PHYSICAL REVIEW D VOLUME 20, NUMBER 8 15 OCTOBER 1979

Magnetic-monopole interactions

Lochlainn O'Raifeartaigh,* Soo Yong Park, and Kameshwar C. Wali
Physics Department, Syracuse University, Syracuse, New York 13210
(Received 12 March 1979)

We study the long-range interactions between an arbitrary number of finite-energy monopoles in non-Abelian gauge theories with spontaneous symmetry breakdown. The Higgs fields may belong to either the adjoint representation or to an arbitrary representation of the gauge group G with residual symmetry group U(1). We use the conservation properties of the stress-energy tensor to calculate the instantaneous force on monopoles starting from given initial field configurations. We show that when the Higgs fields belong to the adjoint representation of G, the stress-energy tensor vanishes everywhere in the Prasad-Sommerfield limit leading to a "no-interaction" result for a system of monopoles (or antimonopoles, but not both). When the monopoles are widely separated, one may picture each of them as consisting of a "core" outside which (the exterior region) the Yang-Mills gauge potentials obey free-field Yang-Mills equations. We find exact solutions in the exterior region (exterior solutions) and show how they determine the desired long-range interactions through the stress-energy tensor. One is led to a very simple physical interpretation of the interactions as consisting of Coulomb-type attractive or repulsive forces due to magnetic charges and Newtonian or Yukawa-type attractive forces due to Higgs fields. We show how these forces differ when we have massive and massless Higgs fields. In the massless case, the Coulomb and Newtonian forces do not have the same strength in general. From this result and also the mass spectrum, we find that the conjectured symmetry between gauge particles and finite-energy monopoles is limited to the case when the Higgs fields belong to the adjoint representation.

I. INTRODUCTION

Monopoles in non-Abelian gauge theories correspond to static, finite-energy classical solutions of a set of coupled nonlinear equations. It is believed that such monopoles play an important role in our understanding of the elementary-particle interactions. For the development of a full quantum dynamics of such monopoles, it will be helpful to understand the interactions between them at the classical level. Towards this end several papers have appeared recently.[1,2] All of these papers, however, consider the 't Hooft-Polyakov[3] monopoles which are spherically symmetric. The purpose of the present paper is to generalize these results to the case when monopoles are not necessarily spherically symmetric, the gauge group is an arbitrary compact group, and the Higgs fields are not necessarily in the adjoint representation. When the Higgs fields do not belong to the adjoint representation, we assume that the spontaneous symmetry breaking is such that there is a unique surviving U(1) symmetry so that there is a unique electromagnetic direction in the gauge space.

In the case of static, spherically symmetric monopole field configurations, the asymptotic behavior of the electromagnetic component corresponds to that of magnetic monopoles with quantized magnetic charges. The quantized magnetic charge can be identified as the topological charge of the underlying Higgs fields. Each monopole or antimonopole in such models can be considered approximately as consisting of a "core" in the interior of which there is a strong coupling between the Yang-Mills and the Higgs fields. Outside this core, which we may call *the exterior region*, the covariant derivatives of the Higgs fields vanish, and thus the gauge potentials obey the free-field Yang-Mills equations. Suppose we now consider a system consisting of an arbitrary number of such monopoles or antimonopoles or monopoles and antimonopoles. The existence of such solutions[4] has not yet been proved, but it is reasonable to suppose that there are such solutions, although in general we do not expect them to be static. When the monopoles are far apart, each individual monopole may be characterized by a core and one may study the long-range interaction forces between such cores. The interactions between extended objects are of course complicated. However, the force acting on a monopole due to others can be defined from the stress-energy tensor by integrating the total force flow through a surface enclosing the monopole. If this surface is chosen far from the core (but excludes cores of other monopoles), then the force flow through the surface is totally insensitive to the shape, and the total force acting on the monopole has a well-defined meaning. In order to calculate this force, the only information that is needed is the initial configuration of the fields in the exterior region. A detailed study of interactions between two widely separated 't Hooft-Polyakov monopoles (antimonopoles or a monopole and an antimonopole) was carried out.

This paper is organized as follows: In Sec. II we define our notations, the Lagrangian, etc., and review how the stress-energy tensor leads to the force equation. In Sec. III we investigate a particularly simple system, namely, the Higgs fields belonging to the adjoint representation and in the Prasad-Sommerfield[5] limit ($V\to 0$). In this case, the stress tensor vanishes everywhere and therefore static solutions for an arbitrary number of monopoles (or antimonopoles, but not both) may exist. This no-interaction result is analogous to a similar result in the case of instantons.[6] In Sec. IV we study asymptotic fields for a single monopole with an arbitrary representation for the Higgs field. In Sec. V we find exact exterior solutions for a multimonopole system in the general case. In Secs. VI and VII we use these solutions to discuss the interactions between monopole plasmas for massive and massless Higgs fields, respectively. The form of the interactions is strongly determined by the mass of the Higgs fields because it determines whether the Higgs fields participate in the long-range interactions or not. In Sec. VIII we discuss the results in the SO(3) model when the Higgs fields belong to an arbitrary isospin representation. Finally, in Sec. IX we discuss the bearing of our investigations on the recent conjecture[7] concerning a possible symmetry between monopoles and the fundamental vector bosons in such theories.

II. BASIC FORMALISM

Let G be a compact group with the generators T^α, $\alpha=1,2,\ldots,k$ obeying the commutation relations

$$[T^\alpha, T^\beta]=C^{\alpha\beta\gamma}T^\gamma\,, \tag{2.1}$$

where $C^{\alpha\beta\gamma}$ are the structure constants of G. We are interested in a Lagrangian $\mathcal{L}(x)$ consisting of a set of gauge fields $A_\mu^\alpha(x)$ which belong to the adjoint representation of G and a set of Lorentz scalar fields $\Phi^a(x)$, $a=1,2,\ldots,k$ which belong to an arbitrary real representation of G. Let

$$\underline{A}_\mu(x)=A_\mu^\alpha(x)\underline{\tau}^\alpha,\quad (\tau^\beta)^{\alpha\gamma}=C^{\alpha\beta\gamma}\,, \tag{2.2}$$

where $\underline{\tau}^\alpha$ denote the matrices of the adjoint representation. The matrices of the representation $\Phi(x)$ are denoted by $\underline{t}^\alpha$, $(\underline{t}^\alpha)^{ab}=-(\underline{t}^\alpha)^{ba}$, $a,b=1,2,\ldots,k$.

The Lagrangian $\mathcal{L}(x)$ invariant under the usual local gauge transformations is given by[8]

$$-\mathcal{L}(x)=\tfrac{1}{4}F^{\mu\nu\alpha}F_{\mu\nu}^\alpha+\tfrac{1}{2}(D^\mu\Phi)^a(D_\mu\Phi)^a+V(\Phi)\,, \tag{2.3}$$

where

$$\underline{F}^{\mu\nu}=F_{\mu\nu}^\alpha\underline{\tau}^\alpha=\partial_\mu\underline{A}_\nu-\partial_\nu\underline{A}_\mu+e[\underline{A}_\mu,\underline{A}_\nu]\,, \tag{2.4}$$

$$D_\mu\Phi^a=\partial_\mu\Phi^a+e(\underline{t}^\alpha)^{ab}A_\mu^\alpha\Phi^b\,, \tag{2.5}$$

and the potential $V(\underline{\Phi})$ is a function of $\underline{\Phi}$, bounded below, invariant under G, and restricted to be a fourth-degree polynomial. The field equations that ensue from the Lagrangian (2.3) are

$$\underline{D}^\mu\underline{D}_\mu\underline{\Phi}=\frac{\partial V}{\partial\underline{\Phi}}\,, \tag{2.6}$$

$$D_\mu\underline{F}^{\mu\nu}=-e\underline{J}^\nu(\underline{\Phi})\,, \tag{2.7}$$

where

$$J^{\nu,\alpha}=(\underline{\Phi},\underline{t}^\alpha\underline{D}^\nu\underline{\Phi})\,. \tag{2.8}$$

The symmetric stress-energy tensor $T^{\mu\nu}$ that follows from the Lagrangian (2.3) and the field equations (2.6) and (2.7) is

$$T^{\mu\nu}=F^{\mu\lambda\alpha}F_\lambda^{\nu\alpha}+(D^\mu\Phi)^a(D^\nu\Phi)^a-g^{\mu\nu}\mathcal{L}\,, \tag{2.9}$$

and it is conserved, that is,

$$\partial_\mu T^{\mu\nu}=0\,, \tag{2.10}$$

by virtue of the field equations. From (2.10),

$$\frac{d}{dt}\int_\Omega d^3x\,P_j=\int_S dS^iT_{ij}\,, \tag{2.11}$$

where

$$P_j=T_{0j}$$

is the momentum density, Ω is an arbitrary volume in space, and S is the surface enclosing this volume.

From (2.11), with the appropriate choice of an initial data set, one can derive the instantaneous force on a monopole (anti monopole).[1] If we choose the initial data set corresponding to a static configuration in the $A_0=0$ gauge at any given instant t, say, $t=0$, then

$$T_{ij}=-F_i{}^\alpha F_j{}^\alpha+\tfrac{1}{2}\delta_{ij}F_k{}^\alpha F_k{}^\alpha+\tfrac{1}{2}(D_i\Phi)^a(D_j\Phi)^a-\tfrac{1}{2}\delta_{ij}(D_k\Phi)^a(D_k\Phi)^a-\delta_{ij}V(\Phi)\,, \tag{2.12}$$

where

$$F_i{}^\alpha=\tfrac{1}{2}\epsilon_{ijk}F_{jk}^\alpha\,.$$

We also note for future reference that the static Hamiltonian H or the total energy of the system at $t=0$ is

$$H=\int T_{00}d^3x=\int d^3x[\tfrac{1}{2}(D_i\Phi)^a(D_i\Phi)^a+\tfrac{1}{2}F_i{}^\alpha F_i{}^\alpha+V(\Phi)]\,, \tag{2.13}$$

and the static field equations are

$$\underline{D}_i\underline{D}_i\Phi=\frac{\partial V}{\partial\Phi}\,, \tag{2.14}$$

$$\vec{\underline{D}}\times\vec{\underline{F}}=-e\vec{\underline{J}}\,. \tag{2.15}$$

After choosing an initial data set, we can calculate[1] the instantaneous force on a monopole due to another monopole or antimonopole by considering the integral of T_{ij} over a sphere surrounding the monopole. Owing to the stress-energy conservation, one does not have to calculate the integral over such a sphere. One can, for example, consider a plane midway between the two objects. In practice even this is not necessary. It turns out that one can make an analogy between the T_{ij} obtained and the T_{ij} for Coulomb and Newtonian fields. From this analogy one can read off the forces from our knowledge of the latter cases. Further, it will turn out, as we shall see, that we shall not need the explicit forms of $\underline{F}_i$ and $\underline{A}_i$ but only their existence, their general forms, and topological properties.

For the purposes of the next section, we note that when $\underline{\Phi}$ belongs to the adjoint representation, we can write

$$D_i\underline{\Phi} = \partial_i\underline{\Phi} + e[\underline{A}_i, \underline{\Phi}], \tag{2.16}$$

where both $\underline{\Phi}$ and A_i are now matrices of the same dimensionality. Consequently, as first noted by Bogomol'nyi,[9] we can write the Hamiltonian density T_{00} in the form

$$T_{00} = \tfrac{1}{2}(D_i\underline{\Phi}^\alpha \mp F_i^\alpha)(D_i\Phi^\alpha \mp F_i^\alpha) \pm \partial_i(\Phi^\alpha F_i^\alpha) + V(\Phi), \tag{2.17}$$

where the second term is a total divergence term whose integral gives the total magnetic flux $4\pi m$ where m is the magnetic or topological charge associated with the field configurations and hence constant. The first term is a positive-definite quantity, and further, it is the norm squared of the vector $(D_i\Phi^\alpha \pm F_i^\alpha)$. This situation gives rise to an interesting no-interaction theorem when the potential $V(\Phi) = 0$. We shall discuss this interesting case in the next section.

III. NO-INTERACTION RESULT IN THE BOGOMOL'NYI-PRASAD-SOMMERFIELD LIMIT

When the Higgs field belongs to the adjoint representation of the gauge group and the Higgs potential is allowed to become zero after the spontaneous breakdown, one obtains an exact result for the stress tensor. It is well known that under the two conditions just stated the minimum of the static energy is reached by those solutions of the second-order field equations which satisfy the first-order Bogomol'nyi condition

$$\underline{F}_i = \pm \underline{D_i\Phi}. \tag{3.1}$$

If we now compare this equation with the expression (2.12) for the static stress-tensor density, we see at once that when (3.1) is satisfied the stress-tensor density vanishes identically:

$$T_{ij} = 0. \tag{3.2}$$

It then follows from (2.11) that the force on any arbitrary volume vanishes:

$$\frac{d}{dt}P_i = \int_S T_{ij}ds_j = 0, \tag{3.3}$$

where S denotes the surface enclosing the volume Ω. We conclude that *any solution of* (3.1) *is in static equilibrium*. This no-interaction result is the analog of the known no-interaction result for self-dual instantons[6] and it suggests that, just as in the instanton case, solutions to (3.1) other than the known spherically symmetric one may exist.

Although the result (3.3) is exact for all spatial configurations, one may perhaps obtain a better intuitive understanding of it by considering the case in which it describes monopoles which are widely separated in space. First we note that since for each separated pole, Eq. (3.1) is known to describe a monopole or antimonopole accordingly as the sign is plus or minus, and the same sign must hold throughout space, Eq. (3.1) can only describe a pure monopole or pure antimonopole system [the mixed systems are presumably described by configurations which do not satisfy (3.1) and hence are not in static equilibrium]. The result (3.3) then shows that such a set of monopoles (or antimonopoles) will not exert any long-range forces on one another—their magnetostatic repulsion will be balanced by the Higgs attraction. This cancellation was first observed by Manton[2] in the case of two monopoles. Note that the cancellation will take place for *any* solution of (3.1), not merely for spherically symmetric or SU(2) monopoles. The long-range cancellation of the magnetic and Higgs fields actually follows directly from (3.1) as will be shown explicitly in Sec. VII.

The physical picture described above for widely separated monopoles suggests that in the more general case where they are not widely separated, the condition (3.1) could be taken as the *definition* of a pure monopole (or pure antimonopole) system. Equation (3.3) then says that there is no interaction between pure monopoles or pure antimonopoles for any spatial configuration once the Bogomol'nyi bound[9] is satisfied.

IV. ASYMPTOTIC FIELDS OF SINGLE MONOPOLES

Since solutions of the field equations in the general case, when the potential is not zero and/or the Higgs fields do not belong to the adjoint representation, are not known, our discussion for that case must be limited to the long-range forces. To discuss these forces one prerequisite will be to find the asymptotic form of the fields for a single static (not necessarily spherically symmetric) monopole, and in this section we find these asymptotic forms. For simplicity, and because it is the most physically relevant case, we shall assume that the little group of the potential minimum is U(1), that is, that the electromagnetic direction in the Lie algebra is unique.

Let us first consider the asymptotic behavior of the fields imposed by the finite-energy conditions. The finite-energy conditions[8] make it natural to assume the leading asymptotic forms

$$\lim_{r\to\infty} r\vec{\underline{A}}(x)=\vec{\underline{a}}(\omega) \text{ and } \lim_{r\to\infty}\underline{\Phi}(x)=c\underline{\phi}(\omega), \tag{4.1}$$

for the static gauge and Higgs fields respectively, where ω are the polar angles, $\vec{\underline{a}}(\omega)$ and $\underline{\phi}(\omega)$ are finite, and c is a constant which allows $\underline{\phi}(\omega)$ to be normalized to unity. Furthermore,

$$\vec{\underline{d}}\phi(\omega)=0 \text{ where } \vec{\underline{d}}=r\vec{\nabla}+\vec{a}^{\alpha}(\omega)\underline{t}^{\alpha}. \tag{4.2}$$

From (4.1) it follows that

$$\lim_{r\to\infty} r^2\vec{\underline{F}}(x)=\vec{\underline{f}}(\omega)\equiv\vec{f}^{\alpha}(\omega)\underline{t}^{\alpha}, \tag{4.3}$$

where $\vec{\underline{f}}(\omega)$ is finite, and the integrability condition for (4.2) implies that

$$\vec{f}^{\alpha}(\omega)\underline{t}^{\alpha}\underline{\phi}(\omega)=0. \tag{4.4}$$

Since the little group of $\phi(\omega)$ is assumed to be U(1), Eq. (4.4) implies that $\vec{f}^{\alpha}(\omega)$ factorizes into

$$\vec{f}^{\alpha}(\omega)=n^{\alpha}(\omega)\vec{f}(\omega), \tag{4.5}$$

where $n^{\alpha}(\omega)$ is the unit vector in the unique electromagnetic direction in the Lie algebra

$$\underline{n}(\omega)\underline{\phi}(\omega)\equiv n^{\alpha}(\omega)\underline{t}^{\alpha}\underline{\phi}(\omega)=0. \tag{4.6}$$

Using (4.2) we also have

$$[\vec{\underline{d}}\underline{n}(\omega)]\underline{\phi}(\omega)=0, \text{ where } \vec{\underline{d}}\underline{n}(\omega)=r\vec{\nabla}\underline{n}(\omega)+[\vec{\underline{a}}(\omega),\underline{n}(\omega)] \tag{4.7}$$

and hence, using the uniqueness and unitarity of $\underline{n}(\omega)$,

$$\vec{\underline{d}}\underline{n}(\omega)=0. \tag{4.8}$$

Equations (4.1)–(4.8) contain all the information on the asymptotic fields which can be obtained from the finite-energy conditions.

To obtain further information we must use the static field Equations (2.14) and (2.15). First we expand the Higgs field in the form

$$\Phi^a(x)=c\phi^a(\omega)+\psi^a(x)h(x), \tag{4.9}$$

where $\psi^a h$ is the remainder which vanishes as $r\to\infty$, $h(x)$ being its magnitude and $\psi^a(x)$ being normalized to unity. Then, inserting (4.9) in the first static field equation (2.14) one obtains

$$(\vec{\underline{D}}^2\psi)^a h+2(\vec{\underline{D}}\psi)^a\cdot\nabla h+\psi^a\nabla^2 h=\left(\frac{\partial^2 V}{\partial\Phi_a\partial\Phi_b}\psi_b\right)_{\underline{\Phi}=c\underline{\phi}} h, \tag{4.10}$$

where on the right-hand side we have used the fact that $\partial V/\partial\Phi=0$, since $\underline{\Phi}=c\underline{\phi}$ is a potential minimum, and we have neglected terms with higher powers of h. Taking the inner product of (4.10) with ψ^a, we obtain

$$\nabla^2 h=\left(\mu+\frac{\sigma}{r^2}\right)h, \tag{4.11}$$

where

$$\mu=\left(\underline{\psi},\frac{\partial^2 V}{\partial\Phi^2}\underline{\psi}\right)_{\Phi=c\phi} \text{ and } \sigma=(rD\underline{\psi})^2=-(\underline{\psi},(r\vec{\underline{D}})^2\psi). \tag{4.12}$$

Since $\underline{\Phi}=c\underline{\phi}$ is a potential minimum both μ and σ are positive. It follows that either

(i) h falls off exponentially, or
(ii) both σ and $\mu\to$ zero as $r\to\infty$.

The physical meaning of the condition $\mu\to 0$ as $r\to\infty$ can be seen by recalling that $\partial^2V/\partial\phi^2$ is the Higgs-Kibble mass matrix and hence $\mu\to 0$ means that the field ψ becomes massless or long range. Note that $\underline{\psi}$ may be any of the physical Higgs fields, not necessarily the field $\underline{\phi}$ in the spontaneous-symmetry-breaking direction, although, of course $\underline{\psi}=\underline{\phi}$ is an important special case. Note also that μ is identically zero in the PS limit $V=0$. Since the cases (i) and (ii) give qualitatively different results for the long-range forces, it will be convenient to discuss them separately.

Case (*i*). In this case the exponential falloff of h implies an exponential falloff of the static matter current since from (4.2)

$$-\vec{J}^{\alpha}=(\underline{t}^{\alpha}\underline{\Phi},\vec{\underline{D}}\underline{\Phi})=(\underline{t}^{\alpha}\underline{\Phi},\vec{\underline{D}}\underline{\psi}h)=O(e^{-\lambda r}), \tag{4.13}$$

where λ is a typical falloff parameter determined by μ and σ. It follows that, up to exponentially decreasing terms, the gauge-field $\vec{F}$ satisfies the matterless field equations (and the Bianchi identities)

$$\vec{\underline{D}}\times\vec{\underline{F}}=0 \text{ and } \vec{\underline{D}}\cdot\vec{\underline{F}}=0. \tag{4.14}$$

The solution of (4.14) compatible with the boundary condition (4.3) is easily verified to be

$$\vec{\underline{F}}=\underline{n}(\omega)m\vec{r}/r^3, \tag{4.15}$$

where m is a constant. Combining (4.15) with (4.9) we have the following result: In case (i) the asymptotic forms of the Higgs and gauge fields are

$$\underline{\Phi}(x)=c\underline{\phi}(\omega)+O(e^{-\lambda r}) \text{ and } \vec{\underline{F}}=\underline{n}(\omega)\frac{m\vec{r}}{r^3}+O(e^{-\lambda r}), \tag{4.16}$$

respectively. That the correction terms are exponential can be verified by direct calculations. It is clear that the constant m will be identified with the strength of the monopole.

Case (*ii*). In this case we have μ, $\sigma \to 0$ as $r \to \infty$ and thus, since $(\underline{\psi}, \underline{\psi}) = 1$, we have

$$\underline{\mathrm{d}}\,\underline{\psi}(\omega) = \lim_{r\to\infty} (r\vec{\underline{\mathrm{D}}})\underline{\psi} = 0 \text{ where } \underline{\psi}(\omega) = \lim_{r\to\infty} \underline{\psi}(x). \tag{4.17}$$

Furthermore, Eq. (4.11) reduces to

$$\nabla^2 h = 0, \tag{4.18}$$

for large r. Since the leading solution of (4.18) subject to the boundary condition $h \to 0$ as $r \to \infty$ is b/r where b is a constant, we see that the leading terms in the Higgs field (4.9) in this case are

$$\Phi^a(x) \simeq c\phi^a(\omega) + b\psi^a(\omega) r^{-1}, \tag{4.19}$$

where

$$\underline{d}\,\underline{\phi} = \underline{d}\,\underline{\psi} = 0. \tag{4.20}$$

Inserting this result into the static matter current (2.8) we obtain

$$-\vec{\mathrm{J}}_\alpha \simeq (\underline{t}_\alpha \underline{\Phi}, \vec{\underline{\mathrm{D}}}\underline{\Phi}) = b(\underline{t}_\alpha \underline{\phi}(\omega), \underline{\psi}(\omega))\vec{\nabla}\frac{1}{r}, \tag{4.21}$$

and hence, in particular from (4.6) and the unitarity of ψ,

$$-\vec{\mathrm{J}}^\alpha \underline{n}^\alpha \simeq b(\underline{n}\,\underline{\Phi}, \psi)\nabla\left(\frac{1}{r}\right) = \frac{b}{r}\vec{\nabla}\left(\frac{1}{r}\right)(\underline{n}\psi, \psi) = 0. \tag{4.22}$$

In other words, $n^\alpha \vec{\mathrm{J}}^\alpha$ vanishes up to and including order $1/r^3$. Hence, if we now consider the field equations (and Bianchi identities) for $\vec{\underline{\mathrm{F}}}$ we obtain the matterless equations

$$\vec{\underline{\mathrm{D}}} \times \vec{\underline{\mathrm{F}}} = 0 \text{ and } \vec{\underline{\mathrm{D}}} \cdot \vec{\underline{\mathrm{F}}} = 0. \tag{4.23}$$

If we now insert the asymptotic form (4.3), (4.5) of F in (4.23), and use (4.8) we obtain

$$\vec{\nabla} \times \vec{\mathrm{f}} = 0 \text{ and } \vec{\nabla} \cdot \vec{\mathrm{f}} = 0 \text{ to order } 1/r^3, \tag{4.24}$$

and the only solution of (4.24) subject to the boundary condition (4.3) is

$$\vec{\mathrm{f}} = \frac{m\vec{\mathrm{r}}}{r^3} + O\left(\frac{1}{r^3}\right). \tag{4.25}$$

Combining the result with that obtained above for $\underline{\Phi}$ we see that in case (ii) the asymptotic form of the fields is given by (4.19), (4.20), and

$$\vec{\mathrm{F}}_\alpha = n_\alpha(\omega)\frac{m\vec{\mathrm{r}}}{r^3} + O\left(\frac{1}{r^3}\right). \tag{4.26}$$

Thus, in case (ii) the leading term in $\vec{\underline{\mathrm{F}}}$ remains the same as in case (i) but the Higgs field picks up a long-range term $b\underline{\psi}/r$ in addition to its leading term $c\underline{\phi}$. It will be seen later that this term cannot be neglected. (We have not investigated whether the next terms in $\underline{\Phi}$ and $\underline{F}$ are really of order $1/r^2$ and $1/r^3$ respectively, but the known results for the spherically symmetric case and the exact exterior solutions of the next section suggest that the next terms may actually fall off exponentially.)

As in case (i) it is clear that the constant m in $\vec{\underline{\mathrm{F}}}$ will be identified with the monopole strength. However, there is a new constant b in the expression for $\underline{\Phi}$, and it will be useful to have a form for b which gives some insight into its physical meaning. Let $E(\Phi)$ denote the Higgs kinetic energy, defined as

$$E(\Phi) = \tfrac{1}{2}\int d^3x(\vec{\underline{\mathrm{D}}}\,\underline{\Phi}, \vec{\underline{\mathrm{D}}}\,\underline{\Phi}). \tag{4.27}$$

Then by partial integration, and using the first field equation (2.6) we have

$$\lim_{r\to\infty}\int_S r^2 d\omega\left(\underline{\Phi}, \frac{\partial\underline{\Phi}}{\partial n}\right) = \int d^3x\,\vec{\underline{\mathrm{D}}}(\underline{\Phi}, \vec{\underline{\mathrm{D}}}\,\underline{\Phi})$$
$$= 2E(\Phi) + \int d^3x\left(\underline{\Phi}, \frac{\partial V}{\partial \underline{\Phi}}\right), \tag{4.28}$$

where n is the normal to the surface S of the unit sphere. Hence, from (4.19)

$$b\int_\Omega d\omega(\underline{\phi}, \underline{\psi}) = 2E(\Phi) + \int d^3x\left(\Phi, \frac{\partial V}{\partial \Phi}\right). \tag{4.29}$$

In particular, in the limit $V = 0$, we have

$$b\int_S d\omega(\underline{\phi}, \underline{\psi}) = 2E(\Phi) > 0, \tag{4.30}$$

which shows that $b \neq 0$ and $(\phi, \psi) \neq 0$. Furthermore, if $\underline{\psi} = \underline{\phi}$ we have

$$b = \frac{1}{2\pi}E(\Phi) > 0, \tag{4.31}$$

which shows that b is just the Higgs kinetic energy. The result (4.31) will be useful for the discussion of duality in Sec. IX.

V. EXACT EXTERIOR SOLUTIONS OF THE YANG-MILLS HIGGS SYSTEM

A second requirement which will be necessary to describe the long-range interaction of monopoles will be to have exact solutions of the field equations which are valid in the region between monopoles characterized by the vanishing of the matter current $\underline{J}_\mu$. In this section we exhibit such "exterior" solutions, and although we are interested primarily in the static case, for simplicity and generality we begin by first considering four-dimensional exterior solutions.

The basic ansatz for finding the exterior solutions is to assume that the Higgs field is of the form

$$\Phi^a(x) = c\phi^a(x) + \psi^a(x)h(x)\,, \tag{5.1}$$

where $\underline{\phi}(x)$ and $\underline{\psi}(x)$ are normalized fields satisfying

$$(\underline{\psi}, \underline{t}^\alpha \underline{\phi}) = 0\,, \tag{5.2}$$

and whose covariant derivatives vanish,

$$\underline{D}_\mu \underline{\phi}(x) = \underline{D}_\mu \underline{\psi}(x) = 0\,. \tag{5.3}$$

The little group of $\underline{\phi}(x)$ is assumed to be U(1), c is a constant such that $\Phi(x) \to c\underline{\phi}(\omega)$ as $r \to \infty$, and $h(x)$ is a function to be determined subject to the boundary condition $h(x) \to 0$ as $r \to \infty$. Note that (5.2) does not exclude $\underline{\psi} = \underline{\phi}$. The motivation for the ansatz (5.1) is that it approaches the asymptotic field (4.9) outside the monopole cores and that it guarantees the vanishing of the matter current, thus justifying the name exterior. In fact, we have

$$-J^\alpha_\mu = (\underline{t}^\alpha \underline{\Phi}, \underline{D}_\mu \underline{\Phi}) = (\underline{t}^\alpha \underline{\Phi}, \underline{\psi})\partial_\mu h = 0 \tag{5.4}$$

using (5.2) and (5.3). In general, however, the ansatz (5.1) is not necessary for the vanishing of the matter current since the latter condition requires only that $\underline{D}_\mu \underline{\Phi}$ be orthogonal to the vectors $\underline{t}^\alpha \Phi$, which, together with Φ, do not span the whole representation space [though for the adjoint representation of SU(2) the ansatz (5.1) is both necessary and sufficient].

Using the ansatz (5.1), the first field equation (2.6) reduces to

$$\Box h = \left(\underline{\psi}, \frac{\partial V}{\partial \Phi}\right) = U(\eta(x); h(x)), \quad \eta(x) = (\underline{\phi}, \underline{\psi}) \tag{5.5}$$

where U is a function that is easily calculated for a polynomial potential V. Any solution of (5.5) gives a solution of the first field equation. In particular, in the limit $V = 0$ any harmonic function h yields a solution.

Of course, in principle, to obtain a complete solution for $\underline{\Phi}(x)$ one has also to solve Eq. (5.3) which couple $\underline{\Phi}$ and ψ to the gauge field, but, as we shall see, all that will be required for discussing the long-range forces will be the *existence* of solutions and the topological properties of $\Phi(x)$. The existence will be guaranteed by the equations for the Yang-Mills field $\underline{F}_{\mu\nu}$ which we shall now discuss.

First, the integrability conditions for (5.3) are

$$\underline{F}_{\mu\nu}\underline{\phi} = F^\alpha_{\mu\nu}\underline{t}^\alpha \underline{\phi} = 0, \quad \underline{F}_{\mu\nu}\underline{\psi} = F^\alpha_{\mu\nu} t^\alpha \underline{\psi} = 0\,. \tag{5.6}$$

Since the little group of $\underline{\phi}$ is assumed to be U(1), Eq. (5.6) implies that all the spatial components of $\underline{F}_{\mu\nu}$ must lie in the little algebra. Hence, we must have the factorization

$$F^\alpha_{\mu\nu}(x) = n^\alpha(x) f_{\mu\nu}(x)\,,$$

where $f_{\mu\nu}(x)$ is an ordinary Maxwell field and $n^\alpha(x)$ is a unit vector in the U(1) direction of the Lie algebra

$$\underline{n}(x)\underline{\phi} = \underline{n}(x)\underline{\psi} = 0\,, \tag{5.7}$$

where $\underline{n}(x) = n^\alpha(x)\underline{t}^\alpha$. Note that (5.3) and (5.4) imply that

$$(\underline{D}_\mu \underline{n})\underline{\phi} = 0 \to \underline{D}_\mu \underline{n} \propto \underline{n} \to \underline{D}_\mu \underline{n} = 0\,, \tag{5.8}$$

the second and third equalities following from the uniqueness of the U(1) direction and the unitarity of n.

We now have to consider the field equation for $\underline{F}_{\mu\nu}$ and the Bianchi identity which guarantees that $\underline{F}_{\mu\nu}$ is related to the gauge potential $\underline{A}_\mu$ of the covariant derivative in the usual way [and therefore guarantees the existence of $\underline{A}_\mu(x)$ given $\underline{F}_{\mu\nu}(x)$]. The Bianchi condition is

$$\sum_{\text{cyclic}} \underline{D}_\lambda \underline{F}_{\mu\nu} = \underline{n}(x) \sum_{\text{cyclic}} \nabla_\lambda f_{\mu\nu} = 0\,, \tag{5.9}$$

and since $\underline{J}_\mu = 0$ the field equation is

$$\underline{D}_\mu \underline{F}_{\mu\nu} = \underline{n}(x)\nabla_\mu f_{\mu\nu} = 0\,, \tag{5.10}$$

where in each case the second equality follows from (5.8). Thus, the Maxwell field $f_{\mu\nu}$ will satisfy the free Maxwell equations and any solution of these equations and (5.8) will yield a solution of the field equation for $\underline{F}_{\mu\nu}$ and $\underline{A}_\mu$. Note that a solution of (5.8) always exists since the integrability condition is the identity $[\underline{n}, \underline{n}] = 0$. As in the case of $\underline{A}_\mu(x)$, $\underline{\phi}(x)$, and $\underline{\psi}(x)$ we shall not need to know the explicit form of $\underline{n}(x)$.

The solutions of the free Maxwell equations most convenient for describing monopoles are obtained by writing the equations in the dual form

$$\sum_{\text{cyclic}} \nabla_\lambda f^\alpha_{\mu\sigma} = 0, \;\; \nabla_\mu f^*_{\mu\lambda} = 0, \;\; \text{with}\, f^*_{\mu\lambda} = \tfrac{1}{2}\epsilon_{\mu\lambda\sigma\tau} f_{\sigma\tau}\,. \tag{5.11}$$

Then the first equation in (5.11) is equivalent to the statement that

$$f^*_{\mu\sigma} = \partial_\mu b_\sigma - \partial_\sigma b_\mu\,, \tag{5.12}$$

where b_μ is an arbitrary pseudovector, and the second is equivalent to

$$\Box b_\mu - \partial_\mu(\partial_\lambda b_\lambda) = 0\,, \tag{5.13}$$

which reduces to the D'Alembertian equation for b_μ in the Landau gauge.

Combining all these results together we see that an exact exterior solution with $\underline{J}_\mu = 0$ is given by

$$\Phi^a(x) = c\phi^a(x) + \psi^a(x)h(x)\,,$$
$$F^\alpha_{\mu\nu}(x) = n^\alpha(x)\epsilon_{\mu\nu\lambda\sigma}\partial_\lambda b_\sigma(x)\,, \tag{5.14}$$

where

$$\Box h(x) = U(h(x), \eta(x)), \quad \Box b_\sigma(x) = \partial_\sigma(\partial_\lambda b_\lambda(x))\,, \tag{5.15}$$

and the unit vectors $\underline{\phi}(x)$, $\underline{\psi}(x)$, and $\underline{n}(x)$ are covariantly constant,

$$\underline{D}\,\underline{\phi}=\underline{D}\,\underline{\psi}=\underline{D}\,\underline{n}=0\,. \tag{5.16}$$

They satisfy the algebraic relations

$$(\underline{t}^{\alpha}\underline{\phi},\underline{\psi})=0 \text{ and } \underline{n}\,\underline{\phi}=\underline{n}\,\underline{\psi}=0\,, \tag{5.17}$$

and the little group of $\underline{\phi}(x)$ is U(1). It will be convenient to let $\underline{\phi}(x)$ in these equations be an arbitrary field whose topological properties will determine the monopole charge configurations. Then (5.13) will be equations for $\underline{A}_{\mu}(x)$, $\underline{\psi}(x)$, and $\underline{n}(x)$ for which solutions are guaranteed to exist because the integrability conditions are satisfied.

In the static limit, with which we shall be concerned from now on, the system of Eqs. (5.14)–(5.17) reduces to the same equations with $\underline{D}_{\mu}\to\vec{\underline{D}}$, $\Box\to\nabla^2$, and

$$F^{\alpha}_{\mu\nu}\to F_i{}^{\alpha}=n^{\alpha}\epsilon_{ijk}\epsilon_{jkl}\partial_l b_0=n^{\alpha}\partial_i U\,, \tag{5.18}$$

where $U=2b_0$ and U satisfies Laplace's equation

$$\nabla^2 U=0\,. \tag{5.19}$$

The pseudoscalar function U is the same function U that was used in Ref. 1 in the treatment of two spherical symmetric SU(2) monopoles.

VI. DILUTE MONOPOLE PLASMA FOR MASSIVE HIGGS FIELDS

We define a dilute monopole plasma as a set of monopoles (centered at points $\vec{r}_q$, $q=1,\ldots,n$, say) such that $d\gg 1/\lambda$ where d is the minimum distance between monopoles and λ is a typical mass which determines the size of the monopole core, i.e., the exponential falloff $e^{-\lambda r}$ of the fields. Then we can surround each monopole with a sphere of radius a where $d\gg a\gg 1/\lambda$ and on the surface of each such sphere the fields are approximately the asymptotic fields discussed in Sec. IV.

In the present section we shall consider only the case when the Higgs fields are massive. Hence, from (4.26), the fields at the surfaces of the spheres are

$$\underline{\phi}(x)\simeq c\underline{\phi}(\omega_q),\quad \vec{F}^{\alpha}(x)\simeq n^{\alpha}(\omega_q)m_q\frac{\vec{a}_q}{|a_q|^3}, \tag{6.1}$$

where $\vec{a}_q=\vec{r}-\vec{r}_q$ for $|\vec{r}-\vec{r}_q|=a_q$, and what we need is an exact exterior solution of the field equations which matches the boundary conditions (6.1) at each sphere surface (and has the correct topological character as $r\to\infty$). It is easy to see that such a solution is given by

$$\underline{\Phi}(x)=c\underline{\phi}(x),\quad \vec{F}_{\alpha}=n_{\alpha}(x)\vec{f}(x)\,, \tag{6.2}$$

where

$$\vec{f}(x)=\sum\frac{m_q(\vec{r}-\vec{r}_q)}{|r-r_q|^3}. \tag{6.3}$$

The functions $\underline{\phi}(x)$, $\underline{n}(x)$ are such that

$$\underline{D}\,\underline{\phi}(x)=0,\quad \underline{n}(x)\underline{\phi}(x)=0 \tag{6.4}$$

and reduce to $\underline{\phi}(\omega),\underline{n}(\omega)$ as $\vec{r}\to\infty$ and $\underline{\phi}(\omega_q),\underline{n}(\omega_q)$ as $\vec{r}\to\vec{r}_q+\vec{a}_q$. Such functions exist, because as discussed in Sec. V, $\underline{\phi}(x)$ can be chosen freely and then (6.4) are simply equations for $\vec{\underline{A}}(x)$ and $\underline{n}(x)$ which can be solved for $\vec{\underline{A}}(x)$ because of the Bianchi identity. The function $\underline{\phi}(x)$ can be chosen to satisfy the correct topological conditions on the surfaces of the spheres $\vec{r}=\vec{a}_q+\vec{r}_q$ and infinity. Explicit expressions for $\vec{\underline{A}}(x)$ and $\underline{\phi}(x)=\underline{n}(x)$ for two SU(2) monopoles with the Higgs field in the adjoint representation have been given in Ref. 11. However, the explicit expressions for $\underline{A}(x)$, $\underline{\phi}(x)$, and $\underline{n}(x)$ will not be needed for the stress-tensor density, which from (2.12) takes the form

$$T_{ij}=\tfrac{1}{2}(\delta_{ij}f_k^2-f_if_j)$$

where

$$\vec{f}=\sum_q\frac{m_q(\vec{r}-\vec{r}_q)}{|\vec{r}-\vec{r}_q|^3}. \tag{6.5}$$

Thus, for massive Higgs fields, only the magnetic field $\vec{f}$ contributes to the stress tensor at long range, and one sees by inspection that the contribution is identical to that of a set of static electric charges of strength m_q. Thus, one has the result that one might expect intuitively, if the Higgs fields are massive a dilute monopole plasma behaves in exactly the same way as a dilute plasma of electrically charged particles. In particular, there is a Coulomb repulsion (or attraction) according to whether the monopole charges are like (or unlike).

VII. DILUTE MONOPOLE PLASMA FOR MASSLESS HIGGS FIELDS

In Sec. IV we saw that when the physical Higgs fields are not all massive, we may have $\mu=(\underline{\psi},(\partial^2 V/\partial\Phi^2)\underline{\psi})$ and then the asymptotic part of the Higgs field picks up an extra term $b\underline{\psi}r^{-1}$. Then at the surface S_q of each sphere in the dilute plasma the boundary condition becomes

$$\underline{\Phi}(\omega_q)=c\underline{\phi}(\omega_q)+\frac{b_q}{a_q}\underline{\psi}(\omega_q)\,, \tag{7.1}$$

where

$$\vec{\underline{d}}\,\underline{\psi}=0,\ \underline{n}\,\underline{\psi}=0,\ \text{and } (\underline{\psi},\underline{\psi})=1 \text{ on } S_q \tag{7.2}$$

and the exact exterior solution must be generalized to fit this boundary condition. It is easy to see that a suitable generalization is given by the solution

$$\underline{\Phi}(x)=c\underline{\phi}(x)+\underline{\psi}(x)h(x)\,,$$

where $\phi(x)$ is as before, $h(x)$ is the solution

$$h(x)=\sum_q \frac{b_q}{|\vec{r}-\vec{r}_q|} \tag{7.3}$$

of the Laplace equation, and $\underline{\psi}(x)$ a function which satisfies the equations

$$\underline{\vec{D}}\,\underline{\psi}=0,\quad \underline{n}\,\underline{\psi}=0 \tag{7.4}$$

with the boundary conditions $\underline{\psi}\to\underline{\psi}(\omega_q)$ on the surfaces of the spheres and $\underline{\psi}\to 0$ as $r\to\infty$. Such a function $\underline{\psi}$ exists because the algebraic condition $\underline{n}\,\underline{\psi}=0$ is just the integrability condition for the differential equation $\underline{\vec{D}}\,\underline{\psi}=0$ both on the surfaces and in the exterior region. However, just as in the case of $\underline{\vec{A}}(x)$, $\underline{n}(x)$, and $\underline{\phi}(x)$, the explicit form of $\underline{\psi}(x)$ will not be needed for the stress tensor, since, from (7.4),

$$\underline{D}_i\underline{\Phi}=\underline{\psi}(x)\nabla_i h\,, \tag{7.5}$$

and hence, the contributions of the Higgs field to the stress-tensor density is simply

$$T_{ij}=\tfrac{1}{2}[(\nabla_i h)(\nabla_j h)-\delta_{ij}(\nabla h)^2]\,, \tag{7.6}$$

where h is given by (7.3). From (7.6) and (7.3) we see that the contribution to the stress-tensor density of the long-range Higgs fields is analogous to that of Newtonian gravitating particles located at $\vec{r}_q$ and with gravitational masses b_q. When the b_q are all positive [as in the case in the PS limit with $\underline{\psi}=\underline{\phi}$, for example, see (4.31)] the analogy is complete and all the long-range Higgs forces are attractive.

Combining the above result with the result obtained in Sec. VI for the magnetostatic forces, we see that when long-range Higgs fields contribute the total stress-tensor density can be written as

$$T_{ij}=\tfrac{1}{2}[(\nabla_i h)(\nabla_j h)-\delta_{ij}(\nabla h)^2] -\tfrac{1}{2}[(\nabla_i f)(\nabla_j f)-\delta_{ij}(\nabla f)^2]\,, \tag{7.7}$$

where

$$h=\sum_q \frac{b_q}{|\vec{r}-\vec{r}_q|}\,,\quad f=\sum_q \frac{m_q}{|\vec{r}-\vec{r}_q|}\,. \tag{7.8}$$

The m_q are the magnetic charges, and b_q are the Higgs constants defined in Sec. IV. In particular, if the monopoles are all of the same kind ($m_q>0$, say) and the b_q are all positive, the magnetostatic Coulomb repulsion is countered by a gravitational attraction. Which force dominates depends on the relative magnitudes of the m_q and b_q, and the forces will exactly balance if, and only if, we have $m_q^{\,2}=b_q^{\,2}$ for each monopole. In general, there is no particular reason why the equality $m_q^{\,2}=b_q^{\,2}$ should hold, and in the next section we shall give an example of an SU(2) monopole with $V=0$ for which it does not hold. However, from the results of Sec. III we know that the equality must hold for $\underline{\Phi}$ in the adjoint representation of any group and $V=0$ since the forces balance at all distances in that case. It is interesting to verify the cancellation directly from the Bogomol'nyi bound (3.1). Using (6.2), (7.3), and (3.1) one obtains

$$\underline{n}(x)\sum_q \frac{m_q(\vec{r}-\vec{r}_q)}{|\vec{r}-\vec{r}_q|^3}=\pm\underline{\psi}(x)\sum_q \frac{b_q(\vec{r}-\vec{r}_q)}{|\vec{r}-\vec{r}_q|^3}\,, \tag{7.9}$$

which establishes that $m_q^{\,2}=b_q^{\,2}$, and also that $\underline{n}(x)=\underline{\psi}(x)$ in this case.

VIII. EXPLICIT EXAMPLE OF MONOPOLE WITH HIGGS FIELD NOT IN THE ADJOINT REPRESENTATION

As monopoles for which the Higgs fields are not necessarily in the adjoint representation have been considered in the previous sections, it may be useful to present an example of such a monopole here. This will help to illustrate the role of the Higgs constant b and will also be useful for the discussion of a conjectured symmetry between monopoles and gauge particles in the next section. The example is the simplest one possible (and so far as we know the only one which has been explicitly worked out) and consists of a spherically symmetric SU(2) monopole of strength unity, with the Higgs field in any integer spin I representation of the group.[8,10] The fields are

$$A_i^\alpha(x)=\epsilon_{\alpha ij}\,x_j\left(\frac{K(r)-1}{r^2}\right),\qquad \Phi^a(x)=\left(\frac{4\pi}{2I+1}\right)^{1/2}Y_a^I(\Omega)\frac{H(r)}{r}\,, \tag{8.1}$$

where $Y_a^I(\Omega)$ are real linear combinations of the spherical harmonics of order I, the functions $K(r)$ and $H(r)$ satisfy the field equations

$$r^2\frac{\partial^2 H}{\partial r^2}=2K^2H\sigma+\frac{\partial V}{\partial H}\,,\qquad r^2\frac{\partial^2 K}{\partial r^2}=\sigma H^2K+K(K^2-1)\,, \tag{8.2}$$

where V is a potential which vanishes in the PS limit, and σ is the Casimir variable $2\sigma=I(I+1)$. The boundary conditions for K and H are

$$K(r)-1=O(r^2),\quad H(r)=O(r^{I+1})\ \text{for}\ r\to 0 \tag{8.3}$$

and

$$K(r)\to 0,\quad H(r)\to cr-b\ \text{for}\ r\to\infty\,. \tag{8.4}$$

Here the constants c and b are as in the previous sections, and since the solutions to the field equations are determined by the boundary conditions at the origin and by $K\to 0, H\to cr$ at infinity, the constant b cannot be chosen freely, but is determined by the field equations. Furthermore, since b is then determined by the boundary conditions at the origin as well as infinity, it is a global functional of K and H and cannot be determined from the asymptotic equations as $r\to\infty$. Thus, to actually determine b one needs to solve Eqs. (8.2) and to estimate it one needs a global estimate. There are only two cases in which Eqs. (8.2) have been solved, namely for the adjoint representation[5] $\sigma=1$ and the limiting case[10] $\sigma\to\infty$, and the results for b in these two cases are

$$b(1)=1 \quad b(\infty)=\tfrac{7}{4}. \tag{8.5}$$

However, (8.5) is already sufficient to show that b varies with σ as one might expect from the σ dependence of the coupling in Eqs. (8.2). The variation of b with σ in turn is sufficient to show that the equality $b=m$ of the previous section cannot be maintained for $\sigma\neq 1$, because m, the magnetic charge, is fixed to be unity for all σ in this model on account of the spherical symmetry. [m must in any case be quantized whereas from (8.5) b is evidently not quantized.]

Using the expression (4.29) for b, which in this case reduces to (4.31), we can obtain the global bounds

$$0<b=2E(H)<2M, \tag{8.6}$$

where M is the mass of the monopole. The upper bound (8.6) is useful because it has been shown[10] that $M(\sigma)$ is an increasing function of σ which is bounded above by $3M(1)$ as $I\to\infty$. Since $M(1)=2b(1)$ from the Bogomol'nyi bound for $I=1$, Eq. (8.6) therefore gives the uniform bounds

$$0<b(\sigma)<6b(1) \tag{8.7}$$

on $b(\sigma)$ for all σ. Together with (8.5) this bound suggests that $b(\sigma)$ is a slowly varying function of σ. This fixed upper bound for all σ in (8.7) will play a crucial role in our discussion in the next section.

IX. SYMMETRY BETWEEN MONOPOLES AND GAUGE PARTICLES

As an application of the foregoing results on the long-range interactions of monopoles we consider the concept of a symmetry between the charged vector meson forming a monopole and the monopoles themselves (in the PS limit $V=0$) proposed by Montonen and Olive.[7] This proposal is based on a number of analogies between the charged vector mesons and the monopoles, in particular the following two:

(i) The masses of the charged vector mesons and monopoles are given by dual formulas, obtainable from one another by inverting the coupling constant.

(ii) For both the charged vector mesons and the monopoles the long-range electromagnetic repulsion is canceled by the Higgs attraction.

By considering the example of the previous section we wish to show that the analogies (i) and (ii) no longer hold when the Higgs fields are not in the adjoint representation, and hence the Montonen-Olive conjecture holds, if at all, only if the Higgs field is in that representation.

Let us consider first the mass formulas. For general integer isospin I the mass matrix for SU(2) Yang-Mills fields is given by the Higgs-Kibble formula

$$M_{\alpha\beta}(\omega)=e^2c^2(\underline{\phi}(\omega),\, t_\alpha t_\beta\underline{\phi}(\omega)), \tag{9.1}$$

where $c\underline{\phi}(\omega)$ is the usual limit of the Higgs field as $r\to\infty$ [and the apparent ω dependence of $M_{\alpha\beta}(\omega)$ does not appear in the physical spectrum]. Assuming that $\underline{\phi}(\omega)$ has a little group U(1), with generator $t_3(\omega)=n^\alpha(\omega)t^\alpha$, say, and letting $t^\pm(\omega)$ be the usual step operators for $t_3(\omega)$ one sees that the charged Yang-Mills field masses are

$$\begin{aligned}M^2(I)&=\frac{e^2c^2}{2}(\underline{\phi}(\omega),\{t^+(\omega),t^-(\omega)\}\underline{\phi}(\omega))\\&=\frac{e^2c^2}{2}(\underline{\phi}(\omega),[\{t^+(\omega),t^-(\omega)\}+t_3{}^2(\omega)]\underline{\phi}(\omega))\\&=e^2c^2\frac{I(I+1)}{2}(\phi(\omega),\phi(\omega))=e^2c^2\sigma.\end{aligned} \tag{9.2}$$

Thus the mass of the charged vector mesons is a linear function of the Casimir variable σ. But, in contrast, in Ref. 10 it was shown that *the mass of the monopoles is a slowly increasing function of* σ *which is bounded above by* $3M(1)$. Hence, the dependence of the mass spectrum on σ is quite different for the charged vector mesons and the monopoles, and so the duality between the two fails for $\sigma\neq 1$.

Next let us consider the long-range forces. For two like charged vector mesons, they are given by the exchange of the photon and the Higgs field, with vertices given by the terms

$$\tfrac{1}{2}\vec{F}^2=eA_\mu W^+_\lambda W^-_{\lambda,\mu}+\cdots\to eM_W A_0\vec{W}^+\cdot\vec{W}^-$$

and

$$\begin{aligned}\tfrac{1}{2}(\vec{D}\Phi)^2&=\tfrac{1}{2}e(\vec{D}\Phi, A\Phi^0)+\cdots\to\tfrac{1}{2}e^2(\Phi,\{t_+,t_-\}\Phi^0)\vec{W}^+\cdot\vec{W}^-\\&\quad+\cdots=e^2|\Phi^0|\sigma\vec{W}^+\cdot\vec{W}^-(\Phi,\Phi^0)\end{aligned}$$

in the standard Yang-Mills-Higgs Lagrangian, where Φ^0 is the vacuum value of Φ and σ is the Casimir. Since $M_W=e|\Phi^0|$ by the spontaneous breakdown, we see that there is then a net attractive force of the form $e^2M_W(\sigma-1)/r^2$ for like W's and that it vanishes if, and only if, $\sigma=1$. Furthermore, the reason for the imbalance for $\sigma\neq 1$ is clear: The photon remains in the adjoint representation no matter which representation the Higgs field is in. Hence, while the coupling of the Higgs field increases with σ, the coupling of the photon remains fixed. The result is that the long-range attraction of like charged vector meson forces increases quadratically with σ as $\sigma \to \infty$.

For the monopoles, on the other hand, we have seen in Sec. VII that the long-range attractive force is $(b^2 - m^2)/r^2$ where b is the Higgs constant and m the monopole strength. Since, as shown in the previous section, $b(\sigma)$ varies with σ and m does not, there is again an imbalance for $\sigma \neq 1$. But since it was also shown in Sec. VIII that $b(\sigma)$ is bounded above by $6b(1)$ the attraction of the monopoles cannot increase quadratically with σ and hence the correspondence with the attraction for the vector mesons is lost. We have thus shown that for the model of Sec. VIII the symmetry conjecture fails for both the mass spectra and the long-range forces once the Higgs field is no longer in the adjoint representation.

ACKNOWLEDGMENT

This work was supported by the U. S. Department of Energy, under Contract No. EY-76-S-02-3533.

---

*Permanent address: Dublin Institute for Advanced Studies, School of Theoretical Physics, 10 Burlington Road, Dublin 4, Ireland.

[1]J. N. Goldberg, P. S. Jang, S. Y. Park, and K. C. Wali, Phys. Rev. D 18, 542 (1978).

[2]N. Manton, Nucl. Phys. B126, 525 (1977); J. Jercák, M. Kiera, and M. Magg, Nuovo Cimento 40A, 269 (1977).

[3]G. 't Hooft, Nucl. Phys. B79, 276 (1974); A. M. Polyakov, Zh. Eksp. Teor. Fiz. Pis'ma Red. 20, 430 (1974) [JETP Lett. 20, 194 (1974)].

[4]P. S. Jang, S. Y. Park, and K. C. Wali, Phys. Rev. D 17, 1641 (1978).

[5]M. K. Prasad and C.M. Sommerfield, Phys. Rev. Lett. 35, 760 (1975). Henceforth we shall refer to this limit as PS limit.

[6]E. Witten, Phys. Rev. Lett. 38, 121 (1977); G. 't Hooft, unpublished; R. Jackiw, C. Nohl, and C. Rebbi, Phys. Rev. D 15, 1642 (1977).

[7]G. Montonen and D. Olive, Phys. Lett. 72B, 117 (1977); P. G. O. Freund, Int. J. Theor. Phys. 17, 301 (1978); Phys. Lett. 76B, 73 (1978); F. A. Bais, Phys. Rev. D 18, 1206 (1978).

[8]Our metric is $-g_{00}=g_{11}=g_{22}=g_{33}=1$. For further details concerning notations, etc., see L. Michel, L. O'Raifeartaigh, and K. C. Wali, Phys. Rev. D 15, 3641 (1977).

[9]E. B. Bogomol'nyi, Yad. Fiz. 24, 861 (1976) [Sov. J. Nucl. Phys. 24, 449 (1976)]; S. Coleman, S. Parke, A. Neveu, and C. M. Sommerfield, Phys. Rev. D 15, 544 (1977).

[10]L. O'Raifeartaigh and J.Rawnsley, Phys. Lett. 72B, 465 (1978).