Gauge field configurations in curved spacetimes. II

H. Boutaleb-Joutei* and A. Chakrabarti
Centre de Physique Théorique † de l'École Polytechnique, 91128 Palaiseau Cedex, France

A. Comtet
Division de Physique Théorique, ‡ Institut de Physique Nucléaire, Université Paris XI, 91406 Orsay Cedex, France

(Received 8 June 1979)

We continue the study of gauge field configurations in curved spaces, using the formalism and results of the preceding paper. A class of static, finite-action, self-dual solutions of SU(2) gauge fields on a Euclidean section of de Sitter space is presented. The action depends on a *continuous* parameter. The spin-connection solution is obtained as a particular case and a certain passage to the limiting case of a flat space is shown to reproduce the Euclidean Prasad-Sommerfield solution. The significance and possible interest of such solutions are discussed. The results are then generalized to a non-Einstein but conformally flat space, including de Sitter space as an Einstein limit. Next, Bäcklund-type transformations are constructed starting from self-duality constraints for such curved spaces. These transformations are applied to the above-mentioned solutions. The last two sections contain remarks on solutions with a background Robinson-Bertotti metric and on static, axially symmetric solutions, respectively.

I. INTRODUCTION

In the preceding paper[1] we studied classical gauge field configurations in curved spacetimes. There, among other things, we formulated the self-duality constraints for SU(2) gauge fields for static, spherically symmetric metrics (Sec. V of Ref. 1). The self-dual fields are always considered on the Euclidean section obtained via Kruskal-type coordinates (Appendix B of Ref. 1). This leads to a periodic time for such metrics and hence to the possibility of the existence of static, finite-action solutions. Moreover, for Euclidean signature the self-dual gauge fields have vanishing energy-momentum tensor and hence do not perturb the metric.

For Einstein spaces ($R_{\mu\nu}=\lambda g_{\mu\nu}$ with a constant λ) the spin connections, decomposed into 2×2 block-diagonal form, furnish an explicit example of one self-dual (and one anti-self-dual) SU(2) field.[1] The natural question arises: Can one construct other solutions of the self-duality equations?

In Sec. II we present, for positive-curvature de Sitter metric, a very simple class of finite-action, static solutions depending on a *continuous* parameter α such that the action is

$$S=8\pi^2(\alpha-1)\quad(\alpha>1). \tag{1.1}$$

The starting point, before continuation, is the metric

$$ds^2=-Ndt^2+N^{-1}dr^2+r^2d\Omega\,, \tag{1.2}$$

where

$$N=\left(1-\frac{\Lambda}{3}r^2\right). \tag{1.2$'$}$$

For $\alpha=2$ one recovers immediately the standard *spin-connection solution* as a particular case. When $\Lambda\to 0$ and $\alpha\to\infty$ such that

$$\alpha(\Lambda/3)^{1/2}\to \text{a constant } C,\ \text{say}\,, \tag{1.3}$$

one obtains, as a limiting case, the flat-space Euclidean *Prasad-Sommerfield solution* (with A_0 replacing the scalar field of Ref. 2).

From this point of view the familiar hyperbolic functions of the Prasad-Sommerfield (PS) solution are seen as a "memory" of the curved space, where they emerge naturally (see Sec. II) via the well-known variable

$$r_*=\int\frac{dr}{N}\,.$$

It will be noticed immediately that for (1.1) the Pontryagin integral is

$$P_y=\frac{S}{8\pi^2}=(\alpha-1)\quad(\alpha>1)\,. \tag{1.4}$$

Thus, instead of the discrete, integer spectrum for flat Euclidean space, one has a continuous one. This aspect is discussed at the end of Sec. II. There it is shown that, apart from factors due to integrations over angles and a time period, the entire contribution to (1.1) arises from the surface $r=(3/\Lambda)^{1/2}$ (which is desingularized in the Kruskal coordinates). *But whereas in the asymptotic surface integration (over an S_3) for flat-space instantons one has a pure gauge form of the A_μ's [leading to a Haar integral on SU(2)], here one has rather a point-dyon form as $r\to(3/\Lambda)^{1/2}$.* This is a basic difference. (Other comments can be found at the end of Sec. II.)

Evidently, (1.4) does not correspond to distinct stable vacuums. Also, it is known that for periodic time the tunneling interpretation (in analogy with flat-space instantons) is not applicable. We intend to study in a following paper the possible roles of our solutions. As an interesting possibility one should examine whether such a background potential can furnish a confining force between heavy quarks. Adler[3] has found that a Euclidean Prasad-Sommerfield background potential brings about a partial convergence of color flux lines. Our solutions must include this result as a limiting case due to (1.3). So, generalizing to nonzero curvature, one can study quarks in a background de Sitter metric (whose "strong microcosmological constant" may induce some baglike features) along with such a gauge potential which does not perturb the metric. Such a study, of course, need not be confined specifically to Adler's type of algebraic quark statics.

In Sec. III we generalize our results to a metric of the form (1.2), where now

$$N=(1-2ar-b^2r^2)\,. \qquad (1.5)$$

This is no longer an Einstein space and certain singular features arise at $r=0$ for $a\neq 0$ (see Sec. III). But we show explicitly that it is (for $a\neq 0$) still *conformally flat.* We are able to generalize our solution to include this case, and the result (1.1) is again obtained.[4]

Though we make a departure from Einstein spaces we retain spherical symmetry and conformal flatness. [In fact, given (1.2), (1.5) is even a necessary condition for the vanishing of the Weyl tensor.] Moreover, the fact that one obtains again (1.1) may indicate that non-Einstein spaces should not always be neglected in considering the consequences of metric fluctuations,[5] at least so far as effects on gauge field actions are concerned.

In Sec. IV we formulate Bäcklund-type transformations for self-duality equations in curved spacetimes given by (1.2′) or (1.5). These are applied to the static solutions mentioned above. We discuss also the special features that arise due to our choice of coordinates in formulating the self-duality equations.[1]

Remarks on solutions for the Robinson-Bertotti metric and a certain approach to static, axially symmetric solutions are given in Secs. V and VI, respectively.

II. SELF-DUAL, STATIC de SITTER SOLUTIONS

Let us consider the metric

$$ds^2=-Ndt^2+N^{-1}dr^2+r^2d\Omega\,, \qquad (2.1)$$

where

$$N=\left(1-\frac{\Lambda r^2}{3}\right)\quad(\Lambda>0)$$

and (2.1′)

$$d\Omega=(d\theta^2+\sin^2\theta\, d\phi^2)\,.$$

Passing to the Euclidean section via the Kruskal-type coordinates,[6] one gets a Euclidean time with a period

$$T=2\pi(3/\Lambda)^{1/2} \qquad (2.2)$$

and the domain $0\leq r<(3/\Lambda)^{1/2}$.

Our ansatz for the SU(2) gauge field is

$$\vec{A}=(e^{\chi}-1)\,i[\hat{\Phi},\vec{\nabla}\hat{\Phi}]\,,$$
$$A_0=\left(N\frac{d\chi}{dr}\right)\hat{\Phi}=N\chi'\hat{\Phi}\,, \qquad (2.3)$$

where

$$\hat{\Phi}=\frac{\vec{x}}{r}\cdot\frac{\vec{\tau}}{2}\equiv\hat{r}\,\frac{\vec{\tau}}{2}\,,$$

$$i[\hat{\Phi},\vec{\nabla}\hat{\Phi}]=\frac{1}{2r}(\hat{r}\times\vec{\tau})\,,$$

and χ is a function of r only $[\chi(r)]$. Let

$$r_*=\int\frac{dr}{N}\,,\quad\text{i.e.,}\quad\left(\frac{\Lambda}{3}\right)^{1/2}r=\tanh\left[\left(\frac{\Lambda}{3}\right)^{1/2}r^*\right] \qquad (2.4)$$

when

$$N\chi'=\frac{d\chi}{dr^*}\equiv\chi_*\,. \qquad (2.5)$$

On the Euclidean section, the self-duality condition can be shown to reduce to[7]

$$\frac{d^2}{dr_*^{\,2}}\chi\equiv\chi_{**}=\frac{N}{r^2}(e^{2\chi}-1)\,. \qquad (2.6)$$

Let

$$\chi=\rho+f\,, \qquad (2.7)$$

where

$$f=\int\frac{dr}{Nr}=\frac{1}{2}\ln\frac{r^2}{N}$$
$$=\ln\left[\left(\frac{3}{\Lambda}\right)^{1/2}\sinh\left(\frac{\Lambda}{3}\right)^{1/2}r_*\right]. \qquad (2.8)$$

Then from (2.6) and (2.8)

$$\rho_{**}=\frac{\Lambda}{3}e^{2\rho}\,. \qquad (2.9)$$

This is satisfied if

$$\rho=\ln\left(\frac{\alpha}{\sinh\alpha\,[(\Lambda/3)^{1/2}r_*+\beta]}\right), \qquad (2.10)$$

where α,β are continuous parameters.

We will put $\beta=0$ (which will be seen to be necessary for a finite action) and hence obtain

$$\chi=\rho+f=\ln\left(\frac{\alpha\sinh[(\Lambda/3)^{1/2}r_*]}{\sinh[\alpha(\Lambda/3)^{1/2}r_*]}\right). \tag{2.11}$$

This gives the explicit self-dual form of (2.3). For $|\alpha|<1$ the action will turn out to be divergent and the solution is invariant under $\alpha\to-\alpha$. So henceforth we will only consider

$$\alpha>1. \tag{2.12}$$

(For $\alpha=1$ one gets the trivial solution $A_\mu=0$.) For

$$\alpha=2 \tag{2.13}$$

one gets the solution corresponding to the spin connection.[8]

Again if

$$\Lambda\to 0 \text{ and } \left[\alpha\left(\frac{\Lambda}{3}\right)^{1/2}\right]\to C, \text{ a constant},$$

then

$$e^\chi\to\left(\frac{Cr}{\sinh Cr}\right)$$

and (2.14)

$$\chi_*\to\left(\frac{1}{r}-C\coth Cr\right).$$

This is the *Prasad-Sommerfield solution*[2] (with the Euclidean A_0 replacing the scalar field).

The action density can be shown[9] to be

$$S_d=\frac{1}{r^2}\frac{d}{dr}[(e^{2\chi}-1)\chi_*]. \tag{2.15}$$

After the angular and the time integrations [remembering (2.2)] the total action is then

$$S=4\pi\left[2\pi\left(\frac{3}{\Lambda}\right)^{1/2}\right]\int_0^{(3/\Lambda)^{1/2}}\left\{\frac{d}{dr}[(e^{2\chi}-1)\chi_*]\right\}dr$$
$$=8\pi^2\left(\frac{3}{\Lambda}\right)^{1/2}\int_0^\infty\left\{\frac{d}{dr_*}[(e^{2\chi}-1)\chi_*]\right\}dr_*. \tag{2.16}$$

We have

$$e^\chi=\frac{\alpha\sinh[(\Lambda/3)^{1/2}r_*]}{\sinh[\alpha(\Lambda/3)^{1/2}r_*]}$$

and

$$\chi_*=\left(\frac{\Lambda}{3}\right)^{1/2}\left\{\coth\left[\left(\frac{\Lambda}{3}\right)^{1/2}r_*\right]-\alpha\coth\left[\alpha\left(\frac{\Lambda}{3}\right)^{1/2}r_*\right]\right\}. \tag{2.17}$$

As $r_*\to 0$ (i.e., $r\to 0$) the singular terms are seen to cancel out (due to our choice $\beta=0$) and at $r_*\to\infty$ there is not divergence due to the condition $\alpha>1$.

Thus, finally

$$S=8\pi^2\left(\frac{3}{\Lambda}\right)^{1/2}[(e^{2\chi}-1)\chi_*]_{r_*=0}^{\infty}$$
$$=8\pi^2(\alpha-1). \tag{2.18}$$

Hence the Pontryagin integral

$$P_y=\frac{S}{8\pi^2}=(\alpha-1). \tag{2.19}$$

For $\alpha=2$ we thus indeed get back the spin-connection results.[8] But we see that it is possible in this space to obtain a continuous spectrum.

Let us note the following points. As

$$r\to\left(\frac{3}{\Lambda}\right)^{1/2} \text{ for } \alpha>1,$$
$$A_0\to\left(\frac{\Lambda}{3}\right)^{1/2}(1-\alpha)\hat{\Phi}=-\left(\frac{\Lambda}{3}\right)^{1/2}(\alpha-1)\hat{r}\cdot\frac{\vec{\tau}}{2},$$
$$\vec{A}\to -i[\hat{\Phi},\vec{\nabla}\hat{\Phi}]=\left(\frac{\Lambda}{3}\right)^{1/2}\frac{1}{[(\Lambda/3)^{1/2}r]}\left(\frac{\vec{\tau}}{2}\times\hat{r}\right). \tag{2.20}$$

[The factors $(\Lambda/3)^{1/2}$ can be absorbed by a change of scale.] It is thus seen that the limit $r\to(3/\Lambda)^{1/2}$ $(r_*\to\infty)$ does not correspond to a pure gauge potential as does the asymptotic limit for flat-space instantons. Here one has rather an "Euclidean point-dyon" form.

On this limiting form, if we make a gauge transformation given by

$$U=e^{-i(\Lambda/3)^{1/2}(\alpha-1)t\hat{\Phi}},$$
$$A_\mu\to UA_\mu U^{-1}+i(\partial_\mu U)U^{-1} \tag{2.21}$$

we obtain a limiting form with $A_0=0$, which is useful, in certain cases, for studying limiting contributions. The resulting, time-dependent A_μ has the periodicity of the Euclidean time $[T=2\pi(3/\Lambda)^{1/2}]$ only for integer α. Such a simple result holds only for the limiting form (2.20). The formula (2.18) for the gauge-invariant action holds for all $\alpha>1$.

Since the domains of integration are based on the criterion of positive definiteness of the metric expressed in Kruskal coordinates, let us examine χ directly in their terms.

One has, on the Euclidean section,

$$e^{-2(\Lambda/3)^{1/2}r_*}=\tfrac{1}{4}(\xi^2+\eta^2)=\zeta^2, \tag{2.22}$$

say,

$$e^{i(\Lambda/3)^{1/2}t}=\left(\frac{\eta-i\xi}{\eta+i\xi}\right)^{1/2} \tag{2.23}$$

and

$$ds^2=\frac{3}{\Lambda}\frac{1}{(1+\zeta^2)^2}[d\xi^2+d\eta^2+(1-\zeta^2)^2d\Omega]. \tag{2.24}$$

One obtains

$$\left(\frac{\Lambda}{3}\right)^{1/2} r = \left(\frac{1-\zeta^2}{1+\zeta^2}\right) \tag{2.25}$$

as

$$r \to 0, \quad r_* \to 0, \quad \zeta \to 1$$

and as

$$r \to (3/\Lambda)^{1/2}, \quad r_* \to \infty, \quad \zeta \to 0. \tag{2.26}$$

Thus one has

$$e^{\chi} = \alpha \zeta^{(\alpha-1)} \frac{(1-\zeta^2)}{(1-\zeta^{2\alpha})}. \tag{2.27}$$

Setting $\zeta = 1-\epsilon$, e^{χ} and χ_* are seen to be free from divergence as $\epsilon \to 0$. If ζ is made complex, e^{χ} has branch points for noninteger α.

We have pointed out two rather different contexts [periodicity of the transformed A_μ on $r=(3/\Lambda)^{1/2}$ and complexification], where integral values of α lead to distinguishing features. We are unable at present to make more precise the possible significances, if any, of these facts.

Finally, one may compare the result (2.19) to the result obtained, in a previous work, by singular transformations of meron solutions.[10] The situation here is, however, quite different, since we are dealing with finite-action solutions. The result (2.29) depends crucially on the fact that the integration not only on θ and ϕ but also on the time period factors out trivially reducing the rest to an integration on a single variable r or r_*.

III. A GENERALIZATION

Let us consider the metric

$$ds^2 = -N dt^2 + N^{-1} dr^2 + r^2 d\Omega,$$

where

$$N = (1 - 2ar - b^2 r^2). \tag{3.1}$$

For $a=0$, we get back the de Sitter case (Sec. II), setting $b=(\Lambda/3)^{1/2}$. For $a \neq 0$, we no longer have an Einstein space and some new, singular, features arise. But the property of being *conformally flat* persists. In fact, defining

$$t_\pm \equiv t \pm \int \frac{dr}{N} \equiv t \pm r_*, \quad \tau_\pm \equiv \tau \pm \rho$$
$$k \equiv (a^2+b^2)^{1/2}, \tag{3.2}$$

and setting

$$\tau_+ = \left[\frac{\alpha(k-a)e^{kt_+} - \beta(k+a)}{\gamma(k-a)e^{kt_+} + \delta(k+a)}\right], \tag{3.3}$$

$$\tau_- = \left[\frac{\alpha e^{kt_-} - \beta}{\gamma e^{kt_-} + \delta}\right], \tag{3.4}$$

where $\alpha, \beta, \gamma, \delta$ are constants such that

$$\alpha\delta + \beta\gamma \neq 0, \tag{3.5}$$

one obtains

$$ds^2 = \frac{r^2}{\rho^2}(-d\tau^2 + d\rho^2 + \rho^2 d\Omega) \tag{3.6}$$

with

$$\rho = (\alpha\delta + \beta\gamma) \frac{e^{kt}(k \sinh kr_* - a \cosh kr_*)}{[\gamma(k-a)e^{kt_+} + \delta(k+a)](\gamma e^{kt_-} + \delta)}. \tag{3.7}$$

This generalizes the formula (4.1) of Ref. 1. Let us note that for (3.1) $[N' \equiv (d/dr)N]$,

$$G_0^0 = G_r^r = \frac{N'}{r} + \frac{1}{r^2}(N-1) = -\left(\frac{4a}{r} + 3b^2\right),$$

$$G_\theta^\theta = G_\phi^\phi = \frac{1}{2}\left(N'' + \frac{2N'}{r}\right) = -\left(\frac{2a}{r} + 3b^2\right),$$

and

$$R = -\left(N'' + \frac{4N'}{r} + \frac{2(N-1)}{r^2}\right) = 12\left(\frac{a}{r} + b^2\right). \tag{3.8}$$

Thus, the scalar curvature now diverges at the origin (for $a \neq 0$, at $r=0$). Also, the derivatives of N with respect to the cartesian components are not well defined at $r=0$, though N itself is not singular at the origin.

However, a solution of the type (2.11) can be still obtained formally. Generalizing the techniques of Sec. II for $a \neq 0$, it can be shown that corresponding to (2.11) one now should take (instead of $\beta=0$) $\beta=\omega$, where

$$\tanh\omega = \frac{a}{(a^2+b^2)^{1/2}} \equiv \frac{a}{k} \tag{3.9}$$

and finally obtain

$$\chi = \ln\left(\frac{\alpha \sinh(kr_* - \omega)}{\sinh\alpha(kr_* - \omega)}\right). \tag{3.10}$$

It is useful to note that (3.1) leads to

$$\frac{r^2}{N} = \frac{1}{k^2}\sinh^2(kr_* - \omega) \quad \left(r_* = \int \frac{dr}{N}\right), \tag{3.11}$$

and as $r \to 0$, $kr_* \to \omega$.

The action density is calculated as before. To obtain the time periodicity let us note that, considering the case $a>0$ and $b>0$,

$$N = (1 - 2ar - b^2r^2) = (r_1 - br)(r_2 + br), \tag{3.12}$$

where

$$r_1 = r_2^{-1} = \left(\frac{k-a}{k+a}\right)^{1/2} = e^{-\omega}$$

with $k=(a^2+b^2)^{1/2}$ and $\tanh\omega = a/k$. Also,

$$e^{2kr_*} = \left(\frac{r_2 + br}{r_1 - br}\right) \quad \left(\frac{\omega}{k} < r_* < \infty\right). \tag{3.13}$$

Hence

$$Ne^{-2cr_*}=(r_1-br)^{1+c/k}(r_2+br)^{1-c/k}. \qquad (3.14)$$

Choosing

$$c=-k,$$

the singularity at $r=r_1/b$ is eliminated and the Kruskal coordinates are then (after passage to the Euclidean section)

$$e^{-2kr_*}=\tfrac{1}{4}(\eta^2+\xi^2), \quad e^{ikt}=\left(\frac{\eta-i\xi}{\eta+i\xi}\right)^{1/2}. \qquad (3.15)$$

Hence the Euclidean t now has a period

$$T=\frac{2\pi}{k}=\frac{2\pi}{(a^2+b^2)^{1/2}}. \qquad (3.16)$$

Integrating over the relevant domains of r_* and t, one again obtains

$$S=8\pi^2(\alpha-1) \quad (\alpha\geq 1). \qquad (3.17)$$

[Since the metric is not well behaved at $r=0$ one may define the action as the limit $\delta\to 0$ after integrating down to a lower bound δ. The result (3.17) is obtained.] Setting $a=0$ all the results of Sec. II are obtained immediately.

The case $b=0$ is best studied directly by starting with

$$N=(1-2ar)=e^{-2ar_*} \quad \left(r_*=\int\frac{dr}{N}\right). \qquad (3.18)$$

The calculations are quite simple. One obtains the solution

$$\chi=\ln\left(\frac{\alpha\sinh(ar^*)}{\sinh(\alpha ar^*)}\right). \qquad (3.19)$$

With a time period $T=2\pi/a$, one again obtains

$$S=8\pi^2(\alpha-1). \qquad (3.20)$$

IV. BÄCKLUND TRANSFORMATIONS

In this section we will generalize the Bäcklund-type transformations found by Corrigan *et al.*[11] to self-dual solutions in curved spacetimes of Sec. III, namely to a metric (on the Euclidean section):

$$ds^2=N\,dt^2+N^{-1}dr^2+r^2d\Omega,$$

where

$$N=(1-2ar-b^2r^2). \qquad (4.1)$$

(If one chooses to restrict oneself to the de Sitter case with $a=0$, the discussion remains formally unaltered.)

The notation of Ref. 1 will be utilized. In Yang's R gauge[12] we write

$$\lambda A_\mu=-i\begin{pmatrix}-\frac{\lambda_\mu}{2} & 0\\ \rho_\mu & \frac{\lambda_\mu}{2}\end{pmatrix},$$
$$\lambda A_{\bar\mu}=i\begin{pmatrix}-\frac{\lambda_{\bar\mu}}{2} & \bar\rho_{\bar\mu}\\ 0 & \frac{\lambda_{\bar\mu}}{2}\end{pmatrix}, \qquad (4.2)$$

where

$$\mu=y,z,\quad \bar\mu=\bar y,\bar z \quad (\lambda_\mu\equiv\partial_\mu\lambda),$$
$$y=\tan\tfrac{1}{2}\theta e^{i\phi},\quad \bar y=\tan\tfrac{1}{2}\theta e^{-i\phi}, \qquad (4.3)$$
$$z=\tfrac{1}{2}(r_*+it),\quad \bar z=\tfrac{1}{2}(r_*-it)\quad \left(r_*=\int\frac{dr}{N}\right).$$

Here t is the Euclidean time and θ, ϕ are the spherical angles. We will restrict ourselves to real (t,r,θ,ϕ). Then λ is real and $\bar\rho$ is the complex conjugate of ρ for Hermitic A_μ's.

(*i*) β *transformations*. We now adapt the β transformations of Ref. 11 to spherical coordinates and generalize them to

$$\rho_y/\lambda^2=\left(\frac{r^2}{N}\right)\bar\eta_{\bar z},\quad \rho_z/\lambda^2=-(1+y\bar y)^2\bar\eta_{\bar y},$$
$$\bar\rho_{\bar y}/\lambda^2=\left(\frac{r^2}{N}\right)\eta_z,\quad \bar\rho_{\bar z}/\lambda^2=-(1+y\bar y)^2\eta_y, \qquad (4.4)$$

and

$$\lambda^2(1+y\bar y)^2\left(\frac{r^2}{N}\right)=\zeta^{-2}.$$

We maintain the convention that $\bar\eta$ is the complex conjugate of η.

It can be shown that if $(\lambda,\rho,\bar\rho)$ of (4.2) satisfies the self-duality constraints[13]

$$g^{\mu\bar\mu}\left[\frac{1}{\lambda^2}(\ln\lambda)_{\mu\bar\mu}+\left(\frac{\rho_\mu}{\lambda^2}\right)\left(\frac{\bar\rho_{\bar\mu}}{\lambda^2}\right)\right]=0,$$
$$g^{\mu\bar\mu}\left(\frac{\rho_\mu}{\lambda^2}\right)_{\bar\mu}=0, \qquad (4.5)$$
$$g^{\mu\bar\mu}\left(\frac{\bar\rho_{\bar\mu}}{\lambda^2}\right)_\mu=0,$$

then $(\zeta,\eta,-\bar\eta)$ satisfies [for N given by (4.1)] the same equations. Namely,

$$\zeta A_\mu=-i\begin{pmatrix}-\frac{\zeta_\mu}{2} & 0\\ \eta_\mu & \frac{\zeta_\mu}{2}\end{pmatrix},$$
$$\zeta A_{\bar\mu}=i\begin{pmatrix}-\frac{\zeta_{\bar\mu}}{2} & -\bar\eta_{\bar\mu}\\ 0 & \frac{\zeta_{\bar\mu}}{2}\end{pmatrix} \qquad (4.6)$$

satisfy the self-duality conditions.

The *violation of the reality constraints* is exhibited explicitly by the negative sign before $\bar{\eta}$. It is easily verified that, as for the flat case, (4.6) represents an SU(1, 1) gauge field with real components.

(ii) γ transformations. The algebraic γ transformation[11] remains unmodified. In our notations we define the matrix

$$P=\begin{pmatrix}\rho & \lambda\\ \lambda & -\bar{\rho}\end{pmatrix}. \tag{4.7}$$

Let

$$Q=(aP+b)(cP+d)^{-1}, \tag{4.8}$$

where a, b, c, d are diagonal matrices, namely

$$a=\begin{pmatrix}a_1 & 0\\ 0 & a_2\end{pmatrix}$$

and so on, satisfying

$$(ad-bc)=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}. \tag{4.9}$$

It can be shown that $(\eta, \bar{\eta}, \zeta)$ where

$$Q=\begin{pmatrix}\eta & \zeta\\ \zeta & -\bar{\eta}\end{pmatrix} \tag{4.10}$$

again satisfies the self-duality constraints corresponding to (4.5). This is perhaps easiest to see as follows. Let us first consider a simple inversion,

$$Q=P^{-1}. \tag{4.11}$$

In the Hermitian K gauge of Yang,[12] considering the gauge transformation properties of K^+K (which coincides with K^2) and its connection with the R-gauge parameters, one finds that a simple gauge rotation (π) about the iso 1-axis corresponds to the inversion (4.11). This is easily seen [in terms of the $\tau_{(i)}$ parameters of Yang] from such relations as

$$\frac{1-\tau_{(3)}}{1+\tau_{(3)}}=(\rho\bar{\rho}+\lambda^2)=\frac{1}{(\eta\bar{\eta}+\zeta^2)}. \tag{4.12}$$

Thus (4.11) is evidently a symmetry.

Now, *when (4.9) is satisfied*, one has

$$\begin{aligned}Q&=(aP+b)(cP+d)^{-1}\\ &=ac^{-1}-c^{-1}(P+c^{-1}d)^{-1}c^{-1}.\end{aligned} \tag{4.13}$$

Now along with (4.11), evident symmetries [of (4.2)] in the R gauge such as constant translations of $\rho, \bar{\rho}$ and dilations such as[14]

$$(\rho, \bar{\rho}, \lambda)\to(\alpha_1\rho, \alpha_2\bar{\rho}, (\alpha_1\alpha_2)^{1/2}\lambda) \tag{4.14}$$

establish γ. In fact, even the restriction (4.1) on the explicit form of N need not be invoked for γ.

(iii) α transformations. Combining the β and γ transformations one can now obtain the α transformations[11] as

$$\begin{aligned}&(\rho_y/\lambda^2)=\left(\frac{r^2}{N}\right)\left(\frac{\eta}{\eta\bar{\eta}+\zeta^2}\right)_{\bar{z}},\\ &\left(\frac{\rho_{\bar{z}}}{\lambda^2}\right)=-(1+y\bar{y})^2\left(\frac{\eta}{\eta\bar{\eta}+\zeta^2}\right)_{\bar{y}},\\ &(\bar{\rho}_{\bar{y}}/\lambda^2)=\left(\frac{r^2}{N}\right)\left(\frac{\bar{\eta}}{\eta\bar{\eta}+\zeta^2}\right)_z,\\ &\left(\frac{\bar{\rho}_{\bar{z}}}{\lambda^2}\right)=-(1+y\bar{y})^2\left(\frac{\bar{\eta}}{\eta\bar{\eta}+\zeta^2}\right)_y,\\ &\lambda^2(1+y\bar{y})^2\left(\frac{r^2}{N}\right)=\left(\frac{\zeta}{\eta\bar{\eta}+\zeta^2}\right)^{-2}.\end{aligned} \tag{4.15}$$

(iv) Transformations of the static self-dual solutions. Let us illustrate the effects of these transformations by starting with the solutions of Sec. II and III. These solutions are already in the K gauge.[1,12]

In the R gauge one obtains

$$\begin{aligned}&\lambda=\frac{e^{\chi}(1+y\bar{y})}{(e^{2\chi}+y\bar{y})},\\ &\rho=\frac{y(e^{2\chi}-1)}{(e^{2\chi}+y\bar{y})},\quad \bar{\rho}=\frac{\bar{y}(e^{2\chi}-1)}{(e^{2\chi}+y\bar{y})}.\end{aligned} \tag{4.16}$$

As a check one may note at this point that the action density for (4.16) turns out to be,[15] including the factor $|g|^{1/2}$,

$$S_d=\frac{2[(e^{\chi}-1)\chi_*]_*}{(1+y\bar{y})^2}. \tag{4.17}$$

Now

$$S=\int S_d\, dy_1dy_2dr^*dt,$$

where

$$y_1=\tan\tfrac{1}{2}\theta\cos\phi,\quad y_2=\tan\tfrac{1}{2}\theta\sin\phi.$$

Using the proper limits and the periodicity of t gives again

$$S=8\pi^2(\alpha-1)\quad(\alpha>1)$$

as it should.

Let us now perform a transformation β on (4.16). One obtains

$$\begin{aligned}&\eta=\bar{\eta}=\frac{\chi_*}{(1+y\bar{y})^2},\\ &\zeta=\frac{(e^{2\chi}+y\bar{y})e^{-\chi}}{(1+y\bar{y})^2}\left(\frac{N}{r^2}\right)^{1/2},\end{aligned} \tag{4.18}$$

where (r^2/N) is given by (3.11) [or (2.8) for $a=0$].

Let us now perform a particular γ transformation γ_e, defined as (with a constant e)

$$\begin{pmatrix} \eta & \zeta \\ \zeta & -\bar{\eta} \end{pmatrix} \xrightarrow{\gamma_e} \begin{pmatrix} \eta + ie & \zeta \\ \zeta & -(\bar{\eta} - ie) \end{pmatrix} \frac{1}{(\eta^2 + \zeta^2 + e^2)} \equiv \begin{pmatrix} \xi & \omega \\ \omega & -\bar{\xi} \end{pmatrix}, \tag{4.19}$$

such that

$$\xi = \frac{\eta + ie}{\eta^2 + \zeta^2 + e^2}, \quad \omega = \frac{\zeta}{\eta^2 + \zeta^2 + e^2}.$$

(For $e = 0$ we get a simple inversion.)

Let us now perform another β transformation:

$$\begin{pmatrix} \xi & \omega \\ \omega & -\bar{\xi} \end{pmatrix} \xrightarrow{\beta} \begin{pmatrix} K & \epsilon \\ \epsilon & -\bar{K} \end{pmatrix} \tag{4.20}$$

such that

$$\bar{K}_{\bar{z}} = \left(\frac{N}{r^2}\right)\left(\frac{\xi_y}{\omega^2}\right), \quad \bar{K}_{\bar{y}} = -(1 + y\bar{y})^2 \left(\frac{\xi_z}{\omega^2}\right) \tag{4.21}$$

with complex-conjugate relations for K_z and K_y and

$$\epsilon^2 = \left[(1 + y\bar{y})^2 \left(\frac{r^2}{N}\right) \omega^2\right]^{-1}. \tag{4.22}$$

The transformation $(\beta\gamma_e\beta)$ restores the reality conditions.

The action still seems to be divergent, though we have not fully analyzed the complicated expressions involved. Instead of attempting it we now propose to illustrate certain striking features of our transformations through a very simple but somewhat artificial example.

Instead of Cartesian coordinates[11, 12] we have used (4.3) to formulate the self-duality constraints and the transformations. [Equations (4.16) and (4.17) show how one can treat well-behaved solutions in these coordinates though y and $\bar{y} \to \infty$ as $\theta \to \pi$.] These coordinates were introduced to exploit the spherical symmetry of the metrics considered. Now we will show how this involves notable differences as compared to the Cartesian choice.

Let us start with the *trivial* case

$$\rho = \bar{\rho} = 0, \quad \lambda = \lambda_0 \text{ a constant.} \tag{4.23}$$

Evidently,

$$A_\mu = 0, \quad F_{\mu\nu} = 0.$$

Let us now apply the transformation β (4.4). We obtain as a solution

$$\eta = \bar{\eta} = 0, \quad \zeta^{-2} = \lambda_0^2 (1 + y\bar{y})^2 \frac{r^2}{N}. \tag{4.24}$$

In fact, one may set, instead of zero,

$$\bar{\eta} = f(y, z) \quad (f_{\bar{y}} = f_{\bar{z}} = 0)$$

and

$$\eta = \bar{f}(\bar{y}, \bar{z}). \tag{4.25}$$

But this leaves the transformed A_μ's unaffected. They are now

$$A_\mu = i(\ln\zeta)_\mu \frac{\tau_3}{2}, \quad A_{\bar{\mu}} = -i(\ln\zeta)_{\bar{\mu}} \frac{\tau_3}{2} \quad (\mu = y, z) \tag{4.26}$$

and the nonzero components of $F_{\mu\nu}$ are

$$F_{\mu\bar{\nu}} = i(2\ln\zeta)_{\mu\bar{\nu}} \frac{\tau_3}{2}. \tag{4.27}$$

Thus [for $N = (1 - 2ar - b^2 r^2)$]

$$F_{y\bar{y}} = i \frac{2}{(1 + y\bar{y})^2} \frac{\tau_3}{2}, \quad F_{z\bar{z}} = -i\left(\frac{2N}{r^2}\right)\frac{\tau_3}{2}. \tag{4.28}$$

Since $\eta = \bar{\eta} = 0$, the reality conditions are not violated in this Abelian example.

The action density[15] is, including the factor $|g|^{1/2}$,

$$S_d = 2\,\mathrm{Tr} F_{y\bar{y}} F_{z\bar{z}} = \frac{4N}{r^2(1 + y\bar{y})^2}. \tag{4.29}$$

Thus *even for flat space* ($N = 1$) we obtain a nonzero $F_{\mu\nu}$ (with however, evidently, a divergent action).

For the "Cartesian choice," one has instead of (4.24) (for flat-space transformations of Ref. 11) simply

$$\zeta = \lambda_0^{-1}, \tag{4.30}$$

and hence even after β,

$$A_\mu = F_{\mu\nu} = 0. \tag{4.31}$$

The difference is now manifest. There is no equivalence between (4.28) and (4.31).

For curved spaces with a time period T on the Euclidean section, (4.29) leads to a total action

$$S = 16\pi T \int \frac{dr}{r^2}. \tag{4.32}$$

Thus, it still diverges linearly at $r = 0$. If

$$N = (r_0 - r)(r_0^{-1} - r) \quad (r_0 \text{ a positive constant}) \tag{4.33}$$

(for which the results still hold formally) the point $r = 0$, is avoided by considering the region

$$r_0 > r > r_0^{-1} \quad (\text{for } r_0 > 1, \text{ say}). \tag{4.34}$$

But in this region N is negative, so that to start with t is no longer a timelike Killing vector. We will not further consider such a possibility here, though in case 2 of Sec. V a similar change of sign is present.

To compare with (4.22), we note that the sequence of transformations $(\beta\gamma_0\beta)$, where γ_0 is a pure inversion, leads to

$$K=\overline{K}=0\,,\quad \epsilon=\lambda_0\xi^2\,, \tag{4.35}$$

doubling the action density. Such a sequence can be repeated and in fact a more general multiplicative factor is possible trivially due to the Abelian nature of the solution. Replacing γ_0 by γ_e [see (4.19)] does not change the action density.

As in (2.14) it would probably be interesting to consider the PS limit after the transformation $(\beta\gamma_e\beta)$, i.e., after arriving at (4.21) and (4.22). But a comparison with such a work as that of Löhe[16] (where the scalar field replaces the Euclidean A_0) is not easy. We have already emphasized nontrivial differences between "Cartesian" and "spherical" formulations.

V. REMARKS ON THE ROBINSON-BERTOTTI METRIC

Certain special features of the RB metric were already noted in Sec. VI of Ref. 1 (where other references are quoted). Since again we have conformal flatness, solutions similar to those of the preceding sections can be formally written down. But firstly, conformal equivalence is not the whole story (the domains of the coordinates in each case finally determining the values of the invariants, such as the action). Besides we would like to compare the somewhat different situations in the two regions

$$|x|>1 \text{ and } |x|<1\,, \tag{5.1}$$

where

$$ds^2=Q^2[-(x^2-1)dt^2+(x^2-1)^{-1}dx^2+d\Omega]\,. \tag{5.2}$$

For these reasons we briefly state certain results.

Case 1 ($|x|>1$). Setting

$$x=1/r\,, \tag{5.3}$$

we have from (5.1)

$$ds^2=\frac{Q^2}{r^2}[-(1-r^2)dt^2+(1-r^2)^{-1}dr^2+r^2d\Omega] \tag{5.4}$$

with a corresponding expression for the Maxwell field. [Compare this to (6.1)–(6.4) of Ref. 1.]

Comparing (5.4) and (2.1), the relation to the de Sitter metric is displayed in the most explicit fashion and the necessary results can be deduced in this way [including a formula of the type of (4.1) of Ref. 1].

In terms of the variable x, directly defining

$$x_*=-\int\frac{dx}{x^2-1} \tag{5.5}$$

one has

$$x=\coth x_*\,, \tag{5.6}$$

$$N=(x^2-1)=(\sinh x_*)^{-2}\,. \tag{5.7}$$

Defining

$$e^{2cx_*}=\left(\frac{x+1}{x-1}\right)^c=\tfrac{1}{4}(-\xi^2+\eta^2)\,, \tag{5.8}$$

$$e^{ct}=\left(\frac{\eta+\xi}{\eta-\xi}\right)^{1/2} \tag{5.9}$$

and setting

$$c=-1 \tag{5.10}$$

removes the singularity at $x=1$.

Continuing to $t\to-it$, $\xi\to-i\xi$, $Q\to-iQ$ [see (6.19) of Ref. 1], one has

$$ds^2=-Q^2\left(\frac{(1+x)^2}{4}(d\xi^2+d\eta^2)+d\Omega\right) \tag{5.11}$$

with a period 2π for t. With an ansatz of the form (2.3) (Ref. 17) one is led again to [with $b=e^{\chi}$, $d=(d/dx^*)\chi\equiv\chi_*$ in Ref. 17]

$$\chi_{**}=(\sinh x_*)^{-2}(e^{2\chi}-1) \tag{5.12}$$

and hence to

$$e^{\chi}=\frac{\alpha\sinh x_*}{\sinh\alpha x_*}\,. \tag{5.13}$$

For $1\leqslant x<\infty$ one again obtains

$$S=8\pi^2(\alpha-1)\quad(\alpha>1)\,. \tag{5.14}$$

Including the domain $-1-\delta>x>-\infty$ $(\delta>0)$ and taking the limit $\delta\to0$ for the action integral, the expression (5.14) is simply doubled.

Case 2 ($|x|<1$). Here one can write

$$ds^2=Q^2[(1-x^2)dt^2-(1-x^2)^{-1}dx^2+d\Omega]\,. \tag{5.15}$$

Defining

$$x_*=\int\frac{dx}{1-x^2}\,, \tag{5.16}$$

$$x=\tanh x_*\,,\quad N=(1-x^2)=(\cosh x_*)^{-2}\,. \tag{5.17}$$

As was already noted in Ref. 1, for $t\to-it$, $Q\to-iQ$

$$ds^2=Q^2[(1-x^2)dt^2+(1-x^2)^{-1}dr^2-d\Omega)] \tag{5.18}$$

$$=Q^2\left[\frac{(1+x^2)}{4}(d\eta^2+d\xi^2)-d\Omega\right] \tag{5.19}$$

in terms of Kruskal-type coordinates.[18]

It can be shown that due to the signs $(++--)$ in (5.19) and the absence of a factor x^2 before $d\Omega$, the self-duality constraints (5.4) of Ref. 1 now become (with an evident generalization of the ansatz of Ref. 17)

$$\begin{aligned}(\dot{a}-db)&=N(b'+ca)\,,\\(\dot{b}+da)&=-N(a'-cb)\,,\\(\dot{c}-d')&=-(a^2+b^2-1)\,.\end{aligned}\tag{5.20}$$

(Note the change of sign in the first two equations and the absence of a factor $1/r^2$ in the third.)

We will thus have for our particular case ($a=c=0$ and vanishing time derivatives)

$$\begin{aligned}db&=-Nb'\,,\\d'&=(b^2-1)\,.\end{aligned}\tag{5.20$'$}$$

We set

$$b=e^{\chi}\quad\text{and}\quad d=-N\chi'=-\chi_*\tag{5.21}$$

where N and x_* are given by (5.17). One is led to the equation

$$\chi_{**}=-(\cosh x_*)^{-2}(e^{2\chi}-1)\tag{5.22}$$

with the solution

$$e^{\chi}=\left(\frac{\alpha\cosh x_*}{\cosh\alpha x_*}\right).\tag{5.23}$$

As a check we have verified directly the second-order equations of motion.

It can be shown that for $(0\leqslant x\leqslant 1)$ the action is again

$$S=8\pi^2(\alpha-1)\quad(\alpha>1)\,,\tag{5.24}$$

provided it is defined as

$$S=-\tfrac{1}{2}\,\mathrm{Tr}\int(|g|)^{1/2}F_{\mu\nu}F^{\mu\nu}d^4x\,.\tag{5.25}$$

(Note the sign.) This is again an effect of the signature $(++--)$ of (5.18) and (5.19). Including the domain $(0\geqslant x>-1)$ again doubles the action. It may be verified directly that $T_{\mu\nu}$ is again zero for self-dual solutions for this signature.

It should also be remarked that the spin-connection solution [(6.12) of Ref. 1 and the ensuing discussion] is not included in the class of solutions obtained here. This is a notable difference as compared to the de Sitter case. Even for $|x|>1$ the effect of the extra conformal factor in (5.4) thus manifests itself.

VI. REMARKS ON STATIC, AXIALLY SYMMETRY SOLUTIONS

Witten has drawn attention[19] to the relation between static, axially symmetric solutions for SU(2) gauge fields in *flat* Euclidean space and static, axially symmetric Einstein metrics.

In our opinion, static, self-dual solutions are of particular interest in curved spacetimes. So, maintaining a close analogy with Witten's work, we will briefly note certain features which arise for our generalization (4.5) of Yang's equations, when one looks for axially symmetric solutions.

We will again assume [as in (2.1)]

$$ds^2=-N\,dt^2+N^{-1}dr^2+r^2d\Omega\tag{6.1}$$

but we will not restrict ourselves to the de Sitter case [(2.1$'$) or its generalization (4.1)]. In fact, eventually, as an illustration, we will consider the Schwarzschild case. We will assume in what follows the continuation of (2.1) to the *Euclidean section*, as discussed before.

Let us note to start with that in (4.5) if we set

$$\rho=\sigma e^{i\alpha}\tag{6.2}$$

with a constant α and assume that λ *and* σ *depend on* r_* *and* θ *only*, the equations reduce to

$$\frac{\lambda}{N}\frac{\partial^2\sigma}{\partial r_*{}^2}+\frac{\lambda}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\sigma}{\partial\theta}\right)-\frac{2}{N}\frac{\partial\sigma}{\partial r_*}\frac{\partial\lambda}{\partial r_*}-\frac{2}{r^2}\frac{\partial\sigma}{\partial\theta}\frac{\partial\lambda}{\partial\theta}=0\tag{6.3}$$

and

$$\frac{\lambda}{N}\frac{\partial^2\lambda}{\partial r_*{}^2}+\frac{\lambda}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\lambda}{\partial\theta}\right)-\frac{1}{r^2}\left(\frac{\partial\lambda}{\partial\theta}\right)^2-\frac{1}{N}\left(\frac{\partial\lambda}{\partial r_*}\right)^2+\frac{1}{r^2}\left(\frac{\partial\sigma}{\partial\theta}\right)^2+\frac{1}{N}\left(\frac{\partial\sigma}{\partial r_*}\right)^2=0\,.\tag{6.4}$$

Corresponding to Witten's Eq. (8), setting

$$\lambda=\frac{1}{\cosh2\psi+\cos\beta\sinh2\psi}\tag{6.5}$$

and

$$\sigma=\frac{-\sin\beta\sinh2\psi}{\cosh2\psi+\cos\beta\sinh2\psi}\,,$$

Eqs. (6.3) and (6.4) reduce to a single equation, namely

$$\frac{1}{N}\frac{\partial^2\psi}{\partial r_*{}^2}+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)=0\,.\tag{6.6}$$

(In fact, one may set $\beta=0$. A nonzero β presumably represents the effect of a gauge transformation. Thus, this discussion will be restricted to effectively Abelian solutions.)

Setting

$$\psi=f_l(r)Y_{l0}(\theta)\tag{6.7}$$

we obtain

$$r^2\frac{d}{dr}\left(N\frac{df_l}{dr}\right)-l(l+1)f_l=0\,.\tag{6.8}$$

As an illustration, the choice

$$N=\left(1-\frac{2M}{r}\right)\equiv\left(1-\frac{1}{\rho}\right) \quad \text{(Schwarzschild)} \tag{6.9}$$

leads to

$$\rho(1-\rho)\frac{d^2f}{d\rho^2}-\frac{df}{d\rho}+l(l+1)f=0\,. \tag{6.10}$$

For $l>1$ the two independent solutions are, in terms of hypergeometric functions,

$$f_l=\rho^{-l}F[l+2,\,l,\,2l+2,\,1/\rho] \tag{6.11}$$

and

$$\tilde{f}_l=\rho^2F[-l+1,\,l+2,\,3,\,\rho]\,. \tag{6.12}$$

The polynomial-type solution $\tilde{f}$ is evidently not suitable for the exterior region.

On the other hand, we note that formally setting $l=1$. The series f_1 can easily be summed and we will now examine this relatively simple case as an illustration. One obtains

$$f_1=-\rho^2\ln\left(1-\frac{1}{\rho}\right)-\rho-\tfrac{1}{2} \tag{6.13}$$

and this can be directly verified to be a solution.

The action density [Eq. (5.37) of Ref. 1] becomes for (6.5), independently of β,

$$S_d=16\left(\frac{\partial^2\psi}{\partial y\partial\bar{z}}\frac{\partial^2\psi}{\partial\bar{y}\partial z}-\frac{\partial^2\psi}{\partial y\partial\bar{y}}\frac{\partial^2\psi}{\partial z\partial\bar{z}}\right). \tag{6.14}$$

The total action is (transforming to the coordinates t, r, θ, ϕ)

$$S=\tfrac{1}{4}\int S_d\,\frac{\sin\theta}{\cos^4\theta/2}\,d\theta\,d\varphi\,\frac{dr}{N}\,dt\,. \tag{6.15}$$

For $l=1$,

$$S_d=16\cos^4\frac{\theta}{2}\,\frac{1}{4M^2}\times\left(\left(\frac{df_1}{d\rho}\right)^2\left(1-\frac{1}{\rho}\right)^2\sin^2\theta+\cos^2\theta\left(1-\frac{1}{\rho}\right)\rho^2\left\{\frac{d}{d\rho}\left[\left(1-\frac{1}{\rho}\right)\frac{df_1}{d\rho}\right]\right\}\right). \tag{6.16}$$

The integrations over the (periodic) time and θ and φ can be performed easily and involve no divergence. The problem is thus reduced finally to the two radial integrals

$$I_1=\int_1^\infty\left(\frac{df_1}{d\rho}\right)^2\left(1-\frac{1}{\rho}\right)d\rho \tag{6.17}$$

and

$$I_2=\int_1^\infty\rho^2\left\{\frac{d}{d\rho}\left[\left(1-\frac{1}{\rho}\right)\frac{df_1}{d\rho}\right]\right\}^2d\rho\,. \tag{6.18}$$

It can be shown that I_2 converges while I_1 *diverges logarithmically* at the bound $\rho=1$. Thus, our simple explicit solution is a divergent one, but only logarithmically so.

We intend to explore elsewhere more adequately the possibilities of axial symmetry.

VII. DISCUSSION

Setting $\alpha=2$ in (2.20), i.e., for the spin-connection solution, we obtain that as $r\to(3/\Lambda)^{1/2}$,

$$A_0\to-\frac{2\pi}{T}\hat{\Phi}$$

and (7.1)

$$\vec{A}\to-i[\hat{\Phi},\vec{\nabla}\hat{\Phi}]\,,$$

where T is the time period.

This feature is more general. We have studied in detail in Ref. 8 (see also Appendix B of Ref. 1) the case

$$N=\left(1-\frac{2M}{r}-\frac{\Lambda}{3}r^2\right). \tag{7.2}$$

For $M=0$ we get (7.1). For $\Lambda=0$, the Schwarzschild case, we again get, as $r\to 2M$, for A_μ obtained through the spin connections,

$$A_0\to\frac{2\pi}{T}\hat{\Phi}\,,\quad \vec{A}\to-i[\hat{\Phi},\vec{\nabla}\hat{\Phi}]\quad(T=8\pi M)\,. \tag{7.3}$$

For the general case (7.2) there are two horizons[8] and the Pontryagin index is noninteger. An associated feature is that (7.3) (with the appropriate T) holds at the horizon regularized by the choice of Kruskal-type coordinates but not at the other. Hence passing to the temporal gauge ($A_0=0$) in a way analogous to (2.21) one cannot have A_μ simultaneously periodic on both horizons. The other case with integral index ($9M^2\Lambda=1$) is precisely the one where the two horizons coincide. The absence of an extra singular horizon was taken in Ref. 8 as a "quantization condition" for the index. Here we see that periodicity (for $A_0=0$) at each surface present might also have been taken as an equivalent condition. In (2.21) we found an analogous condition in a different situation with an extra parameter (generalizing the spin-connection solution) instead of an extra singular horizon. But we repeat that one obtains such a criterion only by restricting ones attention to the surface $r=(3/\Lambda)^{1/2}$. To obtain $A_0=0$ throughout one has to replace (2.21) by

$$U=e^{i\chi_* t\Phi}\,. \tag{7.4}$$

The transformed A_μ's cannot in any case be simultaneously periodic throughout.

Finally, we would like to recapitulate and reemphasize briefly certain points concerning the interest of solutions in de Sitter space. Certain exchanges of views have persuaded us that, in doing so, we may not be uselessly belaboring the obvious. (See also Ref. 4.) One may ask, since the conformal invariance of the gauge fields assure a corresponding solution in flat space, why not work there directly? Without going into detail we note that a transformation such as (4.1) of Ref. 1 permits us to convert (2.11) to the corresponding flat-space solution. (Only then the conformal properties of gauge fields will assure flat-space self-duality.) This involves the substitution $(t, r) \to (\tau, \rho)$ say, where in particular

$$\left(\frac{3}{\Lambda}\right)^{1/2} r_* = \tfrac{1}{2} \ln\left[\left(\frac{b+d(\tau+\rho)}{a-c(\tau+\rho)}\right)\left(\frac{a-c(\tau-\rho)}{b+d(\tau-\rho)}\right)\right], \quad (7.5)$$

where $ad+bc \neq 0$; $a=b=c=d=1$ for example.

Substituting (7.5) in (2.11) one sees that due to formal complications alone such a solution could hardly have been directly discovered in flat space. The relevant domains and boundary conditions in flat space (necessary to ensure finite action) would also seem complicated and arbitrary unless one refers back to de Sitter space. Here the domains of t and r are determined by one single, simple criterion, namely the positive definiteness of the metric in the continued Kruskal coordinates.

The point that, instead of complicating things, de Sitter space can, in certain cases, lead to a gain in formal simplicity was already, previously made (Sec. IV of Ref. 1) with respect to meron-type solutions. Here the reason for working in de Sitter space is much deeper. As stated in the introduction one may hope to give these finite-action self-dual solutions a physical relevance in the context of de Sitter bag models. To what extent such a program can finally be carried out, of course, remains to be seen.

---

*On leave from Laboratoire de Physique Théorique de la Faculté des Sciences de Rabat (Maroc).

†Équipe de Recherche associée au C.N.R.S.

‡Laboratoire associé au C.N.R.S.

[1]H. Boutaleb-Joutei, A. Chakrabarti, and A. Comtet, preceding paper, Phys. Rev. D 20, 1884 (1979).

[2]M. K. Prasad and C. M. Sommerfield, Phys. Rev. Lett. 35, 760 (1975).

[3]S. L. Adler, Phys. Rev. D 18, 411 (1978); 19, 1168 (1979).

[4]Since in this paper we mostly study solutions for conformally flat metrics, let us mention one point which should probably be evident. Expressing (1.2) or (1.5) by suitable changes of variables in an explicitly conformally flat form, one can then (thanks to the invariance properties of the gauge fields) immediately utilize all the known flat-space solutions for these cases. This often leads to undesirable properties and is not of interest to us here (see, however, Secs. III and IV of Ref. 1). What we exploit here are the specific features that arise due to a particular choice of the Euclidean section for metrics in question. The resulting possibilities and properties are very different from those of the flat case. The result (1.4) is a typical example.

[5]S. Hawking, Nucl. Phys. B144, 349 (1978).

[6]See Appendix B of Ref. 1.

[7]See Sec. V of Ref. 1. The χ of the present paper corresponds to $\chi/2$ of Ref. 1.

[8]See Appendix B of Ref. 1 and H. Boutaleb-Joutei and A. Chakrabarti, Phys. Rev. D 19, 457 (1979).

[9]The easiest way is to exploit self-duality to avoid the explicit use of $g^{\mu\nu}$ in calculating this action. Thus, one can directly use, for example, the formulas (A10) and (A11) of Appendix A of A. Chakrabarti and A. Comtet, Phys. Rev. D 19, 3050 (1979). They now reduce to (since $a=c=0$ and time derivatives vanish) $\mathrm{Tr}(\vec{E}\cdot\vec{B}+\vec{B}\cdot\vec{E}) = (1/r^2)\,\{\partial_r[(b^2-1)d]\}$, where $b=e^{\chi}$, $d=N\chi' = \chi_*$. This is our (2.15). A direct calculation of $\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})$, of course, gives the same result. For yet another check using the R gauge, see (4.17) of the present paper.

[10]See the paper quoted in Ref. 9.

[11]E. Corrigan, D. B. Fairlie, R. G. Yates, and P. Goddard, Commun. Math. Phys. 58, 223 (1978).

[12]C. N. Yang, Phys. Rev. Lett. 38, 1377 (1977).

[13]See Eq. (5.34′) of Ref. 1.

[14]Not only constant factors as in (4.14) but also the transformation $\rho \to \rho\bar{f}$, $\bar{\rho} \to \bar{\rho} f$, $\lambda \to \lambda(f\bar{f})^{1/2}$, where $f=f(y, z)$ and $\bar{f}=\bar{f}(\bar{y};\bar{z})$ such that $f_{\bar{y}}=f_{\bar{z}}=0$ and so on, is evidently a symmetry of Eqs. (4.5). The effect of such a transformation on the A_μ's turns out to be the same as that of gauge transformation by the operator $S=e^{i\beta\tau_3/2}$, where $f=|f|\,e^{i\beta}$.

[15]See Eqs. (5.36) and (5.37) of Ref. 1.

[16]M. A. Löhe, Nucl. Phys. B142, 236 (1978).

[17]The ansatz (2.3) adapted to the RB space [specifically to the absence of the factor r^2 before $d\Omega$ in (5.2)] can be written explicitly in terms of the components corresponding to x, θ, φ as

$$A_0 = d\frac{\tau_x}{2}, \quad A_x = 0,$$

$$A_\theta = -(b-1)\frac{\tau_\varphi}{2\sin\theta},$$

$$\frac{A_\varphi}{\sin\theta} = (b-1)\frac{\tau_\theta}{2},$$

where

$$\tau_x = \begin{pmatrix} \cos\theta & \sin\theta e^{-i\varphi} \\ \sin\theta e^{+i\varphi} & -\cos\theta \end{pmatrix}, \quad \tau_\theta = \begin{pmatrix} -\sin\theta & \cos\theta e^{-i\varphi} \\ \cos\theta e^{i\varphi} & \sin\theta \end{pmatrix},$$

$$\tau_\varphi = \sin\theta \begin{pmatrix} 0 & -ie^{-i\varphi} \\ ie^{i\varphi} & 0 \end{pmatrix}.$$

The generalization for a and $c \neq 0$ [as in (5.2) of Ref. 1] is evident. In (2.3) τ_r (corresponding to τ_x) is unchanged while τ_θ, τ_φ each have an extra factor r.

[18]See Eq. (6.20) of Ref. 1.

[19]L. Witten, Phys. Rev. D 19, 718 (1979).