correction to $\vec{F}^{(0,1)}$ that makes the force point in the direction of $-(\vec{r}-\vec{r}')$ rather than $-\vec{r}$. But when $b_0=\frac{1}{2}$, the dipole correction $\vec{F}^{(1,-1)}$ is of the same magnitude as the radial force $\vec{F}^{(0,1)}$. Thus the force for the background field $A_a^i(\vec{x})$ is very different from the ordinary Coulomb force.

The ICP $u(\vec{r},\vec{r}')$ is not translation invariant because the background field $A_a^i(\vec{x})$ is centered at the origin. Of course the potential $V(\vec{r}-\vec{r}')$ of Eq. (1.6) is translation invariant because it is the average of $u(\vec{r},\vec{r}')$ over all possible vacuum fluctuations. To restore translation invariance as was done in I [see Eq. (2.11)] it would be necessary to calculate $u(\vec{r},\vec{r}')$ for a background field that is an arbitrary superposition of fields of the form (1.4) centered at an arbitrary position. This problem has not been solved for a long-range background field. Simply shifting the position of the monopole field in Eq. (1.4) to the point $\vec{c}$ and integrating $u(\vec{r}-\vec{c},\vec{r}'-\vec{c})$ over $\vec{c}$ is not a correct way to restore translation invariance.

The discussion of the nature of the problem for $b_0>\frac{9}{16}$ has been deferred until now. Equations (2.24) for the exterior solutions of $\varphi_\pm(x)$ are not valid for $b_0>\frac{9}{16}$ because then ν is complex. Rather the exterior solutions ($x\geq\rho$) should be written

$$\varphi_\pm(x)=c_\pm\left(\frac{\rho}{x}\right)^{1/2}\cos\left[\delta\ln\frac{x}{\rho}+\eta_\pm\right]\quad(x\geq\rho) \tag{2.37}$$

where

$$\delta=(4b_0-\tfrac{9}{4})^{1/2}, \tag{2.38}$$

and $\eta_\pm$ and $c_\pm$ are constants. However, *both* of these solutions are singular at $x=\infty$; thus a radial Green's function $g_1^{-1}(x,x')$ regular at $x=\infty$ cannot be defined by Eq. (2.21). If the radial Green's function is not regular at $x=\infty$, quantities such as the ICP that involve an integral of the Green's function over all space are not well defined. For instance, the radial function $X_1^{-1}(r,r')$ that determines $u^{(1,-1)}(\vec{r},\vec{r}')$ is divergent unless $\varphi_+(x)$ decreases to zero more rapidly than $x^{-1/2}$ as $x\to\infty$.

The exterior solutions $\varphi_\pm(x)$ for $b_0\leq\frac{9}{16}$ are smooth functions of x and have no nodes; while for $b_0>\frac{9}{16}$ the solutions oscillate rapidly as $x\to\infty$ and have an infinite number of nodes. The origin of this difference is that when $b_0\leq\frac{9}{16}$ the differential operator in Eq. (2.20) has no normalizable eigenfunctions with negative eigenvalue, while for $b_0>\frac{9}{16}$ the operator has an infinite number of such bound-state eigensolutions. The negative eigenvalues for $b_0>\frac{9}{16}$ accumulate at the value zero.[4] This property of the spectrum of the differential operator in Eq. (2.20) depends only on the value of $b_0=\lim_{x\to\infty}b(x)$, and is independent of the precise behavior of $b(x)$ for finite x. This property was proven by Bender *et al.*[4] who used an argument involving the WKB approximation. In the Appendix a simple general proof of this result is given.

For the specific choice of $b(x)$ in Eq. (2.19) the bound-state eigensolutions can be calculated explicitly. The bound-state problem under consideration is

$$-\mu^2R(x)=\left[-\frac{d^2}{dx^2}-\frac{2}{x}\frac{d}{dx}+\frac{2}{x^2}-\frac{4b(x)}{x^2}\right]R(x)\,, \tag{2.39}$$

where μ is real. It is a simple matter to find the interior and exterior solutions $R(x)$ and to match

them at the boundary $x=\rho$. This leads to an eigenvalue condition for μ:

$$\frac{\mu\rho K'_{\nu+1/2}(\mu\rho)}{K_{\nu+1/2}(\mu\rho)}=\tfrac{1}{2}+\frac{Q\rho j_1'(Q\rho)}{j_1(Q\rho)}, \tag{2.40}$$

where

$$Q\rho=[4b_0-(\mu\rho)^2]^{1/2}, \tag{2.41}$$

and ν is given in Eq. (2.26); also $K_{\nu+1/2}(z)$ denotes the modified cylindrical Bessel function regular at $|z|=\infty$.[17] For $b_0<\frac{9}{16}$ the order $\nu+\frac{1}{2}$ of $K_{\nu+1/2}$ is real and the left-hand side of Eq. (2.40) is a slowly varying function of μ; then it can be shown that Eq. (2.40) has no solution. But for $b_0>\frac{9}{16}$ the order $\nu+\frac{1}{2}$ is imaginary:

$$\nu+\tfrac{1}{2}=i\delta=i(4b_0-\tfrac{9}{4})^{1/2}. \tag{2.42}$$

The Bessel function $K_{i\delta}(\mu\rho)$ has an essential singularity at $\mu\rho=0$; as $\mu\rho\to 0$, $K_{i\delta}(\mu\rho)$ is asymptotically

$$K_{i\delta}(\mu\rho)\sim N\cos(\delta\ln\tfrac{1}{2}\mu\rho+\theta), \tag{2.43}$$

where

$$N=|\Gamma(i\delta)|,\qquad \theta=\arg\Gamma(i\delta). \tag{2.44}$$

The right-hand side of Eq. (2.40) is a smooth function of μ while the left-hand side varies from $-\infty$ to $+\infty$ an infinite number of times as $\mu\to 0$ because of the essential singularity. Thus in this case the eigenvalue condition (2.40) has an infinite number of solutions which accumulate at $\mu=0$.

The parameter b_0 has been left unspecified. However, it can be argued that the value $b_0=\frac{1}{2}$ is of special significance. First, for that value the field $A_a^i(\vec{x})$ is precisely the Wu-Yang monopole field, which is a static solution of the field equations. Second, it has been shown that in the path integral formula for the vacuum functional [Eq. (1.8)] meron paths produce vacuum fluctuations that are of the form of $A_a^i(\vec{x})$ with $b_0=\frac{1}{2}$.[11]

Finally, and most importantly, a field $A_a^i(\vec{x})$ of the form (1.4) with $\lim_{x\to\infty}b(x)=b_0$ can be gauge transformed to another field $A_a'^i(\vec{x})$ of the same angular form but with $\lim_{x\to\infty}b'(x)=1-b_0$. Let $g(\vec{x})$ be the gauge function[18]

$$g(\vec{x})=e^{i\vec{\sigma}\cdot\hat{x}(\pi/2)}=i\vec{\sigma}\cdot\hat{x}. \tag{2.45}$$

If $A_a^i(\vec{x})$ is transformed by $g(\vec{x})$ the resulting field $A_a'^i(\vec{x})$ is

$$A_a'^i(\vec{x})=\frac{2}{g}\epsilon_{aij}\frac{x^j}{x^2}(1-b(x)). \tag{2.46}$$

Since fields with $b_0=\frac{1}{2}+\epsilon$ and $\frac{1}{2}-\epsilon$ are gauge equivalent, any difference between their effects may be a gauge artifact. In particular, the singularity that occurs in the ICP $u(\vec{r},\vec{r}')$ at $b_0=\frac{9}{16}$ may be a gauge artifact that would not be present in a gauge-invariant quantity. Indeed if $b_0=\frac{9}{16}$ the differential operator $-D_{abi}\partial_i$ has a zero mode signaling the ambiguity in the definition of the Coulomb-gauge field that was pointed out by Gribov.[10]

The contribution of the angular mode with quantum numbers $(n,\sigma)=(1,-1)$ has been discussed in some detail. Other modes can be discussed qualitatively by examining the radial potential $[n(n+1)-2b(x)f_n^\sigma]/x^2$ that occurs in Eq. (2.5) for the radial Green's function $g_n^\sigma(x,x')$. For $\sigma=0$ or -1 the field-dependent term is attractive (for $b_0>0$) so $g_n^\sigma(x,x')$ falls off less rapidly at large distances than it would for $b_0=0$. For $\sigma=+1$ the field-dependent term is repulsive so $g_n^\sigma(x,x')$ falls off more rapidly. Also, since the centrifugal term is proportional to n^2 while the field-dependent term is proportional to n or 1, the effect of $b(x)$ is less important for larger values of n. The modes discussed above, $(n,\sigma)=(0,1)$ and $(1,-1)$, are thus the most interesting ones in that they are the most slowly decreasing at large distances. The mode $(1,-1)$ is affected most by the long-range part of $b(x)$ because the ratio of the field-dependent term to the centrifugal term is largest for that mode.

It is not possible to say whether long-range vacuum fluctuations would produce quark confinement, i.e., a $q\bar{q}$ potential $V(\vec{r}-\vec{r}')$ that diverges as $|\vec{r}-\vec{r}'|\to\infty$, because the problem of restoring translation invariance (i.e., of computing the effects of a superposition of long-range fields) has not been solved. Calculations in which translation invariance is restored artificially[4,5] suggest that confinement might result.

Another approach to the calculation of the $q\bar{q}$ static potential is the semiclassical approach. In the next section a derivation of the instanton contribution to the $q\bar{q}$ potential will be given and the result compared to the ICP discussed here and in I.

III. A SEMICLASSICAL CALCULATION OF THE QUARK-ANTIQUARK STATIC POTENTIAL

The purpose of this section is to derive the instanton contribution to the $q\bar{q}$ static potential and to compare the result to the instantaneous Coulomb potential (ICP) calculated in Sec. II and Ref. 1. The derivation of the instanton contribution to be given here differs from that of Callan *et al.*[6] in two respects: It is done in the temporal gauge ($A_a^0=0$) formulation of the theory, and it starts from a generalization of the Wilson-loop formula[7] that avoids the introduction of a singular string operator to approximate the $q\bar{q}$ state.[8] The extension of the derivation from instantons to other

classical configurations such as meron pairs[6] would be straightforward.

It is interesting to derive the instanton contribution in the temporal gauge because that is the gauge in which the instanton is most naturally interpreted as a tunneling configuration.[19]

The starting point for the derivation is a device for computing the energy of a static $q\bar{q}$ pair that is a generalization of the Wilson-loop formula. Consider the matrix element $W(T)$ defined in the temporal-gauge formulation of the theory by

$$W(T)=\mathrm{tr}\langle\Omega|K^\dagger(A)e^{-HT}K(A)|\Omega\rangle, \tag{3.1}$$

where $K(A)$ is *any* functional of the field operator $A_a^i(\vec{x})$ that transforms according to Eq. (1.12) under a gauge transformation of $A_a^i(\vec{x})$; tr means trace over SU(2) indices of $K(A)$. If the complete set of eigenstates of H is inserted in Eq. (3.1), $W(T)$ can be rewritten

$$W(T)=\sum_n \mathrm{tr}\langle\Omega|K^\dagger(A)|n\rangle\langle n|K(A)|\Omega\rangle e^{-TE_n}. \tag{3.2}$$

Because $K(A)$ obeys Eq. (1.12) only gauge-invariant states that contain a quark at $\vec{r}$ and antiquark at $\vec{r}\,'$ in an SU(2) singlet combination contribute to the sum over $|n\rangle$. When $T\to\infty$ the lowest-energy state $|n\rangle$ for which $\langle n|K(A)|\Omega\rangle$ is nonzero is the dominant term in the sum; that state is the exact $q\bar{q}$ state. Thus as $T\to\infty$

$$\begin{aligned}W(T)&\sim \mathrm{tr}\langle\Omega|K^\dagger(A)|q\bar{q}\rangle\langle q\bar{q}|K(A)|\Omega\rangle e^{-TE(R)}\\&=c^2e^{-TE(R)},\end{aligned} \tag{3.3}$$

where $E(R)$ is the energy of the $q\bar{q}$ pair and $R=|\vec{r}-\vec{r}\,'|$.

Equation (3.3) provides a formula that can be used to compute $E(R)$. If $K(A)$ is taken to be the string operator $K_\Gamma(A)$ [Eq. (1.16)], then Eq. (3.3) is the temporal-gauge version of the Wilson-loop formula.[7] But since $K_\Gamma(A)$ is singular the matrix element $\langle q\bar{q}|K_\Gamma(A)|\Omega\rangle$ is actually zero.[8] The formula (3.3) could still be used to compute $E(R)$ if some momentum cutoff were introduced. Alternatively a nonsingular functional $K(A)$ could be used. In the following calculation, however, this problem turns out to be irrelevant: It will be shown that to calculate the instanton contribution to $E(R)$ it is not necessary to specify $K(A)$ beyond the requirement that it obey Eq. (1.12).

The matrix element $W(T)$ can be viewed in two different ways. First, $K(A)$ creates a coherent color-electric field $\bar{E}_a^i(\vec{x})$; the energy $E(R)$ is something like the energy of this mean field. This is the view taken in Sec. II and Ref. 1, where $K(A)$ is the Coulomb-gauge functional (1.20). On the other hand, the factor e^{-HT} is the Euclidean-time translation operator, so $W(T)$ can be rewritten

$$W(T)=\mathrm{tr}\langle\Omega|K^\dagger[A(\vec{x},T/2)]K[A(\vec{x},-T/2)]|\Omega\rangle, \tag{3.4}$$

where

$$A_a^i(\vec{x},x_4)=e^{Hx_4}A_a^i(\vec{x})e^{-Hx_4}. \tag{3.5}$$

This view is more suitable for deriving the instanton contribution to $E(R)$. In addition it is useful to write $W(T)$ in terms of the vacuum wave functional $\langle A_a^i|\Omega\rangle$ defined in Eqs. (1.7), (1.8) as

$$\begin{aligned}W(T)=\int dA_a^i(\vec{x})\langle\Omega|A_a^i\rangle\\ \times\mathrm{tr}K^\dagger[A(T/2)]K[A(-T/2)]\langle A_a^i|\Omega\rangle.\end{aligned} \tag{3.6}$$

In the semiclassical approximation the path integral (1.8) for $|\langle A_a^i|\Omega\rangle|^2$ is dominated by paths that are near the local minima of the action $S(A)$; these paths are the instanton solutions.[20] Then $\langle A_a^i|\Omega\rangle$ is negligible unless $A_a^i(\vec{x})$ lies near such a path. Furthermore, in the semiclassical approximation the time dependence of the field $A_a^i(\vec{x},x_4)$ in the integrand of Eq. (3.6) is just that dictated by the classical equations of motion. Therefore the instanton contribution to $W(T)$ is obtained by replacing $K[A(\vec{x},\pm T/2)]$ in Eq. (3.6) by $K[\bar{A}(\vec{x};\pm T/2)]$ where $\bar{A}_a^i(\vec{x},x_4)$ is an arbitrary multi-instanton solution, and by approximating the functional integral over all field configurations by an integral over the collective coordinates that determine $\bar{A}_a^i(\vec{x},x_4)$. As usual the dilute-gas approximation[6] must be used. Then $\bar{A}_a^i(\vec{x},x_4)$ is a superposition of instantons (and anti-instantons) and the collective coordinates are a position c ($\bar{c}$), scale ρ ($\bar{\rho}$), and global SU(2) orientation ω ($\bar{\omega}$) for each instanton (anti-instanton).[21] The matrix element $W(T)$ is, in this approximation,

$$W(T)=\frac{1}{N}\sum_{n,\bar{n}}\frac{1}{n!}\frac{1}{\bar{n}!}\int\prod_{i=1}^{n}d^4c_i\,d\rho_i\,\frac{d^3\omega_i}{2\pi^2}\prod_{j=1}^{\bar{n}}d^4\bar{c}_j\,d\bar{\rho}_j\,\frac{d^3\bar{\omega}_j}{2\pi^2}\,\xi(\bar{A})\,\mathrm{tr}K^\dagger[\bar{A}(T/2)]K[\bar{A}(-T/2)], \tag{3.7}$$

where the normalization factor N is, in the same approximation,

$$\begin{aligned}N&=\int dA_a^i(\vec{x})\,|\langle A_a^i|\Omega\rangle|^2\\&=\sum_{n,\bar{n}}\frac{1}{n!}\frac{1}{\bar{n}!}\int\prod_{i=1}^{n}d^4c_i\,d\rho_i\,\frac{d^3\omega_i}{2\pi^2}\prod_{j=1}^{\bar{n}}d^4\bar{c}_j\,d\bar{\rho}_j\,\frac{d^3\bar{\omega}_j}{2\pi^2}\,\xi(\bar{A}).\end{aligned} \tag{3.8}$$

In these expressions the factor $1/n!\bar{n}!$ is a Boltzmann counting factor and $2\pi^2$ is the volume of the group SU(2): $2\pi^2 = \int d^3\omega$. The density parameter $\xi(\bar{A})$ measures the relative probability of the multi-instanton solution $\bar{A}_a^i(\vec{x},x_4)$, and is proportional to the volume of Gaussian fluctuations around $\bar{A}_a^i(\vec{x},x_4)$ in the space of Euclidean field configurations. To be precise, the path integral formula (1.8) for $|\langle A_a^i|\,\Omega\rangle|^2$ implies that $\xi(\bar{A})$ is

$$\xi(\bar{A}) = e^{-S(\bar{A})}\det{}^{-1/2}M(\bar{A})\,, \tag{3.9}$$

where $M(\bar{A})$ is the differential operator for the quadratic field fluctuations, defined schematically by the relation

$$S(\bar{A}+\delta A) = S(\bar{A}) + \delta A M(\bar{A})\delta A + \text{higher order}\,.$$

In the dilute-gas approximation the parameter $\xi(\bar{A})$ factorizes into the product of contributions from each instanton

$$\xi(\bar{A}) = \prod_{i=1}^{n}\xi(\rho_i)\prod_{j=1}^{\bar{n}}\xi(\bar{\rho}_j)\,, \tag{3.10}$$

where $\xi(\rho)$ is the value of $\xi(\bar{A})$ for a single instanton with scale parameter ρ, which was computed by 't Hooft[22] and which can be written[6]

$$\xi(\rho) = c\rho^{-5}\left[\frac{8\pi^2}{\bar{g}^2(1/\mu\rho)}\right]^4\exp\left[\frac{-8\pi^2}{\bar{g}^2(1/\mu\rho)}\right], \tag{3.11}$$

where c is a numerical factor of order 1, μ is the renormalization point, and $\bar{g}^2(1/\mu\rho)$ the running coupling constant of the renormalization group.[23]

The evaluation of the right-hand side of Eq. (3.7) will be described in some detail. The result of the calculation is given in Eqs. (3.35), (3.36).

In the temporal gauge, the single instanton field $A_{1a}^i(\vec{x},x_4)$ tunnels between pure-gauge configurations with gauge functions in different homotopy classes[19]:

$$A_1^i(\vec{x},x_4) = \begin{cases} g_1(\vec{x})\partial_i g_1^\dagger(\vec{x}), & \text{for } x_4 = +\infty\,, \\ 0, & \text{for } x_4 = -\infty\,, \end{cases} \tag{3.12}$$

where

$$\begin{aligned} g_1(\vec{x}) &= e^{i\vec{\sigma}\cdot\hat{x}\alpha(r)}\,, \\ \alpha(r) &= \frac{\pi r}{(r^2+\rho^2)^{1/2}}\,. \end{aligned} \tag{3.13}$$

The strict equality in Eq. (3.12) holds at $x_4 = \pm\infty$, but this form is a good approximation provided only $|x_4 - c_4| \gg \rho$ where c_4 is the time coordinate of the instanton. Any time-independent gauge transformation of $A_1^i(\vec{x},x_4)$ is also a temporal-gauge version of the single-instanton field. Similarly the anti-instanton field $\bar{A}_{1a}^i(\vec{x},x_4)$ has

$$\bar{A}_1^i(\vec{x},x_4) = \begin{cases} g_1^\dagger(\vec{x})\partial_i g_1(\vec{x}), & \text{for } x_4 = +\infty\,, \\ 0, & \text{for } x_4 = -\infty\,. \end{cases} \tag{3.14}$$

In the temporal gauge a multi-instanton configuration $\bar{A}_a^i(\vec{x},x_4)$ with instanton positions c_i ($i = 1,2,\ldots,n$) and anti-instanton positions $\bar{c}_j$ ($j = 1,2,\ldots,\bar{n}$) tunnels successively from one homotopy class to another, the number of tunneling events being $n+\bar{n}$. In the dilute-gas approximation the positions c_i and $\bar{c}_j$ are widely separated so they can be ordered in (Euclidean) time, $c_1^4 < c_2^4 < \cdots < c_{n+\bar{n}}^4$ [where now c_k ($k = 1,2,\ldots,n+\bar{n}$) stands for either instanton or anti-instanton positions]. During the time intervals between tunneling events the field $\bar{A}_a^i(\vec{x},x_4)$ is a pure-gauge configuration. For example, if $c_i^4 \ll x_4 \ll c_{i+1}^4$, where c_{i+1} is the position of the instanton (or anti-instanton) that follows c_i, the field is approximately

$$\bar{A}^i(\vec{x},x_4) \simeq G(\vec{x},x_4)\partial_i G^\dagger(\vec{x},x_4)\,, \tag{3.15}$$

where

$$G(\vec{x},x_4) = g_1(\vec{x}-\vec{c}_1)g_1(\vec{x}-\vec{c}_2)\cdots g_1(\vec{x}-\vec{c}_i) \tag{3.16}$$

[$g_1(\vec{x}-\vec{c}_k)$ should be replaced by $g_1^\dagger(\vec{x}-\vec{c}_k)$ for anti-instantons]. In particular, in the limits $x_4 \to \pm\infty$ the multi-instanton solution $\bar{A}_a^i(\vec{x},x_4)$ tunnels between the pure-gauge configurations

$$\bar{A}^i(\vec{x},x_4) = \begin{cases} G(\vec{x})\partial_i G^\dagger(\vec{x}) & \text{for } x_4 = +\infty\,, \\ 0 & \text{for } x_4 = -\infty\,, \end{cases} \tag{3.17}$$

where

$$G(\vec{x}) = Tg_1(\vec{x}-\vec{c}_1)\cdots g_1(\vec{x}-\vec{c}_n)g_1^\dagger(\vec{x}-\vec{\bar{c}}_1)\cdots g_1^\dagger(\vec{x}-\vec{\bar{c}}_{\bar{n}})\,, \tag{3.18}$$

and the symbol T indicates a time ordering of the product of g_1's and $g_1^\dagger$'s: $g_1(\vec{x}-\vec{c}_i)$ stands to the left of $g_1(\vec{x}-\vec{c}_{i'})$ if $c_i^4 < c_{i'}^4$, i.e., if c_i occurs earlier than $c_{i'}$. Note that the field $\bar{A}_a^i(\vec{x},x_4)$ is continuous in x_4.

Now what is needed in Eq. (3.7) for $W(T)$ are the fields $\bar{A}^i(\vec{x},\pm T/2)$. In the dilute-gas approximation, $\bar{A}^i(\vec{x},x_4)$ is a pure-gauge configuration for most values of the position coordinates c_i and $\bar{c}_j$, i.e., except when c_i^4 or $\bar{c}_j^4$ is within a distance ρ of x_4. Therefore in this approximation $\bar{A}^i(x,\pm T/2)$ can be approximated by

$$\bar{A}^i(\vec{x},\pm T/2) = G(\vec{x},\pm T/2)\partial_i G^\dagger(\vec{x},\pm T/2)\,, \tag{3.19}$$

where by Eqs. (3.15), (3.16)

$$\begin{aligned} G(x,\pm T/2) &= T\prod_i g_1(\vec{x}-\vec{c}_i)\prod_j g_1^\dagger(\vec{x}-\vec{\bar{c}}_j)\,, \\ & c_i^4 \lesseqgtr \pm T/2\,, \quad \bar{c}_j^4 \lesseqgtr \pm T/2\,, \end{aligned} \tag{3.20}$$

where again T stands for time ordering. The products over i and j only include values of i and j for which the instanton positions c_i and $\bar{c}_j$ lie earlier in (Euclidean) time than $\pm T/2$.

If $A_a^i(\vec{x})$ is a pure-gauge configuration

$$A^i(\vec{x}) = g(\vec{x})\partial_i g^\dagger(\vec{x}) , \tag{3.21}$$

then by Eq. (1.12) the value of the functional $K(A)$ is

$$K(A) = g(\vec{r})g^\dagger(\vec{r}') . \tag{3.22}$$

This fact, combined with the approximation (3.19) for $\bar{A}_a^i(\vec{x},\pm T/2)$, makes it possible to evaluate the factors $K[\bar{A}(\pm T/2)]$ that occur in the integrand in Eq. (3.7) as

$$K[\bar{A}(\pm T/2)] = G(\vec{r};\pm T/2)G^\dagger(\vec{r}';\pm T/2) . \tag{3.23}$$

The integrand in Eq. (3.7) is thus

$$\begin{aligned}&\mathrm{tr}K^\dagger[\bar{A}(T/2)]K[\bar{A}(-T/2)]\\ &\quad = \mathrm{tr}G^\dagger(\vec{r}';-T/2)G(\vec{r}';T/2)[G^\dagger(\vec{r},-T/2)G(\vec{r},T/2)]^\dagger ,\end{aligned} \tag{3.24}$$

where by Eq. (3.20)

$$G^\dagger(\vec{x};-T/2)G(\vec{x};T/2) = T\prod_{i,j}{}' g_1(\vec{x}-\vec{c}_i)g_1^\dagger(\vec{x}-\vec{\bar{c}}_j) \tag{3.25}$$

and the product $\prod'_{i,j}$ includes only those terms for which $-T/2 \le c_i^4 \le T/2$ and $-T/2 \le \bar{c}_j^4 \le T/2$. At this point the functional $K(A)$ drops out of the problem entirely; thus the precise choice of $K(A)$ is unimportant so long as it obeys the condition (1.12) so that Eq. (3.22) holds. In the semiclassical approximation the matrix element $W(T)$ does not distinguish between singular functionals such as the string functional and smooth ones. The distinction comes about when quantum fluctuations of the fields are included.[8]

With these simplifications the matrix element $W(T)$ becomes

$$W(T) = \frac{1}{N}\sum_{n,\bar{n}}\frac{1}{n!}\frac{1}{\bar{n}!}\int\prod_{i=1}^{n} d^4c_i\,d\rho_i\,\xi(\rho_i)\frac{d^3\omega_i}{2\pi^2}\prod_{j=1}^{\bar{n}} d^4\bar{c}_j\,d\bar{\rho}_j\,\xi(\bar{\rho}_j)\frac{d^3\bar{\omega}_j}{2\pi^2}$$
$$\times \mathrm{tr}T\prod_{i,j}{}' g_1(\vec{r}'-\vec{c}_i)g_1^\dagger(\vec{r}'-\vec{\bar{c}}_j)T^*\prod_{i,j}{}' g_1^\dagger(\vec{r}-\vec{c}_i)g_1(\vec{r}-\vec{\bar{c}}_j) , \tag{3.26}$$

where T^* means antitime ordering (later times stand to the left of earlier times).

In the discussion so far, and in particular in Eq. (3.26), the dependence on the SU(2) global orientations of the instantons has been suppressed. This dependence must be restored as follows. Let $g(\omega)$ denote the global SU(2) rotation matrix that determines the SU(2) orientation of an instanton. For a single instanton, the gauge function $g_1(\vec{x}-\vec{c})$ becomes, for the rotated instanton, $g(\omega)g_1(\vec{x}-\vec{c})$; for an anti-instanton, $g_1^\dagger(\vec{x}-\vec{\bar{c}})$ becomes $g(\omega)g_1^\dagger(\vec{x}-\vec{\bar{c}})$. For the multi-instanton field $\bar{A}^i(\vec{x},x_4)$, the instantons can be rotated separately, and have orientations $\omega_i,\bar{\omega}_j$; but $\bar{A}^i(\vec{x},x_4)$ must remain continuous in x_4. Thus if any given instanton is rotated, then all those that occur later must be rotated by the same amount. Specifically, the factors $g_1(\vec{x}-\vec{c}_i)$ and $g_1^\dagger(\vec{x}-\vec{\bar{c}}_j)$ that occur in Eq. (3.26) must be replaced by

$$g_1(\vec{x}-\vec{c}_i) \to \left[T\prod_{i',j'}{}^{(i)} g(\omega_{i'})g(\bar{\omega}_{j'})\right]g(\omega_i)g_1(\vec{x}-\vec{c}_i) ,$$
$$g_1^\dagger(\vec{x}-\vec{\bar{c}}_j) \to \left[T\prod_{i',j'}{}^{(j)} g(\omega_{i'})g(\bar{\omega}_{j'})\right]g(\bar{\omega}_j)g_1^\dagger(\vec{x}-\vec{\bar{c}}_j) , \tag{3.27}$$

where the product $\prod^{(i)}_{i',j'}$ (or $\prod^{(j)}_{i',j'}$) is only over terms for which $c_{i'}$ and $\bar{c}_{j'}$ occur earlier than c_i (or $\bar{c}_j$).

Now the averages over SU(2) orientations can be done. The integrand is independent of the orientations of instantons that occur at times later than $+T/2$; for these instantons

$$\int\frac{d^3\omega_i}{2\pi^2} = 1 . \tag{3.28}$$

For the instanton with the largest time coordinate less than $T/2$, the rotation matrix $g(\omega_i)$ occurs in the form

$$\int\frac{d^3\omega_i}{2\pi^2}g(\omega_i)g_1(\vec{r}'-\vec{c}_i)g_1^\dagger(\vec{r}-\vec{c}_i)g^\dagger(\omega_i)$$
$$= I\tfrac{1}{2}\,\mathrm{tr}g_1(\vec{r}'-\vec{c}_i)g_1^\dagger(\vec{r}-\vec{c}_i) , \tag{3.29}$$

where I is the 2×2 unit matrix; this equation follows from the formula

$$\int\frac{d^3\omega}{2\pi^2}g_{\alpha\beta}(\omega)g^\dagger_{\beta'\alpha'}(\omega) = \tfrac{1}{2}\delta_{\alpha\alpha'}\delta_{\beta\beta'} . \tag{3.30}$$

Thus after this group average has been done, the product of gauge functions in (3.26) reduces to the same form but with the last instanton with $c_4 < T/2$ eliminated, and the integral over the orientation of the next to the last instanton can be done in the same way. The time ordering of the product of group elements is such that the matrix product collapses to a scalar product as the group averages are done (see also Ref. 6). After all group averages have been done, $W(T)$ is simply

$$W(T)=\frac{2}{N}\sum_{n,\bar{n}}\frac{1}{n!}\frac{1}{\bar{n}!}\int\prod_{i=1}^{n}d^4c_i\,d\rho_i\,\xi(\rho_i)\prod_k{}'\tfrac{1}{2}\mathrm{tr}g_1(\vec{r}\,'-\vec{c}_k)g_1^\dagger(\vec{r}-\vec{c}_k)$$
$$\times\int\prod_{j=1}^{\bar{n}}d^4\bar{c}_j\,d\bar{\rho}_j\,\xi(\bar{\rho}_j)\prod_l{}'\tfrac{1}{2}\mathrm{tr}g_1^\dagger(\vec{r}\,'-\vec{\bar{c}}_l)g_1(\vec{r}-\vec{\bar{c}}_l)\,. \tag{3.31}$$

The overall factor of 2 comes from the overall trace in Eq. (3.26). The product $\prod_k'$ (and $\prod_l'$) runs over those values of the indices that correspond to instantons (and anti-instantons) that occur in the time interval $(-T/2,T/2)$.

Finally the sum that remains [Eq. (3.31)] is familiar from statistical mechanics; it resembles the partition function of a perfect gas. This sum can be evaluated by partitioning the states into classes for which the number of instantons (anti-instantons) that occur in the time interval $(-T/2,T/2)$ is ν $(\bar{\nu})$. Let L be the total length of the time axis, kept finite for the moment. Then $W(T)$ is

$$W(T)=\frac{2}{N}\sum_{n,\bar{n}}\frac{1}{n!\,\bar{n}!}\sum_{\nu,\bar{\nu}=0}^{n,\bar{n}}\binom{n}{\nu}\binom{\bar{n}}{\bar{\nu}}(L-T)^{n-\nu+\bar{n}-\bar{\nu}}T^{\nu+\bar{\nu}}\left[\int d^3c\,d\rho\,\xi(\rho)\right]^{n-\nu+\bar{n}-\bar{\nu}}$$
$$\times\left[\int d^3c\,d\rho\,\xi(\rho)\tfrac{1}{2}\mathrm{tr}g_1(\vec{r}\,'-\vec{c})g_1^\dagger(\vec{r}-\vec{c})\right]^{\nu}\left[\int d^3c\,d\rho\,\xi(\rho)\tfrac{1}{2}\mathrm{tr}g_1^\dagger(\vec{r}\,'-\vec{c})g_1(\vec{r}-\vec{c})\right]^{\bar{\nu}}; \tag{3.32}$$

the combinational factors $\binom{n}{\nu}$ and $\binom{\bar{n}}{\bar{\nu}}$ are binomial coefficients. The sums exponentiate

$$W(T)=\frac{2}{N}\exp\left\{2(L-T)\int d^3c\,d\rho\,\xi(\rho)+T\int d^3c\,d\rho\,\xi(\rho)[\tfrac{1}{2}\mathrm{tr}g_1(\vec{r}\,'-\vec{c})g_1^\dagger(\vec{r}-\vec{c})+\tfrac{1}{2}\mathrm{tr}g_1^\dagger(\vec{r}\,'-\vec{c})g_1(\vec{r}-\vec{c})]\right\}. \tag{3.33}$$

Similarly the normalization factor N is

$$N=\exp\left[2L\int d^3c\,d\rho\,\xi(\rho)\right]. \tag{3.34}$$

Thus $W(T)$ can be written

$$W(T)=2e^{-TE(R)}, \tag{3.35}$$

where the $q\bar{q}$ energy $E(R)$ is

$$E(R)=\int d^3c\int d\rho\,\xi(\rho)[2-\tfrac{1}{2}\mathrm{tr}g_1(\vec{r}\,'-\vec{c})g_1^\dagger(\vec{r}-\vec{c})-\tfrac{1}{2}\mathrm{tr}g_1^\dagger(\vec{r}\,'-\vec{c})g_1(\vec{r}-\vec{c})]. \tag{3.36}$$

This formula was derived by Callan *et al.*[6] who used the Wilson-loop formula and a Lorentz-invariant background-field gauge-fixing procedure. The temporal-gauge derivation is more complicated but it illustrates explicitly the interpretation of instantons as tunneling configurations and sheds some light on the meaning of the Wilson-loop formula in semiclassical calculations.

The energy $E(R)$ was calculated in the semiclassical approximation. Thus $E(R)$ does not include the usual Coulomb energy of the free theory (i.e., with $g=0$) which arises from the influence of the Gaussian quantum fluctuations of the gauge field around $A_a^i(\vec{x})=0$.[24] The effect of these fluctuations on $K[A(\pm T/2)]$ is absent in the semiclassical approximation because in this approximation $A_a^i(\vec{x})$ is set equal to $\bar{A}_a^i(\vec{x})$ in the integrand, and the quantum fluctuations are ignored. In particular, the gauge field $\bar{A}_a^i(\vec{x})$ is approximated as a pure-gauge configuration [Eq. (3.19)] and $K[A(\pm T/2)]$ is then just evaluated for that value of $\bar{A}_a^i(\vec{x})$ [Eq. (3.23)]. This is why the value of $W(T)$ is independent of the choice of $K(A)$, in marked contrast to the Abelian theory, which can be completely solved and in which $W(T)$ depends on $K(A)$ because of the small fluctuations.[8]

Since $E(R)$ does not include the ordinary Coulomb energy, it should be compared not with the complete ICP but with the correction term [the second term on the right-hand side of Eq. (2.11)] produced by the background field. This comparison can be made by writing $E(R)$ as

$$E(R)=\int d^3c\int\frac{d\rho}{\rho}\rho^2\xi(\rho)\Delta u(\vec{r}-\vec{c},\vec{r}\,'-\vec{c}), \tag{3.37}$$

in accordance with Eq. (2.11); then $\Delta u(\vec{r},\vec{r}\,')$, which is to be compared to the ICP, is

$$\Delta u(\vec{r},\vec{r}\,')=\frac{1}{\rho}[2-\tfrac{1}{2}\mathrm{tr}g_1(\vec{r})g_1^\dagger(\vec{r}\,')-\tfrac{1}{2}\mathrm{tr}g_1^\dagger(\vec{r})g_1(\vec{r}\,')], \tag{3.38}$$

or by Eq. (3.13) for $g_1(\vec{x})$,

$$\Delta u(\vec{r},\vec{r}\,')=\frac{2}{\rho}[1-\cos\alpha(r)\cos\alpha(r')$$
$$-\hat{r}\cdot\hat{r}'\sin\alpha(r)\sin\alpha(r')]. \tag{3.39}$$

Note that $\Delta u(\vec{r},\vec{r}')$ vanishes for $\vec{r}=\vec{r}'$ because then the quark color charges cancel.

At large distances, i.e., for $r,r' \gg \rho$, $\Delta u(\vec{r},\vec{r}')$ is approximately

$$\Delta u(\vec{r},\vec{r}') = \tfrac{1}{4}\pi^2\rho^3\left[\frac{1}{r^4}+\frac{1}{(r')^4}-\frac{2\hat{r}\cdot\hat{r}'}{r^2(r')^2}\right]. \tag{3.40}$$

The first two terms are quark self-energy terms that depend only on the position of one of the quarks; the third term is the interaction potential

$$\Delta u_{\rm int}(\vec{r},\vec{r}') \simeq -\tfrac{1}{2}\pi^2\rho^3\frac{\hat{r}\cdot\hat{r}'}{r^2(r')^2} \tag{3.41}$$

which has the same form as the correction to the ICP for a short-range background field [Eq. (2.10)] that was calculated in I.[25] The large-distance behavior of the translationally invariant potential $\Delta v(\vec{r}-\vec{r}')$ that follows from Eqs. (3.41) and (3.37) is

$$\Delta v(\vec{r}-\vec{r}') = \int\frac{d\rho}{\rho}\rho^2\xi(\rho)\int d^3c\,\Delta u_{\rm int}(\vec{r}-\vec{c},\vec{r}'-\vec{c})$$

$$\simeq -\tfrac{1}{2}\pi^2\int\frac{d\rho}{\rho}\rho^5\xi(\rho)\frac{4\pi}{|\vec{r}-\vec{r}'|} \tag{3.42}$$

which is Coulombic. The fact that the instanton contribution to the $q\bar{q}$ potential is asymptotically proportional to $|\vec{r}-\vec{r}'|^{-1}$ is in accordance with the idea that instantons produce short-range vacuum fluctuations, which affect the large-distance behavior of the potential only to the extent of renormalizing the charge.[1]

The formula (3.36) was derived for the instanton solution. It could be extended to other classical paths such as the meron pair[6] that are sufficiently small at large distances, but not to the isolated meron, which decreases slowly at infinity. Nevertheless, one might naively ask under what circumstances the potential $\Delta u(\vec{r},\vec{r}')$ would decrease more slowly than r^{-2} as $r\to\infty$. Equation (3.38) indicates that $\Delta u(\vec{r},\vec{r}')$ can be large at large r if the gauge function $g(\vec{r})$ does not approach 1 as $r\to\infty$. The classical path $\bar{A}_a^i(\vec{x},x_4)$ that interpolates between 0 at $x_4=-\infty$ and $g\partial_i g^\dagger$ at $x_4=+\infty$ would then be of long range for finite x_4, and produce long-range vacuum fluctuations in the path integral (1.8). Of course such fields are beyond the realm of applicability of Eq. (3.35). But these remarks are meant to suggest that the effect of paths that produce long-range vacuum fluctuations in semiclassical calculations of the $q\bar{q}$ potential might be similar to the effect of the long-range background field on the ICP described in Sec. II.

IV. SUMMARY AND DISCUSSION

In this paper and in I the instantaneous Coulomb interaction has been described for a background field $A_a^i(\vec{x})$ of the spherically symmetric form (1.4). The problem of computing the Coulomb Green's function and the instantaneous Coulomb potential (ICP) for a static quark-antiquark pair in a color-singlet combination was treated by separating radial and angular variables and thus reducing the problem to a radial problem. The radial problem was analyzed for simple choices of the radial function $b(x)$ that defines $A_a^i(\vec{x})$.

The background field $A_a^i(\vec{x})$ considered in I was taken to be of short range, i.e., $A_a^i(\vec{x})$ decreases to zero faster than $|\vec{x}|^{-1}$ as $|\vec{x}|\to\infty$. It was shown that the long-range part of the instantaneous Coulomb interaction is unaffected by the background field. To be specific, the contribution to the ICP due to the background field, denoted $\Delta u(\vec{r},\vec{r}')$ in Eq. (2.9), decreases as $|\vec{r}|^{-2}$ as $|\vec{r}|\to\infty$. In addition, translation invariance was restored in I by averaging the ICP over background fields that are arbitrary superpositions of the short-range spherically symmetric field $A_a^i(\vec{x})$, in the dilute-gas approximation. The resulting translationally invariant potential $v(\vec{r}-\vec{r}')$ is proportional to $|\vec{r}-\vec{r}'|^{-1}$ at large separations. Thus the effect of superpositions of a short-range background field is, at large distances, equivalent to a charge renormalization. This point was illustrated in I by considering a background field not only of short range, but in fact sharply cutoff at an arbitrary range ρ: $A_a^i(\vec{x})=0$ if $|\vec{x}|\geq\rho$. The charge renormalization found in I is positive, consistent with the presumed infrared slavery of QCD.

Now in this paper a long-range background field has been considered. Here it was shown that the asymptotic dependence on $|\vec{x}|$ of the contribution to the Coulomb Green's function $G_{ab}(\vec{x},\vec{x}')$ with given angular form depends on b_0, the magnitude of the long-range field, in contrast to the case of a short-range field. In particular the contribution to $G_{ab}(\vec{x},\vec{x}')$ of the modes with angular quantum numbers $(n,\sigma)=(1,-1)$, which is of order $|\vec{x}|^{-2}$ for a short-range background field, is of order $|\vec{x}|^{-\nu-1}$ with $0\leq\nu\leq 1$ for $0\leq b_0\leq\frac{1}{2}$. For the most interesting case, in which the background field is a Wu-Yang monopole field ($b_0=\frac{1}{2}$), the correction to $G_{ab}(\vec{x},\vec{x}')$ due to the field is asymptotically of order $|\vec{x}|^{-1}$, i.e., comparable to the ordinary Coulomb Green's function with $A_a^i(\vec{x})=0$. Similarly the correction to the ICP $u(\vec{r},\vec{r}')$ due to the background field is of order $|\vec{r}|^{-1}\ln|\vec{r}|$ at large $|\vec{r}|$.

The problem of restoring translation invariance, i.e., of treating a superposition of fields of the form of the long-range field $A_a^i(\vec{x})$, has not been considered in this paper. This problem has two parts: defining a superposition of long-range fields, and computing the ICP in such a configuration. The solutions of these problems are not at hand, so it is

not possible to say whether long-range fields would produce a confining $q\bar{q}$ potential.[4,5] But the fact that superpositions of a short-range field, even one that is sharply cutoff, produce a potential of order $|\vec{r}-\vec{r}'|^{-1}$ suggests at least that superpositions of a long-range field might change the asymptotic dependence on $|\vec{r}-\vec{r}'|$, and perhaps even lead to confinement of the charges.

The background field $A_a^i(\vec{x})$ was identified in these papers as a vacuum fluctuation of the gauge fields. The vacuum expectation value of the ICP was assumed to be a model of the $q\bar{q}$ potential. It was argued in I that the short-range background field considered there is similar to the vacuum fluctuation of the gauge field produced by an instanton solution in the path integral formula for the vacuum functional [Eq. (1.8)]. The fact that instantons do not affect the long-range behavior of the $q\bar{q}$ potential beyond an effective charge renormalization, as shown in Sec. III, can be understood as a consequence of the fact that the vacuum fluctuations that they produce are superpositions of a short-range field.[1]

It has also been argued that the meron-field configurations in an ionized meron phase of the theory[6] would produce long-range magnetic-monopole vacuum fluctuations.[11] The existence of such vacuum fluctuations was also suggested by Mandelstam.[3,26] Thus merons, if they do occur in path integrals, would play the very special role of being the origin of long-range vacuum fluctuations.

The possibility of the existence of a simple ionized meron phase in QCD[6] is suggested by an analogy with the disordered phase of the two-dimensional spin model called the XY model. This analogy was described in detail in Ref. 12. The high-temperature disordered phase of the XY model results from contributions to the partition function from vortices in the spin field.[27] These vortices are analogs of merons in the Yang-Mills theory in that they produce long-range vacuum fluctuations (these terms being suitably defined[12]), which lead to the disorder. These long-range vacuum fluctuations do not occur in perturbation theory, in either the XY model or QCD.[12] Some nonperturbative mechanism is needed to produce them. The analogy between vortices in the XY model and merons in QCD suggests that the mechanism is ionization of vortices or merons, since this is indeed the mechanism that operates in the XY model.

Thus the model of the $q\bar{q}$ static potential that has been advocated in these papers consists of two assumptions. First is an assumption about the nature of the vacuum fluctuations: These are assumed to be of the spherically symmetric form (1.4) with translation invariance restored by averaging over all superpositions of this form. Second is an assumption about the effect of these vacuum fluctuations on the $q\bar{q}$ potential energy, which is assumed to be adequately represented by the instantaneous Coulomb potential (ICP). This model is simple, and can be interpreted in terms of familiar concepts from electrostatics. Although it is naive, the model is a natural one to study for a sign of confinement. For example, the instantaneous Coulomb interaction is precisely the term in the Hamiltonian that accounts for the antiscreening of color charge, which is the origin of asymptotic freedom, in the Coulomb-gauge calculation of the $q\bar{q}$ potential in perturbation theory.[11] Furthermore, this model for the $q\bar{q}$ potential gives a result similar to the potential found in semiclassical calculations, as indicated by the similarity between the potential calculated in I and the instanton contribution calculated in Sec. III.

The instanton calculation in Sec. III is interesting in its own right. It illustrates in an explicit way the temporal-gauge interpretation of instantons as tunneling paths. And it illustrates the fact that the semiclassical calculation ignores the effects of the Gaussian quantum variations of the fields around the classical solutions.

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APPENDIX

The eigenvalue problem in Eq. (2.39) is

$$-\mu^2 R(x)=\left[-\frac{d^2}{dx^2}-\frac{2}{x}\frac{d}{dx}+\frac{2}{x^2}-\frac{4b(x)}{x^2}\right]R(x)\ . \quad \text{(A1)}$$

The purpose of the Appendix is to show that this equation has no solution if $b_0 \leq \frac{9}{16}$ but has an infinite number of solutions if $b_0 > \frac{9}{16}$, where $b_0 = \lim b(x)$ as $x \to \infty$. The eigenvalues μ for $b_0 > \frac{9}{16}$ accumulate at the value $\mu = 0$. This result is independent of the precise nature of $b(x)$ for x finite; for the sake of definiteness it will be assumed that $b(x) \sim cx^2$ as $x \to 0$ and that $b(x)$ increases monotonically from 0 to b_0 as x varies from 0 to ∞.

This property of the spectrum of eigensolutions of Eq. (A1) can be proven by changing the variables that occur in the equation, and examining the resulting alternative equation. Let the independent variable y and dependent variable $\varphi(y)$ be defined by

$$y = \ln \mu x\,, \qquad \varphi(y) = \mu^{-1} x^{1/2} R(x)\,, \tag{A2}$$

where the eigenvalue μ is positive. In terms of y and $\varphi(y)$ the eigenvalue equation is

$$0 = -\frac{d^2\varphi(y)}{dy^2} + V(\mu;y)\varphi(y)\,, \tag{A3}$$

where

$$V(\mu;y) = e^{2y} + \tfrac{9}{4} - 4b(\mu^{-1}e^{y})\,. \tag{A4}$$

The range of the variable x is $0 \leq x \leq \infty$ since x is a radial coordinate; thus the range of y is $-\infty \leq y \leq \infty$. The eigenfunction $R(x)$ is normalizable, so

$$1 = \int_0^\infty x^2 dx R^2(x) = \int_{-\infty}^{\infty} dy\, e^{2y}\varphi^2(y)\,. \tag{A5}$$

Equation (A4) is a one-dimensional Schrödinger equation with the potential $V(\mu;y)$ and energy eigenvalue zero. That is, the original eigenvalue equation (A1) has an eigenvalue μ if the operator $-d^2/dy^2 + V(\mu;y)$ has a normalizable zero mode, i.e., an eigensolution with eigenvalue zero.

The potential $V(\mu;y)$ grows exponentially as $y \to \infty$, and approaches $+\frac{9}{4}$ as $y \to -\infty$ for any positive value of μ, since $b(x) \to 0$ as $x \to 0$. On the other hand, as $\mu \to 0$ with y fixed the potential $V(\mu;y)$ tends to $e^{2y} + \frac{9}{4} - 4b_0$.

If $b_0 \leq \frac{9}{16}$ then $V(\mu;y) \geq 0$ for all μ. Then the eigenvalues of the differential operation $-d^2/dy^2 + V(\mu;y)$ are positive. Thus for $b_0 \leq \frac{9}{16}$ the eigenvalue problem (A1) has no solution.

If $b_0 > \frac{9}{16}$, however, then $V(\mu;y)$ is negative if μ is sufficiently small and y large and negative. For any positive value of μ the operator $h \equiv -d^2/dy^2 + V(\mu;y)$ has a finite number of negative eigenvalues with normalizable eigenfunctions, because $V(\mu;y) \to +\frac{9}{4}$ as $y \to -\infty$. But for $\mu = 0$ it has a continuum of negative eigenvalues extending down to the value $\frac{9}{4} - 4b_0$. Thus when μ is sufficiently large the operator h has no negative eigenvalues. As μ decreases, the largest value of μ for which this operator h has a negative eigenvalue is the largest eigenvalue of Eq. (A1). As μ decreases further, the range of y over which $V(\mu;y)$ is negative increases, and thus the operator h has more negative eigenvalues; each value of μ at which an eigenvalue of h passes from positive to negative is an eigenvalue of Eq. (A1). Finally when μ reaches $\mu = 0$ an infinite number of negative eigenvalues have accumulated, because the operator h with $\mu = 0$ has a continuum of negative eigenvalues. Therefore the number of eigenvalues μ of Eq. (A1) is infinite, and the eigenvalues accumulate at $\mu = 0$, as claimed.[4]

Additional insight into this problem can be gained by a qualitative examination of the solution $\varphi(y)$ of the zero-mode equation (A3) for $\mu \approx 0$. If $b_0 \leq \frac{9}{16}$, the zero-mode solution $\varphi(y)$ for $\mu = 0$ is not regular because $V(0;y) > 0$. If $b_0 > \frac{9}{16}$ then the zero mode for $\mu = 0$ is in the continuum of eigensolutions for the potential $V(0;y) = e^{2y} + \frac{9}{4} - 4b_0$. In that case, the solution $\varphi(y)$ is asymptotically, as $y \to -\infty$, of the form

$$\varphi(y) \sim c\cos[\delta y + \eta] \quad (y \to -\infty)\,, \tag{A6}$$

where $\delta^2 = 4b_0 - \frac{9}{4}$ and η is a constant phase shift. Similarly, for μ small the potential $V(\mu;y)$ is approximately $e^{2y} + \frac{9}{4} - 4b_0$ for a large part of the negative y axis, specifically for $0 \gtrsim y \gtrsim \ln \mu\rho$ where ρ is the characteristic scale in x over which $b(x)$ varies from 0 to b_0. Thus a zero-mode solution with μ small is, over that range of y, approximately

$$\varphi(y) \simeq c_1(\mu)\cos[\delta y + \eta(\mu)]\,. \tag{A7}$$

On the other hand for $y \to -\infty$, $V(\mu;y) \to \frac{9}{4}$ so

$$\varphi(y) \simeq c_2(\mu) e^{3y/2} \tag{A8}$$

for y sufficiently large and negative.

The implication of these equations for the eigenfunction $R(x)$ with eigenvalue μ is, by (A2), that

$$R(x) \simeq \mu c_1(\mu) x^{-1/2}\cos[\delta \ln \mu x + \eta(\mu)] \quad \text{for } \rho \lesssim x \lesssim \frac{1}{\mu}, \tag{A9}$$

and

$$R(x) \simeq c_2(\mu)\mu^{5/2} x \quad \text{for } x \ll \rho\,. \tag{A10}$$

In order for μ to be an eigenvalue of Eq. (A1), the x dependence of these functions must be the same for $x \simeq \rho$. The spacing between eigenvalues decreases to zero as $\mu \to 0$ because of the rapid oscillations of $R(x)$ for $x \gtrsim \rho$ [Eq. (A9)] as $\mu \to 0$, which makes it possible to match the functions in Eqs. (A9) and (A10) at $x \simeq \rho$ with only small changes in μ. This qualitative remark is verified quantitatively in the case of the particular choice of $b(x)$ examined in Sec. II, given by Eq. (2.19).

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[1]D. R. Stump, Phys. Rev. D 20, 1002 (1979).

[2]T. T. Wu and C. N. Yang, in *Properties of Matter Under Unusual Conditions*, edited by H. Mark and S. Fernbach (Interscience, New York, 1969); G. 't Hooft, Nucl. Phys. B79, 276 (1974); A. M. Polyakov, Pis'ma Zh. Eksp. Teor. Fiz. 20, 430 (1974) [JETP Lett. 20, 194 (1974)].

[3]S. Mandelstam, Phys. Rep. 23C, 245 (1976); and talk

presented to the American Physical Society, Washington, D. C., 1977, Berkeley report (unpublished).
[4]C. M. Bender, T. Eguchi, and H. Pagels, Phys. Rev. D 17, 1086 (1978).
[5]K. Cahill and D. R. Stump, Phys. Rev. D 20, 540 (1979).
[6]C. G. Callan, R. F. Dashen, and D. J. Gross, Phys. Lett. 66B, 375 (1977) and Phys. Rev. D 17, 2717 (1978).
[7]K. Wilson, Phys. Rev. D 10, 2445 (1974).
[8]K. Cahill and D. R. Stump, this issue, Phys. Rev. D 20, 2096 (1979).
[9]J. Schwinger, Phys. Rev. 125, 1043 (1962).
[10]V. N. Gribov, lecture at the 12th Winter School of the Leningrad Nuclear Physics Institute 1977, Report No. SLAC-trans-176 (unpublished).
[11]F. L. Feinberg, Phys. Rev. Lett. 39, 316 (1977) and Phys. Rev. D 17, 2659 (1978); T. Appelquist, M. Dine, and I. J. Muzinich, Phys. Lett. 69B, 231 (1977) and Phys. Rev. D 17, 2074 (1978); D. R. Stump (unpublished).
[12]D. R. Stump, Phys. Rev. D (to be published).
[13]J. L. Gervais and B. Sakita, Phys. Rev. D 18, 453 (1978).
[14]The subtle point arising from the multiplicity of choices of $U(\vec{x})$ related to the Gribov ambiguity (Ref. 10) will be ignored. This problem is present in any attempt to define $K(A)$ for all A, because $K(A)$ can be shown to be equivalent to a gauge-fixing map, which must be ambiguous: I. Singer, Commun. Math. Phys. 60, 7 (1978).
[15]Quark self-energy terms arise because the energy must vanish at $\vec{r}=\vec{r}'$ for then the quark color charges cancel. These self-energies diverge if the $q\bar{q}$ potential diverges at $\vec{r}=\vec{r}'$.
[16]P. M. Morse and H. Feshback, *Methods of Theoretical Physics* (McGraw-Hill, New York, 1953) Vol. II, pp. 1801, 1898-1901; J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (Wiley, New York, 1952), p. 796.
[17]M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (U. S. Government Printing Office, Washington, D. C., 1965).
[18]Actually this gauge function $g(\vec{x})$ is singular at $|\vec{x}|=0$, and leads to a gauge field $A'^{i}_{a}(\vec{x})$ that is also singular. If $g(\vec{x})$ is made nonsingular at $\vec{x}=0$ the long-range part of $A'^{i}_{a}(\vec{x})$ would not be changed.
[19]R. Jackiw and C. Rebbi, Phys. Rev. Lett. 37, 172 (1976); C. G. Callan, R. F. Dashen, and D. J. Gross, Phys. Lett. 63B, 334 (1976).
[20]A. A. Belavin, A. M. Polykov, A. S. Schwartz, and Yu. S. Tyupkin, Phys. Lett. 59B, 85 (1975); A. M. Polyakov, Nucl. Phys. B120, 429 (1977).
[21]D. R. Stump, Phys. Rev. D 17, 2126 (1978).
[22]G. 't Hooft, Phys. Rev. D 14, 3432 (1976).
[23]More precisely, if $g_R(t)$ is the running coupling constant, which obeys the Gell-Mann–Low equation $g'_R(t) = -\beta[g_R(t)] \simeq +b_0 g_R{}^3(t)$, then the relation between $\bar{g}^2$ and g_R^2 is $\bar{g}^2(1/\mu\rho) = g_R^2[\ln(1/\mu\rho)]$. See Ref. 6.
[24]For example, in the Abelian theory of the free photon field, the functional $K(A)$ is of the form $\exp(i\int d^3x A^i \epsilon^i)$ where ϵ^i is a c number, and the vacuum functional is Gaussian. Integration over the Gaussian fluctuations of the field leads to the result $W \propto \exp[-TE(R)]$ where $E(R)$ is the Coulomb energy of the charges (see, e.g., Ref. 8).
[25]The extra factor of g^2 in Eq. (2.10) for $\Delta u(\vec{r},\vec{r}')$ should not be considered important because the density factor ξ that appears in Eq. (2.11) is not identical to the instanton density factor $\rho^2\xi(\rho)$ in Eq. (3.37).
[26]See also, S. Mandelstam, Phys. Rev. D 19, 2391 (1979); G. 't Hooft, Nucl. Phys. B138, 1 (1978); Utrecht report 1979 (unpublished).
[27]J. M. Kosterlitz and D. J. Thouless, J. Phys. C 6, 1181 (1973).